FORCED OSCILLATIONS OF SOLUTIONS OF
PARABOLIC EQUATIONS

NORIO YOSHIDA

Parabolic equations with forcing terms are studied and sufficient
conditions are given that all solutions of boundary value problems
are oscillatory in a cylindrical domain.

Recently there has been much interest in studying the oscillatory
behaviour of solutions of parabolic equations with functional arguments.
We refer the reader to Bykov and Kultaev [1], Kreith and Ladas [2] and
the author [3]. However, forced oscillations have not been discussed.

In this paper we are concerned with the forced oscillation of
solutions of the parabolic equation

\[ u_t - a(t)\Delta u + c(x,t,u(x,t),u(x,\sigma(t))) = f(x,t) \]

where \( \Delta \) is the Laplacian in Euclidean \( n \)-space \( \mathbb{R}^n \), \( \mathbb{R}^+_+ = [0,\infty) \) and \( \Omega \)
is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). It is assumed that

\( (A_1) \ a(t) \) is a nonnegative continuous function in \( \mathbb{R}^+_+ \) and \( f(x,t) \)
is a continuous function in \( \bar{\Omega} \times \mathbb{R}^+_+ \);

\( (A_2) \ c(x,t,\xi,\eta) \geq 0 \) for \( (x,t) \in \bar{\Omega} \times \mathbb{R}^+_+ , \ \xi \geq 0 , \ \eta \geq 0 , \) and

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\( \sigma(x,t,\xi,n) \leq 0 \) for \((x,t) \in \Omega \times \mathbb{R}_+^n, \xi \leq 0, n \leq 0; \)

(A3) \( \sigma(t) \) is a continuous function in \( \mathbb{R}_+ \) such that \( \lim_{t \to \infty} \sigma(t) = \infty. \)

Our objective is to present conditions which imply that every (classical) solution \( u \) of (1) satisfying a certain boundary condition is oscillatory in \( \Omega \times \mathbb{R}_+^n \) in the sense that \( u \) has a zero in \( \Omega \times [t,\infty) \) for any \( t > 0 \). We consider three kinds of boundary conditions:

(B1) \( u = \phi \) on \( \partial\Omega \times \mathbb{R}_+^n, \)

(B2) \( \frac{\partial u}{\partial n} = \psi \) on \( \partial\Omega \times \mathbb{R}_+^n, \)

(B3) \( \frac{\partial u}{\partial n} + au = 0 \) on \( \partial\Omega \times \mathbb{R}_+^n, \)

where \( \phi, \psi, a \) are continuous functions on \( \partial\Omega \times \mathbb{R}_+^n, \) \( n \) denotes the unit exterior normal vector to \( \partial\Omega \) and \( a \geq 0 \) on \( \partial\Omega \times \mathbb{R}_+^n. \)

It is known that the first eigenvalue \( \lambda_1 \) of the eigenvalue problem

\[ \begin{align*}
\Delta w + \lambda w &= 0 \quad \text{in} \quad \Omega \\
\frac{\partial w}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega
\end{align*} \]

is positive and the corresponding eigenfunction \( \phi \) is positive in \( \Omega \).

**Theorem 1.** Assume that (A1)-(A3) hold. Every solution \( u \) of the problem (1), (B1) is oscillatory in \( \Omega \times \mathbb{R}_+^n \) if

\[
\begin{align*}
\liminf_{s \to \infty} \int_0^s \exp(\lambda_1 A(t)) \left[ -a(t) \int_{\partial\Omega} \frac{\partial \phi}{\partial n} dx + \int_{\Omega} f(x,t) \phi dx \right] dt &= -\infty, \\
\limsup_{s \to \infty} \int_0^s \exp(\lambda_1 A(t)) \left[ -a(t) \int_{\partial\Omega} \frac{\partial \phi}{\partial n} dx + \int_{\Omega} f(x,t) \phi dx \right] dt &= \infty
\end{align*}
\]

for all large \( s \), where \( A(t) = \int_0^t a(\tau) d\tau \).

**Proof.** Suppose to the contrary that there is a solution \( u \) of the problem (1), (B1) which has no zero in \( \Omega \times [t_0,\infty) \) for some \( t_0 > 0 \). Let \( u > 0 \) in \( \Omega \times [t_0,\infty). \) Since \( \lim_{t \to \infty} \sigma(t) = \infty \), there is a number \( t_1 \) such that \( t_1 > t_0 \) and \( \sigma(t) \geq t_0 (t \geq t_1) \). Hence \( u(x,\sigma(t)) > 0 \) in
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\[ \Omega \times [t_1, \infty) \]. From assumption (A_2) we see that
\[ a(x,t,u(x,t),u(x,\sigma(t))) \geq 0 \]
in \( \Omega \times [t_1, \infty) \), and therefore
\[ u_t - a(t)u \leq f(x,t) \quad \text{in} \quad \Omega \times [t_1, \infty) . \]

Multiplying (2) by \( \phi \) and integrating over \( \Omega \), we obtain

\[ \frac{d}{dt} \int_{\Omega} u\phi \, dx - a(t)\int_{\Omega} (u\phi) \, dx \leq \int_{\Omega} f(x,t)\phi \, dx , \quad t \geq t_1 . \]

It follows from Green's formula that

\[ \int_{\Omega} (u\phi) \, dx = \int_{\partial \Omega} \left( \frac{\partial u\phi}{\partial n} - u \frac{\partial \phi}{\partial n} \right) \, d\omega + \int_{\Omega} u\phi \, dx \]

\[ = - \int_{\partial \Omega} \phi \frac{\partial \phi}{\partial n} \, d\omega - \lambda_1 \int_{\Omega} u\phi \, dx . \]

Combining (3) with (4) yields

\[ \frac{d}{dt} \int_{\Omega} u\phi \, dx + \lambda_1 a(t) \int_{\Omega} u\phi \, dx \leq -a(t) \int_{\partial \Omega} \phi \frac{\partial \phi}{\partial n} \, d\omega + \int_{\Omega} f(x,t)\phi \, dx , \]

which is equivalent to

\[ \exp(\lambda_1 A(t))U(t) ' \leq \exp(\lambda_1 A(t)) \left[ -a(t) \int_{\partial \Omega} \phi \frac{\partial \phi}{\partial n} \, d\omega + \int_{\Omega} f(x,t)\phi \, dx \right] , \]

where \( A(t) = \int_0^t a(\tau) \, d\tau \) and \( U(t) = \int_{\Omega} u\phi \, dx \). Integrating (5) over \([t_1, s] \), we obtain

\[ \exp(\lambda_1 A(s))U(s) - \exp(\lambda_1 A(t_1))U(t_1) \leq \int_{t_1}^s \exp(\lambda_1 A(t)) \left[ -a(t) \int_{\partial \Omega} \phi \frac{\partial \phi}{\partial n} \, d\omega + \int_{\Omega} f(x,t)\phi \, dx \right] \, dt . \]

The hypothesis implies that the right hand side of (6) is not bounded from below, and hence \( \exp(\lambda_1 A(s))U(s) \) cannot be eventually positive.

This contradicts the positivity of \( \exp(\lambda_1 A(s))U(s) \) (s \( \in [t_1, \infty) \)). If \( u < 0 \) in \( \Omega \times [t_0, \infty) \), \( v \equiv -u \) satisfies
\[
\frac{d}{dt} \int_{\Omega} v \phi \, dx + \lambda_1 a(t) \int_{\Omega} v \phi \, dx \leq -a(t) \int_{\Omega} (-\phi) \frac{\partial \psi}{\partial \nu} \, dx + \int_{\Omega} (-f(x,t)) \phi \, dx.
\]

Proceeding as in the case where \( u > 0 \), we are led to a contradiction. The proof is complete.

A special case of the problem (1), (B_1) is the following:

(7) \[ u_t - \Delta u + c(x,t,u(x,t),u(x,c(t))) = f(x,t), \quad (x,t) \in \Omega \times \mathbb{R}_+^1, \]

(8) \[ u = 0 \quad \text{on} \quad \partial \Omega \times \mathbb{R}_+^1. \]

**COROLLARY.** Assume that (A_1)-(A_3) hold. Every solution \( u \) of the problem (7), (8) is oscillatory in \( \Omega \times \mathbb{R}_+^1 \) if

\[
\liminf_{s \to \infty} \int_{s}^{\infty} \exp(\lambda_1 t) \left[ \int_{\Omega} f(x,t) \phi \, dx \right] dt = -\infty,
\]

\[
\limsup_{s \to \infty} \int_{s}^{\infty} \exp(\lambda_1 t) \left[ \int_{\Omega} f(x,t) \phi \, dx \right] dt = \infty
\]

for all large \( s \).

**Proof.** Since \( A(t) = t \) and \( \phi \equiv 0 \), the conclusion follows from Theorem 1.

**THEOREM 2.** Assume that (A_1)-(A_3) hold. Every solution \( u \) of the problem (1), (B_2) is oscillatory in \( \Omega \times \mathbb{R}_+^1 \) if

(9) \[ \liminf_{s \to \infty} \int_{s}^{\infty} \left[ a(t) \int_{\partial \Omega} \psi \, d\omega + \int_{\Omega} f(x,t) \, dx \right] dt = -\infty, \]

(10) \[ \limsup_{s \to \infty} \int_{s}^{\infty} \left[ a(t) \int_{\partial \Omega} \psi \, d\omega + \int_{\Omega} f(x,t) \, dx \right] dt = \infty \]

for all large \( s \).

**Proof.** Suppose that the problem (1), (B_2) has a solution \( u \) which has no zero in \( \Omega \times [t_0, \infty) \) for some \( t_0 > 0 \). We may suppose that \( u > 0 \) in \( \Omega \times [t_0, \infty) \). As in the proof of Theorem 1, we see that the inequality (2) holds. Integration of (2) over \( \Omega \) gives

\[
\frac{d}{dt} \int_{\Omega} u \, dx \leq a(t) \int_{\partial \Omega} \psi \, d\omega + \int_{\Omega} f(x,t) \, dx, \quad t \geq t_1.
\]
Arguing as in the proof of Theorem 1, we are led to a contradiction. The proof is complete.

**THEOREM 3.** Assume that (A 1)-(A 3) hold. Every solution u of the problem (1),(B 3) is oscillatory in \( \Omega \times \mathbb{R}_+ \) if

\[
\lim \inf_{s \to \infty} \int_{\tilde{s}}^{s} \left( \int_{\Omega} f(x,t) \, dx \right) \, dt = -\infty ,
\]

\[
\lim \sup_{s \to \infty} \int_{\tilde{s}}^{s} \left( \int_{\Omega} f(x,t) \, dx \right) \, dt = \infty
\]

for all large \( \tilde{s} \).

**Proof.** Let \( u \) be a solution of (1),(B 3), which has no zero in \( \Omega \times [t_0, \infty) \) for some \( t_0 > 0 \). We may suppose that \( u > 0 \) in \( \Omega \times [t_0, \infty) \).

Integrating (2) over \( \Omega \) and taking into account (B 3), we obtain

\[
\frac{d}{dt} \int_{\Omega} u \, dx \leq a(t) \int_{\partial \Omega} \frac{\partial u}{\partial n} \, d\omega + \int_{\Omega} f(x,t) \, dx
\]

\[
= -a(t) \int_{\partial \Omega} \psi \, d\omega + \int_{\Omega} f(x,t) \, dx \leq \int_{\Omega} f(x,t) \, dx , t \geq t_1 .
\]

The same argument as in the proof of Theorem 1 leads us to a contradiction.

**EXAMPLE 1.** We consider the problem

\[
u_t - u_{xx} + e^{\pi/2} u(x,t-\pi/2) = 2(\cos x)e^t \sin t , \quad (x,t) \in (0,\pi/2) \times \mathbb{R}_+ ,
\]

\[
- u_x(0,t) = 0 , \quad u_x(\pi/2,t) = -e^t \sin t , \quad t \in \mathbb{R}_+ .
\]

Here \( n = 1, a(t) \equiv 1, \Omega = (0,\pi/2), f(x,t) = 2(\cos x)e^t \sin t \) and

\[
\int_{\partial \Omega} \psi \, d\omega = -e^t \sin t . \quad \text{We easily see that}
\]

\[
\int_{\tilde{s}}^{s} \left( \int_{\partial \Omega} \psi \, d\omega + \int_{\Omega} f(x,t) \, dx \right) \, dt
\]

\[
= \int_{\tilde{s}}^{s} \psi \, e^t \sin t \, dt
\]

\[
= 2^{-1/2} e^{s \sin (s-\pi/4)} + 2^{-1} e^s (\cos \tilde{s} - \sin \tilde{s}) .
\]
Hence, we find that conditions (9) and (10) are satisfied. It follows from Theorem 2 that every solution $u$ of (13), (14) is oscillatory in $(0, \pi/2) \times \mathbb{R}_+$. One such solution is $u = (\cos x)e^t \sin t$.

**EXAMPLE 2.** We consider the problem

(15) \[ u_t - u_{xx} + e^{\pi/2}u(x,t-\pi/2) = (2\cos x + 1)e^t \cos t, \quad (x,t) \in (0,\pi) \times \mathbb{R}_+ \]

(16) \[ -u_x(0,t) = u_x(\pi,t) = 0, \quad t \in \mathbb{R}_+ \]

Here $n = 1$, $\alpha(t) \equiv 1$, $\Omega = (0,\pi)$ and $f(x,t) = (2\cos x + 1)e^t \cos t$. Since

\[
\int_0^\pi \left( \int_{\Omega} f(x,t)dx \right) dt = \int_0^\pi \pi e^t \cos t \ dt = 2^{1/2} \pi e^t \sin (\pi/4) - (\pi/2) e^t (\cos \pi/4 + \sin \pi/4),
\]

conditions (11) and (12) are satisfied. Theorem 3 implies that every solution $u$ of (15), (16) is oscillatory in $(0,\pi) \times \mathbb{R}_+$. In fact, there is an oscillatory solution $u = (\cos x + 1)e^t \cos t$ of the problem (15), (16).

**References**


Department of Mathematics
Faculty of Engineering
Iwate University
Morioka, Japan.