Abstract

We study the asymptotic behavior of the survival probability of a multi-type branching process in a random environment. In the one-dimensional situation, the class of processes considered corresponds to the strongly subcritical case. We also prove a conditional limit theorem describing the distribution of the number of particles in the process given its survival for a long time.

Keywords: Branching process; random environment; survival probability; conditional limit theorem

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1. Introduction and statement of results

In this paper we study the asymptotic properties of multi-type branching processes in random environments (BPREs). Informally speaking, such a process serves as a stochastic model for the evolution of a population with \( p \) different types of particle, where each particle may have children of all types and the distribution of the number of descendants changes from generation to generation in some random fashion.

In order to give a rigorous definition of multi-type BPREs and to formulate our results, we must introduce some notation. First, \( p \)-dimensional deterministic vectors with nonnegative coordinates and \( p \times p \) deterministic matrices with nonnegative entries are denoted by bold lower case symbols. In particular, \( \mathbf{0} := (0, 0, \ldots, 0) \) and \( \mathbf{1} := (1, 1, \ldots, 1) \). The standard basis vectors are denoted by \( e_i, i = 1, 2, \ldots, p \). For \( p \)-dimensional vectors \( \mathbf{x} = (x_1, \ldots, x^p) \) and \( \mathbf{y} = (y_1, \ldots, y^p) \), we set

\[
(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^p x^i y^i, \quad |\mathbf{x}| := \sum_{i=1}^p |x^i|, \quad \text{and} \quad \mathbf{x}^\mathbf{y} := \prod_{i=1}^p (x^i)^{y^i}.
\]

For every \( p \)-tuple \( (\mu_1, \mu_2, \ldots, \mu^p) \) of probability measures on \( \mathbb{Z}_+^p \), define its multidimensional generating function \( \mathbf{f} = (f^1, f^2, \ldots, f^p) \) by the relations

\[
f^i(s) := \sum_{\mathbf{z} \in \mathbb{Z}_+^p} s^{|\mathbf{z}|} \mu^i\{\mathbf{z}\}, \quad s \in [0, 1]^p, \quad i = 1, 2, \ldots, p.
\]

We call any sequence \( \{f_n, n \geq 1\} \) of multidimensional generating functions a varying environment, and interpret the corresponding measures \( \mu_{n,i}^i \) as the offspring law for a particle of type \( i \) in generation \( n - 1 \).
We now define a $p$-type branching process $\mathbf{Z}_n = (Z_1^n, Z_2^n, \ldots, Z_p^n)$, $n \geq 0$, where $Z_i^n$ is the number of type-$i$ particles in the process at moment $n$. This process has a deterministic starting point $\mathbf{Z}_0$ and the population sizes of the subsequent generations of the process are specified by the standard recursion

$$Z_i^n = \sum_{j=1}^{p} \sum_{k=1}^{Z_{i-1}^n} X_{n,j,k}^i, \quad i = 1, 2, \ldots, p, \quad n \geq 1,$$

where $X_{n,j,k} = (X_{n,j,k,1}, X_{n,j,k,2}, \ldots, X_{n,j,k,p})$, $k \geq 1$, are independent random vectors distributed according to $\mu_i^n$. Here and in what follows we denote random objects by upper case symbols using the respective bold symbols if the objects are vectors or matrices.

Equipping the set of all $p$-tuples $(\mu_1, \mu_2, \ldots, \mu_p)$ of probability measures with the total variation distance we obtain a metric space. Therefore, we may consider probability measures on this space. Because of the one-to-one correspondence between $p$-tuples of measures and $p$-dimensional generating functions, we can and do assume that we are given a probability measure on the space of generating functions. With this agreement in hand we call a sequence of independent, identically distributed (i.i.d.) random $p$-dimensional generating functions \( \{F_n = (F_{1n}, F_{2n}, \ldots, F_p^n), n \geq 1\} \) a random environment. We say that \( \{\mathbf{Z}_n, n \geq 0\} \) is a $p$-type BPRE, if its conditional distribution is determined by (1) for every fixed realization of the environmental sequence.

Branching processes in a random environment with one type of particle have been intensively investigated during the last two decades and their properties are well understood and described in the literature. The reader can find a modern and unified presentation of the corresponding results in a recent book by Kersting and Vatutin [9]. The multidimensional case is much less studied and many basic questions, e.g. a detailed classification, asymptotics of the survival probability, and the corresponding conditional limit theorems, are still not answered in full generality. For instance, only recently an asymptotic representation for the survival probability in critical multi-type BPREs was found under relatively general conditions (see [10] and [11]).

The purpose of this note is to study asymptotic properties of a class of subcritical multi-type BPREs, which correspond to the so-called strongly subcritical BPREs with one type of particle.

It is a general phenomenon that the asymptotic behavior of BPREs is mainly specified by the properties of certain basic characteristics of the environment. In the case of multi-type processes the crucial role is played by the (random) mean matrices

$$M_n = (M_{ij}^n)_{i,j=1}^p := \left( \frac{\partial F_i^n}{\partial s^j}(1) \right)_{i,j=1}^p, \quad n \geq 1.$$

If the $F_n$ are independent and distributed as a generating function $F = (F^1, \ldots, F^p)$ then, obviously, the $M_n$ are independent probabilistic copies of the random matrix

$$M = (M_{ij}^1)_{i,j=1}^p := \left( \frac{\partial F_i^1}{\partial s^j}(1) \right)_{i,j=1}^p.$$

We assume that the distribution of $M$ satisfies the following conditions.

(H0) $\mathbb{P}(\|M\| > 0) = 1$, where $\|M\|$ is the operator norm of $M$.

(H1) The set $\Theta := \{\theta > 0 : \mathbb{E}[\|M\|^\theta] < \infty\}$ is nonempty.
(H2) The support of the distribution of \( M \) acts strongly irreducibly on the semi-group of matrices with nonnegative entries, i.e. no proper finite union of subspaces of \( \mathbb{R}^p \) is invariant with respect to all elements of the multiplicative semi-group generated by the support of \( M \).

(H3) There exists a positive number \( \gamma > 1 \) such that

\[
1 \leq \max_{i,j} M_{i,j} \leq \gamma.
\]

This last assumption is rather strong. Unfortunately, we do not know how to proceed without this restriction.

We also need to consider the so-called Hessian matrices

\[
B(k) := \left( \frac{\partial^2 F_k}{\partial s^i \partial s^j} (1) \right)_{i,j=1}^p \quad \text{and} \quad B_n(k) := \left( \frac{\partial^2 F_k^n}{\partial s^i \partial s^j} (1) \right)_{i,j=1}^p, \quad k = 1, 2, \ldots, p,
\]

and the random variables

\[
T := \frac{1}{\| M \|^2} \sum_{k=1}^p \| B(k) \| \quad \text{and} \quad T_n := \frac{1}{\| M_n \|^2} \sum_{k=1}^p \| B_n(k) \|, \quad n = 1, 2, \ldots.
\]

Thus, \( T_n \) are independent probabilistic copies of \( T \). Along with conditions (H0)–(H3), we impose the following restriction on the distribution of \( T \).

(H4) There exists \( \epsilon > 0 \) such that

\[
\mathbb{E}[\| M \| \log T]^{1+\epsilon} < \infty.
\]

Using standard subadditivity arguments, we readily deduce that, for every \( \theta \in \Theta \), the limit

\[
\lambda(\theta) := \lim_{n \to \infty} \left( \mathbb{E}[\| M_n \cdots M_1 \|^\theta] \right)^{1/n} < \infty
\]

is well defined. This function is an analogue of the moment generating function for the associated random walk in the case of BPRES with a single type of particle. Set

\[
\Lambda(\theta) := \log \lambda(\theta), \quad \theta \in \Theta.
\]

**Theorem 1.** Assume that conditions (H0)–(H4) hold, that the point \( \theta = 1 \) belongs to the interior of the set \( \Theta \), and that \( \Lambda'(1) < 0 \). Then

(a) there exists a vector \( c = (c^1, \ldots, c^p) \) with strictly positive components such that

\[
P(|Z_n| > 0 \mid Z_0 = e_i) \sim c^i \lambda^n(1) \quad \text{as} \ n \to \infty; \tag{2}
\]

(b) for each \( s \in [0, 1]^P, \ s \neq 1 \),

\[
\lim_{n \to \infty} \mathbb{E}[s^{Z_n} \mid |Z_n| > 0; Z_0 = e_i] = \Phi_1(s),
\]

where \( \Phi_1(s) \) is the probability generating function of a proper distribution on \( \mathbb{Z}^P_+ \) given at (10) below.
If \( p = 1 \) then the assumption \( \Lambda'(1) < 0 \) reduces to \( \mathbb{E}[M \log M] < 0 \); one-type BPREs satisfying this condition are usually called **strongly subcritical**. Guivarch and Liu [7] studied the asymptotic behavior of the survival probability for one-type strongly subcritical processes; Geiger et al. [6] proved the corresponding conditional limit theorem for the distribution of the number of particles given survival. Dyakonova [3] obtained (2) under the assumption that all possible realizations of \( M \) have a common deterministic right eigenvector corresponding to the Perron–Frobenius root \( \varphi(M) \) of \( M \). In this special case the condition \( \Lambda'(1) < 0 \) reduces to the inequality \( \mathbb{E}[\varphi(M) \log \varphi(M)] < 0 \). Dyakonova [4] studied the case \( \mathbb{E}[\varphi(M) \log \varphi(M)] = 0 \) under the assumption that there exists a common left eigenvector corresponding to the Perron–Frobenius root of \( M \).

Our theorem and similar statements for strongly subcritical BPREs with one type of particle are very close to the classical results for subcritical Galton–Watson processes: for all these processes, there is purely geometric decay of the survival probability and a proper limit for the conditional distribution. The main difference between results for random and fixed environments consists in the fact that we cannot characterize the limit distribution in Theorem 1(b) by a functional equation.

2. Proof of Theorem 1

2.1. Representation for the generating function of the process

For every fixed realization of the environmental sequence \( F_n \) and \( k < n \), we define

\[
F_{k,n}(s) := F_{k+1} \circ F_{k+2} \circ \cdots \circ F_n(s), \quad F_{n,k}(s) := F_n \circ F_{n-1} \circ \cdots \circ F_{k+1}(s),
\]

and set

\[
F_{n,n}(s) := s.
\]

It is immediate from the definition of the process \( \{Z_n, n \geq 0\} \) that

\[
\mathbb{E}[s^{Z_n} | Z_0 = e_i, F_1, F_2, \ldots, F_n] = F_{0,n}(s),
\]

where \( F_{0,n}(s) := (F_{0,1}(s), \ldots, F_{0,n}(s)) \). Therefore,

\[
\mathbb{E}[s^{Z_n} | Z_0 = e_i] = \mathbb{E}[F_{0,n}(s)]. \tag{3}
\]

Set \( s = 0 \); then

\[
P(|Z_n| > 0 | Z_0 = e_i) = 1 - \mathbb{E}[F^i_{0,n}(0)] = \mathbb{E}[1 - F^i_{0,n}(0)].
\]

Let \( f \) be the generating function of the \( p \)-tuple \((\mu^1, \mu^2, \ldots, \mu^p)\), and let \( m \) be the corresponding mean matrix, i.e.

\[
m = \left( \frac{\partial f^j_i}{\partial x^j}(1) \right)_{i,j=1}^p.
\]

For any generating function \( f \) and matrix \( a \), define

\[
\psi_{f,a}(s) := \frac{|a|}{|a(1-f(s))|} - \frac{|a|}{|am(1-s)|},
\]

where, with an abuse of notation, \( | \cdot | \) here denotes the \( L_1 \)-norm of matrices.
Let \( a_i \) be the matrix with \( a^{i,i} = 1 \) and \( a^{k,l} = 0 \) for all \((k, l) \neq (i, i)\). Then, clearly,

\[
1 - F_{0,n}^i(s) = |a_i(1 - F_{0,n}(s))|.
\]

Using \( \psi \), we can write

\[
1 - F_{0,n}^i(s) = \frac{|a_i|}{|a_i|} |a_i(1 - F_{0,n}(s))| = \frac{|a_i|}{|a_i|} |a_i M_1(1 - F_{1,n}(s))| + \psi F_i a_i(F_{1,n}(s))
\]

\[
= \frac{1}{|a_i|} \psi F_i a_i(F_{1,n}(s)) + \psi F_i a_i(F_{1,n}(s)).
\]

Iterating this procedure yields

\[
1 - F_{0,n}^i(s) = \frac{1}{|a_i|} \psi F_i a_i R_{i-1}(F_{k,n}(s)),
\]

where \( R_0 := \text{Id} \) and \( R_k := M_1 M_2 \cdots M_k, \quad k \geq 1 \).

Recall now that \( F_k \) are i.i.d. random elements, so we can substitute \( F_k \leftrightarrow F_{n-k+1} \) and obtain

\[
\mathbb{E}[1 - F_{0,n}^i(s)] = \mathbb{E}[1 - F_{n,0}^i(s)]
\]

\[
= \mathbb{E}\left[ \left( \frac{1}{|a_i| L_{n,1}(1 - s)|} + \sum_{k=1}^{n} \frac{1}{|a_i| L_{n,n-k+1}|} \psi F_{n-k+1,a_i L_{n,n-k+2}}(F_{n-k,0}(s)) \right)^{-1} \right]
\]

\[
= \mathbb{E}\left[ \left( \frac{1}{|a_i| L_{n,1}(1 - s)|} + \sum_{k=1}^{n} \frac{1}{|a_i| L_{n,k}|} \psi F_{n,a_i L_{n,k+1}}(F_{k-1,0}(s)) \right)^{-1} \right],
\]

(4)

where

\[
L_{n,1} := \text{Id} \quad \text{and} \quad L_{n,k} := M_n M_{n-1} \cdots M_k, \quad 1 \leq k \leq n.
\]

We show eventually that, as \( n \to \infty \), the asymptotic behavior of \( 1 - F_{0,n}^i(s) \) is determined by the summands corresponding to bounded values of \( k \). To control \( \psi_{F_k,a_i R_k} \), we use Lemma 5 from [11]: under assumption (H3),

\[
0 \leq \psi_{F_k,a_i L_{n,k+1}}(F_{k-1,0}(s)) \leq \gamma p^2 T_k \quad \text{for all} \quad s \in [0, 1]^p,
\]

(5)

where we specify the value of \( \psi_{F_k,a_i L_{n,k+1}}(s) \) at \( s = 1 \) by continuity.

### 2.2. Exponential change of measure

Define

\[
\mathbb{S}_+ := \{ x \in \mathbb{R}_+^p : |x| = 1 \}.
\]

Then, for every matrix \( m \) with positive entries, the vector

\[
m \cdot x := \frac{m x}{|m x|}
\]

is the projection of \( m x \) on the set \( \mathbb{S}_+ \).

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Let $\mathcal{C}(\mathbb{S}_+)$ denote the set of all continuous functions on $\mathbb{S}_+$. For $\theta \in \Theta$, $g \in \mathcal{C}(\mathbb{S}_+)$, and $x \in \mathbb{S}_+$, define the operator

$$P_0 g(x) := \mathbb{E}[|Mx|^\theta g(M \cdot x)].$$

If conditions (H1)–(H3) hold then, according to Proposition 3.1 of [1], $\lambda(\theta)$ is the spectral radius of $P_0$ and there exist a unique strictly positive function $r_0 \in \mathcal{C}(\mathbb{S}_+)$ and a unique probability measure $l_0$ satisfying the scaling condition

$$\int_{\mathbb{S}_+} r_0(x) \, dl_0(x) = 1,$$

and such that

$$l_0 P_0 = \lambda(\theta) l_0, \quad P_0 r_0 = \lambda(\theta) r_0.$$

Following [2], introduce the functions

$$p_n^\theta(x, m) := \frac{|m x|^\theta r_0(m \cdot x)}{\lambda_n(\theta) r_0(x)}, \quad x \in \mathbb{S}_+.$$

It is easy to see that, for every $n \geq 1$, every $x \in \mathbb{S}_+$, and every matrix $m$,

$$\mathbb{E}[p_{n+1}^\theta(x, Mm)] = p_n^\theta(x, m)$$

and, in particular,

$$\mathbb{E}[p_n^\theta(x, L_{n,1})] = 1. \quad \text{(7)}$$

For each $n \geq 1$, denote by $\mathcal{F}_n$ the $\sigma$-algebra generated by random elements $Z_1, Z_2, \ldots, Z_n$ and $F_1, F_2, \ldots, F_n$. Let $\mathbb{1}_A$ be the indicator of the event $A$. It follows from (7) that

$$\mathbb{P}^\theta_n(A) := \mathbb{E}[p_n^\theta(x, L_{n,1}) \mathbb{1}_A]$$

is a probability measure on $\mathcal{F}_n$. Furthermore, (6) implies that $\{\mathbb{P}^\theta_n, n \geq 1\}$ is a sequence of consistent probability measures. It can be extended to a probability measure $\mathbb{P}^\theta$ on our original probability space $(\Omega, \mathcal{F})$.

We now take $\theta = 1$ and apply the corresponding change of measure to representation (3). Since $1 - F_{n,0}(s)$ is measurable with respect to $\mathcal{F}_n$, we can write

$$\mathbb{E}[1 - F_{n,0}^1(s)] = \lambda^n(1) r_1(e_1) \mathbb{E}\left[ p_n^1(e_1, L_{n,1}) \frac{1 - F_{n,0}^1(s)}{|L_{n,1} e_1| r_1(L_{n,1} \cdot e_1)} \right]$$

$$= \lambda^n(1) r_1(e_1) \mathbb{E}\left[ \frac{1 - F_{n,0}^1(s)}{|L_{n,1} e_1| r_1(L_{n,1} \cdot e_1)} \right].$$

Now apply (4), recall the definition of $a_i$, and use the equality $|a_i L_{n,k}| = |e_i L_{n,k}|$. Then

$$\mathbb{E}[1 - F_{n,0}^1(s)] = \lambda^n(1) r_1(e_1) \mathbb{E}\left[ \frac{1}{r_1(L_{n,1} \cdot e_1)} \frac{|a_i L_{n,1}(1 - s)|}{|L_{n,1} e_1|} \Xi_n(s) \right],$$

where

$$\Xi_n(s) := \left( 1 + \sum_{k=1}^n \frac{|a_i L_{n,1}(1 - s)|}{|e_i L_{n,k}|} \psi_{F_{k,n} a_i L_{n,k+1}}(F_{k-1,0}(s)) \right)^{-1}.$$
For brevity, write 
\[ \tilde{\psi}_{n,k}(s) := \psi(F_{n,k}, L_{n,k+1}, F_{k-1,0}(s)). \]

Fix some \( N \geq 1 \); we study the asymptotic behavior of the expectation 
\[ K_n(N; s) := \mathbb{E}_n \left[ \frac{1}{r_1(L_{n,1} \cdot e_i)} \frac{|a_i L_{n,1}(1-s)|}{|L_{n,1} e_i|} \mathbb{E}_n,N(s) \right], \]
where 
\[ \mathbb{E}_n,N(s) := \left( 1 + \sum_{k=1}^{N} \frac{|a_i L_{n,1}(1-s)|}{|e_i L_{n,k}|} \tilde{\psi}_{n,k}(s) \right)^{-1}. \]

For each \( k \leq N \), 
\[ \frac{|a_i L_{n,1}(1-s)|}{|e_i L_{n,k}|} = |a_i L_{n,N} L_{N-1,1}(1-s)| \]

By Theorem 1 of [8], there exist sequences of random numbers \( \{\lambda_n(N) > 0, n \geq 1\} \) and random vectors \( \{U_n(N), V_n(N), n \geq 1\} \) such that, as \( n \to \infty \),
\[ \frac{L_{n,N}}{\lambda_n(N)} = U_n(N) \otimes V_n(N) \to 0 \text{ almost surely (a.s.)}. \]

(Here and in what follows we agree to associate with vectors \( u = (u^1, \ldots, u^p) \) and \( v = (v^1, \ldots, v^p) \) the matrix \( u \otimes v = (u^i v^j)_{i,j=1}^p \).) Besides, as \( n \to \infty \), the sequence of random vectors \( \{U_n(N), V_n(N) / |V_n(N)|, n \geq 1\} \) converges weakly to a vector \( (U_n(N), V_n(N)) \). Consequently, as \( n \to \infty \), the sequence of ratios 
\[ \frac{|a_i L_{n,N} L_{N-1,1}(1-s)|}{|e_i L_{n,N} L_{N-1,k}|} = \frac{|a_i L_{n,N} L_{N-1,1}(1-s)/\lambda_n(N)| |V_n(N)|}{|e_i L_{n,N} L_{N-1,k}/\lambda_n(N)| |V_n(N)|} \]
converges weakly to 
\[ \frac{|a_i (U_n(N) \otimes V_n(N)) L_{N-1,1}(1-s)|}{|e_i (U_n(N) \otimes V_n(N)) L_{N-1,k}|}, \quad k \leq N. \]

By the same arguments, as \( n \to \infty \), the vectors \( \{\tilde{\psi}_{n,1}(s), \ldots, \tilde{\psi}_{n,N}(s)\} \) converge weakly to \( (\tilde{\psi}_{1}(s), \ldots, \tilde{\psi}_{N}(s)) \), and \( \tilde{r}_1(L_{n,1} \cdot e_i) \) converges weakly to a bounded random variable. Therefore, there exists 
\[ \lim_{n \to \infty} K_n(N; s) =: K'(N; s) \]
and the sequence \( K'(N; s) \) is decreasing in \( N \) for each fixed \( s = (s^1, \ldots, s^p) \).

Let \( A(s) = \{1 \leq j \leq p : s^j < 1\} \). Setting \( \Delta(s) := \min_{j \in A(s)} (1 - s^j) \) and using Lemma 2 of [5], it is not difficult to check that 
\[ \frac{\Delta(s)}{p^2 \gamma^2} \leq \frac{|a_i L_{n,1}(1-s)|}{|L_{n,1} e_i|} \leq \frac{|e_i L_{n,1}|}{|L_{n,1} e_i|} \leq \gamma^2 p^2, \quad n \geq 1. \]

Combining these estimates with the fact that \( \tilde{r}_1 \) is bounded away from 0, for some absolute constant \( C = C(s) \), we obtain the estimates 
\[ 0 \leq K_n(N; s) - \mathbb{E}_n \left[ \frac{1}{\tilde{r}_1(L_{n,1} \cdot e_i)} \frac{|a_i L_{n,1}(1-s)|}{|L_{n,1} e_i|} \mathbb{E}_n(s) \right] \]
\[ \leq C \mathbb{E}_n \left[ \frac{\sum_{k=N+1}^n \tilde{\psi}_k(s) |a_i L_{n,1}(1-s)| / |e_i L_{n,k}|}{1 + \sum_{k=N+1}^n \tilde{\psi}_k(s) |a_i L_{n,1}(1-s)| / |e_i L_{n,k}|}. \right] \]
Clearly,
\[ |a_iL_{n,1}(1-s)| = |a_iL_{n,k}L_{k-1,1}(1-s)| \leq |e_iL_{n,k}||L_{k-1,1}|, \quad k \leq n. \]

From this bound and (5), for some constant \( C_1 \), we obtain
\[ 0 \leq K_n(N; s) - E_1[1 - r_1(L_{n,1} \cdot e_i)|e_i| - \Xi_n(s)] \leq C_1 E_1[\sum_{k=N+1}^{\infty} \frac{||L_{k-1,1}\parallel_{T_k}}{1 + \sum_{k=N+1}^{\infty} ||L_{k-1,1}\parallel_{T_k}}] := K(N). \]

Suppose that we can show that the series
\[ \Psi := 1 + \sum_{k=1}^{\infty} ||L_{k-1,1}\parallel_{T_k} < \infty \quad \mathbb{P}^1\text{-a.s.} \quad (9) \]

Then we should have
\[ K(N) \to 0 \quad \text{as } N \to \infty. \]

Combining this with (8) also implies the existence of the limit
\[ \lim_{n \to \infty} E_1[\frac{1}{r_1(L_{n,1} \cdot e_i)} |a_iL_{n,1}(1-s)| - \Xi_n(s)] = \lim_{N \to \infty} K^i(N; s) := \phi_i(s). \]

But the limit on the right-hand side exists, because the sequence \( K^i(N; s) \) is decreasing in \( N \). Taking into account (9) and the estimate \( \Xi_n(s) \geq \Psi^{-1} \), we conclude that this limit is strictly positive for each \( s \in [0, 1]^p \), \( s \neq 1 \). In addition, \( \phi_i(1) = 0 \), since \( \Xi_n(s) \leq 1 \).

With these results to hand we deduce that, as \( n \to \infty \),
\[ \mathbb{P}(|Z_n| > 0 \mid Z_0 = e_i) = E[1 - F_{0,n}^i(0)] \sim \lambda^p(1) r_1(e_i) \phi_i(0) \]
and
\[ \lim_{n \to \infty} \mathbb{E}[|Z_n| \mid |Z_n| > 0; Z_0 = e_i] = 1 - \lim_{n \to \infty} \frac{E[1 - F_{0,n}^i(s)]}{E[1 - F_{0,n}^i(0)]} = 1 - \frac{\phi_i(s)}{\phi_i(0)} =: \Phi_i(s). \quad (10) \]

To complete the proof of the theorem, it remains to check that (9) holds.

Note first that our assumption that the point 1 belongs to the interior of \( \Theta \) proves the finiteness of \( E^1[\log ||M||] \). Moreover, by condition (H3),
\[ \min_{x \in S_+} \frac{|xM|}{|x|} \geq \gamma^{-1} ||M||. \]

Then all the conditions of Theorem 2 of [8] are satisfied and, therefore,
\[ \lim_{k \to \infty} \frac{\log ||L_{k,1}||}{k} = \Lambda'(1) < 0 \quad \mathbb{P}^1\text{-a.s.} \]

In particular, for every \( \varepsilon > 0 \),
\[ ||L_{k,1}|| = O(e^{-k^{1-\varepsilon}/2}) \quad \mathbb{P}^1\text{-a.s.} \quad (11) \]
By Markov’s inequality,
\[ P_1(\log T_k > k^{(1+\delta)/(1+\epsilon)}) \leq \frac{E_1[\log T_k^{1+\epsilon}]}{k^{1+\delta}}. \]

Now using the definition of the measure \( P_1 \), we obtain
\[ P_1(\log T_k > k^{(1+\delta)/(1+\epsilon)}) \leq C_2 \frac{E[\|M\|] \log T_k^{1+\epsilon}}{k^{1+\delta}} \]
for some constant \( C_2 \). Taking into account condition (H4) and applying the Borel–Cantelli lemma, we conclude that
\[ T_k = O(e^{k^{(1+\delta)/(1+\epsilon)}}) \] \( P_1 \)-a.s.

Combining this estimate with (11) and choosing sufficiently small \( \delta \), we obtain (9).

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References


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