

HYPERCYCLIC AND MIXING OPERATOR SEMIGROUPS

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Abstract We describe a class of topological vector spaces admitting a mixing uniformly continuous operator group $\{T_t\}_{t \in \mathbb{C}^n}$ with holomorphic dependence on the parameter t . This result builds on those existing in the literature. We also describe a class of topological vector spaces admitting no supercyclic strongly continuous operator semigroups $\{T_t\}_{t \geq 0}$.

Keywords: hypercyclic operators; supercyclic operators; hypercyclic semigroups; mixing semigroups

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1. Introduction

Unless stated otherwise, all vector spaces in this paper are over the field \mathbb{K} , being either the field \mathbb{C} of complex numbers or the field \mathbb{R} of real numbers, and all topological spaces are assumed to be Hausdorff. As usual, \mathbb{Z} is the set of integers, \mathbb{Z}_+ is the set of non-negative integers, \mathbb{N} is the set of positive integers and \mathbb{R}_+ is the set of non-negative real numbers. $L(X, Y)$ stands for the space of continuous linear operators from a topological vector space X to a topological vector space Y . We write $L(X)$ instead of $L(X, X)$ and X' instead of $L(X, \mathbb{K})$. X'_σ is X' with the weak topology σ , being the weakest topology on X' making the maps $f \mapsto f(x)$ from X' to \mathbb{K} continuous for all $x \in X$. For any $T \in L(X)$, the dual operator $T': X' \rightarrow X'$ is defined as usual: $(T'f)(x) = f(Tx)$ for $f \in X'$ and $x \in X$. Clearly, $T' \in L(X'_\sigma)$. For a subset A of a vector space X , $\text{span}(A)$ stands for the linear span of A . For brevity, we say *locally convex space* for a locally convex topological vector space. A subset B of a topological vector space X is called *bounded* if for any neighbourhood U of zero in X , a scalar multiple of U contains B . The topology τ of a topological vector space X is called *weak* if τ is exactly the weakest topology making each $f \in Y$ continuous for some linear space Y of linear functionals on X separating points of X . An \mathcal{F} -space is a complete metrizable topological vector space. A locally convex \mathcal{F} -space is called a *Fréchet space*. ω denotes the space of all sequences $\{x_n\}_{n \in \mathbb{Z}_+}$ in \mathbb{K} with coordinate-wise convergence topology. We denote by φ the linear subspace of ω consisting of sequences x with finite support $\{n \in \mathbb{Z}_+ : x_n \neq 0\}$. If X is a topological

vector space, then $A \subset X'$ is called *equicontinuous* if there is a neighbourhood U of zero in X such that $|f(x)| \leq 1$ for any $x \in U$ and $f \in A$.

Let X and Y be topological spaces and let $\{T_a : a \in A\}$ be a family of continuous maps from X to Y . An element $x \in X$ is called *universal* for this family if $\{T_ax : a \in A\}$ is dense in Y , and $\{T_a : a \in A\}$ is said to be *universal* if it has a universal element. An *operator semigroup* on a topological vector space X is a family $\{T_t\}_{t \in A}$ of operators from $L(X)$ labelled by elements of an abelian monoid A and satisfying $T_0 = I$, $T_{s+t} = T_t T_s$ for any $t, s \in A$. A *norm* on A is a function $|\cdot| : A \rightarrow [0, \infty)$ satisfying $|na| = n|a|$ and $|a+b| \leq |a| + |b|$ for any $n \in \mathbb{Z}_+$ and $a, b \in A$. An abelian monoid equipped with a norm is a *normed semigroup*. We are mainly concerned with the case when A is a closed additive subsemigroup of \mathbb{R}^k containing 0 with the norm $|a|$ being the Euclidean distance from a to 0. In the latter case A carries the topology inherited from \mathbb{R}^k , and an operator semigroup $\{T_t\}_{t \in A}$ is called *strongly continuous* if the map $t \mapsto T_t x$ from A to X is continuous for any $x \in X$. We say that an operator semigroup $\{T_t\}_{t \in A}$ is *uniformly continuous* if there is a neighbourhood U of zero in X such that for any sequence $\{t_n\}_{n \in \mathbb{Z}_+}$ in A converging to $t \in A$, $T_{t_n} x$ converges to $T_t x$ uniformly on U . Clearly, uniform continuity is strictly stronger than strong continuity. If A is a normed semigroup and $\{T_t\}_{t \in A}$ is an operator semigroup on a topological vector space X , then we say that $\{T_t\}_{t \in A}$ is *mixing* if, for any non-empty open subsets U, V of X , there is $r > 0$ such that $T_t(U) \cap V \neq \emptyset$ provided $|t| > r$. We say that $\{T_t\}_{t \in A}$ is *hypercyclic* (respectively, *supercyclic*) if the family $\{T_t : t \in A\}$ (respectively, $\{zT_t : z \in \mathbb{K}, t \in A\}$) is universal. $\{T_t\}_{t \in A}$ is said to be *hereditarily hypercyclic* if for any sequence $\{t_n\}_{n \in \mathbb{Z}_+}$ in A satisfying $|t_n| \rightarrow \infty$, $\{T_{t_n} : n \in \mathbb{Z}_+\}$ is universal. $T \in L(X)$ is called *hypercyclic*, *supercyclic*, *hereditarily hypercyclic* or *mixing* if the semigroup $\{T^n\}_{n \in \mathbb{Z}_+}$ has the same property. Hypercyclic and supercyclic operators have been intensely studied over the last few decades (see [2] and references cited therein). Recall that a topological space X is called a *Baire space* if the intersection of countably many dense open subsets of X is dense in X . By the classical Baire Theorem, complete metric spaces are Baire.

Proposition 1.1. *Let X be a topological vector space and let A be a normed semigroup. Then any hereditarily hypercyclic operator semigroup $\{T_a\}_{a \in A}$ on X is mixing. If X is Baire separable and metrizable, then the converse implication holds: any mixing operator semigroup $\{T_a\}_{a \in A}$ on X is hereditarily hypercyclic.*

The above proposition is a combination of well-known facts, and it appears in the literature in various modifications. We prove the proposition in the next section for the sake of completeness. It is worth noting that for any subsemigroup A_0 of A , not lying in the kernel of the norm, $\{T_t\}_{t \in A_0}$ is mixing if $\{T_t\}_{t \in A}$ is mixing. In particular, if $\{T_t\}_{t \in A}$ is mixing, then T_t is mixing whenever $|t| > 0$.

The question of existence of supercyclic or hypercyclic operators or semigroups on various types of topological vector spaces has been intensely studied. The fact that there are no hypercyclic operators on any finite-dimensional topological vector space goes back to Rolewicz [22]. The last result in this direction is due to Wengenroth [26], who proved that a hypercyclic operator on any topological vector space (locally convex or not) has no closed invariant subspaces of positive finite codimension, while any supercyclic operator

has no closed invariant subspaces of finite \mathbb{R} -codimension greater than 2. In particular, his result implies the (already well known by then) fact that there are no supercyclic operators on a finite-dimensional topological vector space of \mathbb{R} -dimension greater than 2. Herzog [18] proved that there is a supercyclic operator on any separable infinite-dimensional Banach space. Ansari [1] and Bernal-González [5], answering a question raised by Herrero, showed independently that any separable infinite-dimensional Banach space supports a hypercyclic operator. Using the same idea as in [1], Bonet and Peris [9] proved that there is a hypercyclic operator on any separable infinite-dimensional Fréchet space and demonstrated that there is a hypercyclic operator on the inductive limit X of a sequence $\{X_n\}_{n \in \mathbb{Z}_+}$ of separable Banach spaces provided X_0 is dense in X . Grivaux [16] observed that hypercyclic operators T in [1, 5, 9] are mixing and therefore hereditarily hypercyclic. They actually come from the same source. Namely, according to Salas [23], an operator of the shape $I + T$, where T is a backward-weighted shift on ℓ_1 , is hypercyclic. Virtually the same proof demonstrates that these operators are mixing. Moreover, all operators constructed in the above-cited papers are hypercyclic or mixing because of a quasisimilarity with an operator of the shape identity plus a backward-weighted shift. A similar idea was used by Bermúdez *et al.* [4] and Bernal-González and Grosse-Erdmann [6], who proved that any separable infinite-dimensional Banach space supports a hypercyclic strongly continuous semigroup $\{T_t\}_{t \in \mathbb{R}_+}$. Bermúdez *et al.* [3] proved that on any separable infinite-dimensional complex Banach space X , there is a mixing strongly continuous semigroup $\{T_t\}_{t \in \mathbb{C}}$ such that the map $t \mapsto T_t$ is holomorphic. Finally, Conejero [11] proved that any separable infinite-dimensional complex Fréchet space X that is non-isomorphic to ω supports a mixing operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ such that $T_{t_n}x$ uniformly converges to $T_t x$ for x from any bounded subset of X whenever $t_n \rightarrow t$.

Definition 1.2. We say that a topological vector space X belongs to the class \mathfrak{M}_0 if there is a dense subspace Y of X admitting a topology τ stronger than the one inherited from X and such that (Y, τ) is a separable \mathcal{F} -space. We say that X belongs to \mathfrak{M}_1 if there is a linearly independent equicontinuous sequence $\{f_n\}_{n \in \mathbb{Z}_+}$ in X' . Finally, $\mathfrak{M} = \mathfrak{M}_0 \cap \mathfrak{M}_1$.

Remark 1.3. Obviously, $X \in \mathfrak{M}_1$ if and only if there exists a continuous seminorm p on X such that $\ker p = p^{-1}(0)$ has infinite codimension in X . In particular, a locally convex space X belongs to \mathfrak{M}_1 if and only if its topology is not weak.

1.1. Results

The following theorem extracts the maximum from the method in terms of both the class of spaces and the class of semigroups. Although the general idea remains the same, the proof requires dealing with a number of technical details of various kinds.

Theorem 1.4. *Let $X \in \mathfrak{M}$. Then for any $k \in \mathbb{N}$, there exists a uniformly continuous hereditarily hypercyclic (and therefore mixing) operator group $\{T_t\}_{t \in \mathbb{K}^k}$ on X such that the map $z \mapsto f(T_z x)$ from \mathbb{K}^k to \mathbb{K} is analytic for each $x \in X$ and $f \in X'$.*

Since for any hereditarily hypercyclic semigroup $\{T_t\}_{t \in \mathbb{K}^k}$ and any non-zero $t \in \mathbb{K}^k$, T_t is hereditarily hypercyclic, Theorem 1.4 provides a hereditarily hypercyclic operator

on each $X \in \mathfrak{M}$. Obviously, any separable \mathcal{F} -space belongs to \mathfrak{M}_0 . It is well known [24] that the topology on a Fréchet space X differs from the weak topology if and only if X is infinite dimensional and it is non-isomorphic to ω . Thus any separable infinite-dimensional Fréchet space non-isomorphic to ω belongs to \mathfrak{M} . The latter fact is also implicitly contained in [9]. Similarly, an infinite-dimensional inductive limit X of a sequence $\{X_n\}_{n \in \mathbb{Z}_+}$ of separable Banach spaces belongs to \mathfrak{M} provided X_0 is dense in X . Thus all the above-mentioned existence theorems are particular cases of Theorem 1.4. The following proposition characterizes \mathcal{F} -spaces in the class \mathfrak{M} .

Proposition 1.5. *Let X be an \mathcal{F} -space. Then X belongs to \mathfrak{M} if and only if X is separable and the algebraic dimension of X' is uncountable.*

Proposition 1.5 ensures that Theorem 1.4 can be applied to a variety of \mathcal{F} -spaces (including ℓ_p with $0 < p < 1$) that are not locally convex. We briefly outline the main idea of the proof of Theorem 1.4 because it is barely recognizable in the main text, where the intermediate results are presented in much greater generality than is strictly necessary. Consider the completion of the k th projective tensor power of ℓ_1 : $X = \ell_1 \hat{\otimes} \cdots \hat{\otimes} \ell_1$ and $T_1, \dots, T_k \in L(X)$ of the shape $T_j = I \otimes \cdots \otimes I \otimes S_j \otimes I \otimes \cdots \otimes I$, where $S_j \in L(\ell_1)$ is a backward-weighted shift sitting in j th place. Since T_j are pairwise commuting, we have got a uniformly continuous operator group $\{e^{\langle z, T \rangle}\}_{z \in \mathbb{K}^k}$ on X , where $\langle z, T \rangle = z_1 T_1 + \cdots + z_k T_k$. We show that $\{e^{\langle z, T \rangle}\}_{z \in \mathbb{K}^k}$ is hereditarily hypercyclic. The class \mathfrak{M} turns out to be exactly the class of topological vector spaces to which such a group can be transferred by means of quasisimilarity.

The following theorem is kind of an opposite of Theorem 1.4.

Theorem 1.6. *There are no supercyclic strongly continuous operator semigroups $\{T_t\}_{t \in \mathbb{R}_+}$ on a topological vector space X if either $2 < \dim_{\mathbb{R}} X < 2^{\aleph_0}$ or $2 < \dim_{\mathbb{R}} X' < 2^{\aleph_0}$.*

Since $\dim \omega' = \aleph_0$, Theorem 1.6 implies that there are no supercyclic strongly continuous operator semigroups $\{T_t\}_{t \in \mathbb{R}_+}$ on ω , which is a stronger version of a result in [11]. This observation along with Theorem 1.4 imply the following curious result.

Corollary 1.7. *For a separable infinite-dimensional Fréchet space X , the following are equivalent:*

- (1) *for each $k \in \mathbb{N}$, there is a mixing uniformly continuous operator group $\{T_t\}_{t \in \mathbb{R}^k}$ on X ;*
- (2) *there is a supercyclic strongly continuous operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ on X ;*
- (3) *X is non-isomorphic to ω .*

2. Extended backward shifts

Godefroy and Shapiro [15] introduced the notion of a generalized backward shift. Namely, a continuous linear operator T on a topological vector space X is called a *generalized backward shift* if the union of $\ker T^n$ for $n \in \mathbb{N}$ is dense in X and $\ker T$ is one dimensional.

We say that T is an *extended backward shift* if the linear span of the union of $T^n(\ker T^{2n})$ is dense in X . Using an easy dimension argument [15] one can show that any generalized backward shift is an extended backward shift. It is worth noting [2, Theorem 2.2] that for any extended backward shift T , $I + T$ is mixing. We need a multi-operator analogue of this concept.

Let X be a topological vector space. We say that $T = (T_1, \dots, T_k) \in L(X)^k$ is an extended backward shift k -tuple (*EB S_k -tuple*) if $T_m T_j = T_j T_m$ for $1 \leq j, m \leq k$ and $\ker^\dagger(T)$ is dense in X , where

$$\ker^\dagger(T) = \text{span} \bigcup_{n \in \mathbb{N}^k} \mathcal{X}(n, T) \quad \text{and} \quad \mathcal{X}(n, T) = T_1^{n_1} \cdots T_k^{n_k} \left(\bigcap_{j=1}^k \ker T_j^{2n_j} \right). \tag{2.1}$$

2.1. Shifts on finite-dimensional spaces

The following two lemmas are implicitly contained in the proof of Theorem 5.2 in [13]. For the sake of convenience, we provide their proofs.

Lemma 2.1. *For each $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \{0\}$, the matrix*

$$A_{n,z} = \left\{ \frac{z^{j+k-1}}{(j+k-1)!} \right\}_{j,k=1}^n$$

is invertible.

Proof. Invertibility of $A_{n,1}$ is proved in [2, Lemma 2.7]. For $z \in \mathbb{C}$, consider the diagonal $n \times n$ matrix $D_{n,z}$ with the entries $(1, z, \dots, z^{n-1})$ on the main diagonal. Clearly,

$$A_{n,z} = z D_{n,z} A_{n,1} D_{n,z} \quad \text{for any } z \in \mathbb{C}. \tag{2.2}$$

Since $A_{n,1}$ and $D_{n,z}$ for $z \neq 0$ are invertible, $A_{n,z}$ is invertible for any $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \{0\}$. □

Lemma 2.2. *Let $n \in \mathbb{N}$, e_1, \dots, e_{2n} be the canonical basis of \mathbb{K}^{2n} , let $S \in L(\mathbb{K}^{2n})$ be defined by $Se_1 = 0$ and $Se_k = e_{k-1}$ for $2 \leq k \leq 2n$ and let P be the linear projection of \mathbb{K}^{2n} onto $E = \text{span}\{e_1, \dots, e_n\}$ along $F = \text{span}\{e_{n+1}, \dots, e_{2n}\}$. Then, for any $z \in \mathbb{K} \setminus \{0\}$ and $u, v \in E$, there exists a unique $x^z = x^z(u, v) \in \mathbb{K}^{2n}$ such that*

$$Px^z = u \quad \text{and} \quad Pe^{zS}x^z = v. \tag{2.3}$$

Moreover, for any bounded subset B of E and any $\varepsilon > 0$, there is $c = c(\varepsilon, B) > 0$ such that

$$\sup_{u,v \in B} |(x^z(u, v))_{n+j}| \leq c|z|^{-j} \quad \text{for } 1 \leq j \leq n \text{ and } |z| \geq \varepsilon, \tag{2.4}$$

$$\sup_{u,v \in B} |(e^{zS}x^z(u, v))_{n+j}| \leq c|z|^{-j} \quad \text{for } 1 \leq j \leq n \text{ and } |z| \geq \varepsilon. \tag{2.5}$$

In particular, $x^z(u, v) \rightarrow u$ and $e^{zS}x^z(u, v) \rightarrow v$ as $|z| \rightarrow \infty$ uniformly for u and v from any bounded subset of E .

Proof. Let $u, v \in E$ and $z \in \mathbb{K} \setminus \{0\}$. For $y \in \mathbb{K}^{2n}$ we set $\bar{y} = (y_{n+1}, \dots, y_{2n}) \in \mathbb{K}^n$. One easily sees that (2.3) is equivalent to the vector equation

$$A_{n,z} \bar{x}^z = w^z, \tag{2.6}$$

where $A_{n,z}$ is the matrix from Lemma 2.1 and $w^z = w^z(u, v) \in \mathbb{K}^n$ is defined as

$$w_j^z = v_{n-j+1} - \sum_{k=n-j+1}^n \frac{z^{k+j-n-1} u_k}{(k+j-n-1)!} \quad \text{for } 1 \leq j \leq n, \tag{2.7}$$

provided we set $x_j = u_j$ for $1 \leq j \leq n$. By Lemma 2.1, $A_{n,z}$ is invertible for any $z \neq 0$ and therefore (2.6) is uniquely solvable. Thus there exists a unique $x^z = x^z(u, v) \in \mathbb{K}^{2n}$ satisfying (2.3). It remains to verify (2.4) and (2.5). By (2.7), for any bounded subset B of E and any $\varepsilon > 0$, there is $a = a(\varepsilon, B) > 0$ such that

$$|(w^z(u, v))_j| \leq a|z|^{j-1} \quad \text{if } u, v \in B, |z| \geq \varepsilon \text{ and } 1 \leq j \leq n. \tag{2.8}$$

By (2.8), $\{D_{n,z}^{-1} w^z(u, v) : |z| \geq \varepsilon, u, v \in B\}$ and therefore

$$Q = \{A_{n,1}^{-1} D_{n,z}^{-1} w^z(u, v) : |z| \geq \varepsilon, u, v \in B\}$$

are bounded in \mathbb{K}^n . Since, by (2.6) and (2.2), $\bar{x}^z = A_{n,z}^{-1} w^z = z^{-1} D_{n,z}^{-1} A_{n,1}^{-1} D_{n,z}^{-1} w^z$, we have

$$(x^z(u, v))_{n+j} = \bar{x}_j^z \subseteq \{z^{-1} (D_{n,z}^{-1} y)_j : y \in Q\} \quad \text{if } |z| \geq \varepsilon \text{ and } u, v \in B.$$

Boundedness of Q implies that (2.4) is satisfied with some $c = c_1(\varepsilon, B)$. Finally, since, for $1 \leq j \leq n$, we have

$$(e^{zS} x^z)_{n+j} = \sum_{l=n+j}^{2n} \frac{z^{l-n-j} x_l^z}{(l-n-j)!},$$

there exists $c = c_2(\varepsilon, B)$ for which (2.5) is satisfied. Hence (2.5) and (2.4) hold with $c = \max\{c_1, c_2\}$. □

Corollary 2.3. *Let $n \in \mathbb{N}$, $E \subseteq \mathbb{K}^{2n}$ and $S \in L(\mathbb{K}^{2n})$ be as in Lemma 2.2. Then, for any $u, v \in E$ and any sequence $\{z_j\}_{j \in \mathbb{Z}_+}$ in \mathbb{K} satisfying $|z_j| \rightarrow \infty$, there exists a sequence $\{x_j\}_{j \in \mathbb{Z}_+}$ in \mathbb{K}^{2n} such that $x_j \rightarrow u$ and $e^{z_j S} x_j \rightarrow v$.*

We need the following multi-operator version of Corollary 2.3.

Lemma 2.4. *Let $k \in \mathbb{N}$, $n_1, \dots, n_k \in \mathbb{N}$; for each $j \in \{1, \dots, k\}$ let $e_1^j, \dots, e_{2n_j}^j$ be the canonical basis in \mathbb{K}^{2n_j} , let $E_j = \text{span}\{e_1^j, \dots, e_{2n_j}^j\}$ and let $S_j \in L(\mathbb{K}^{2n_j})$ be the backward shift: $S_j e_1^j = 0$ and $S_j e_l^j = e_{l-1}^j$ for $2 \leq l \leq 2n_j$. Also, let $X = \mathbb{K}^{2n_1} \otimes \dots \otimes \mathbb{K}^{2n_k}$, $E = E_1 \otimes \dots \otimes E_k$ and*

$$T_j \in L(X), \quad T_j = I \otimes \dots \otimes I \otimes S_j \otimes I \otimes \dots \otimes I \quad \text{for } 1 \leq j \leq k,$$

where S_j sits in the j th place. Finally, let $\{z_m\}_{m \in \mathbb{Z}_+}$ be a sequence in \mathbb{K}^k satisfying $|z_m| \rightarrow \infty$. Then, for any $u, v \in E$, there exists a sequence $\{x_m\}_{m \in \mathbb{Z}_+}$ in X such that $x_m \rightarrow u$ and $e^{(z_m, T)} x_m \rightarrow v$, where $\langle s, T \rangle = s_1 T_1 + \dots + s_k T_k$.

Proof. Let $\bar{\mathbb{K}} = \mathbb{K} \cup \{\infty\}$ be the one-point compactification of \mathbb{K} . Clearly, it is enough to show that any sequence $\{w_m\}$ in \mathbb{K}^k satisfying $|w_m| \rightarrow \infty$ has a subsequence $\{z_m\}$ for which the statement of the lemma is true. Since $\bar{\mathbb{K}}^k$ is compact and metrizable, we can, without loss of generality, assume that $\{z_m\}$ converges to $w \in \bar{\mathbb{K}}^k$. Since $|z_m| \rightarrow \infty$, the set $C = \{j : w_j = \infty\}$ is non-empty. Without loss of generality, we may also assume that $C = \{1, \dots, r\}$ with $1 \leq r \leq k$.

Denote by Σ the set of $(u, v) \in X^2$ for which there is a sequence $\{x_m\}_{m \in \mathbb{Z}_+}$ in X such that $x_m \rightarrow u$ and $e^{(z_m, T)} x_m \rightarrow v$. In this notation, the statement of the lemma is equivalent to the inclusion $E \times E \subseteq \Sigma$. Let $u_j \in E_j$ for $1 \leq j \leq k$ and $u = u_1 \otimes \dots \otimes u_k$. By Corollary 2.3, there exist sequences $\{x_{j,m}\}_{m \in \mathbb{Z}_+}$ and $\{y_{j,m}\}_{m \in \mathbb{Z}_+}$ in \mathbb{K}^{2n_j} such that, for $1 \leq j \leq r$,

$$x_{j,m} \rightarrow 0, \quad e^{(z_m)_j S_j} x_{j,m} \rightarrow u_j, \quad y_{j,m} \rightarrow u_j \quad \text{and} \quad e^{(z_m)_j S_j} y_{j,m} \rightarrow 0.$$

We put $x_{j,m} = e^{-w_j S_j} u_j$ and $y_{j,m} = u_j$ for $r < j \leq k$ and $m \in \mathbb{Z}_+$. Consider the sequences $\{x_m\}_{m \in \mathbb{Z}_+}$ and $\{y_m\}_{m \in \mathbb{Z}_+}$ in X defined by $x_m = x_{1,m} \otimes \dots \otimes x_{k,m}$ and $y_m = y_{1,m} \otimes \dots \otimes y_{k,m}$. By definition of x_m and y_m and the above display, $x_m \rightarrow 0$ and $y_m \rightarrow u$. For instance, $x_m \rightarrow 0$ because the sequences $\{x_{j,m}\}$ are bounded and $x_{1,m} \rightarrow 0$. Similarly, taking into account that $(z_m)_j \rightarrow w_j$ for $j > r$, we see that $e^{(z_m, T)} x_m \rightarrow u$ and $e^{(z_m, T)} y_m \rightarrow 0$. Hence $(u, 0) \in \Sigma$ and $(0, u) \in \Sigma$. Thus $(\{0\} \times E_0) \cup (E_0 \times \{0\}) \subseteq \Sigma$, where $E_0 = \{u_1 \otimes \dots \otimes u_k : u_j \in E_j, 1 \leq j \leq k\}$. On the other hand, $\text{span}(\{0\} \times E_0) \cup (E_0 \times \{0\}) = E \times E$. Since Σ is a linear space, $E \times E \subseteq \Sigma$. \square

For applications it is more convenient to reformulate the above lemma in the coordinate form.

Corollary 2.5. *Let $k \in \mathbb{N}$, $n_1, \dots, n_k \in \mathbb{N}$, $N_j = \{1, \dots, 2n_j\}$ and $Q_j = \{1, \dots, n_j\}$ for $1 \leq j \leq k$. Consider $M = N_1 \times \dots \times N_k$ and $M_0 = Q_1 \times \dots \times Q_k$, let $\{e_m : m \in M\}$ be the canonical basis of $X = \mathbb{K}^M$ and let $E = \text{span}\{e_m : m \in M_0\}$. For $1 \leq j \leq k$, let $T_j \in L(X)$ be defined by $T_j e_m = 0$ if $m_j = 1$ and $T_j e_m = e_{m'}$ if $m_j > 1$, where $m'_l = m_l$ if $l \neq j$, $m'_j = m_j - 1$. Then, for any sequence $\{z_m\}_{m \in \mathbb{Z}_+}$ in \mathbb{K}^k satisfying $|z_m| \rightarrow \infty$ and any $u, v \in E$, there is a sequence $\{x_m\}_{m \in \mathbb{Z}_+}$ in X such that $x_m \rightarrow u$ and $e^{(z_m, T)} x_m \rightarrow v$, where $\langle s, T \rangle = s_1 T_1 + \dots + s_k T_k$.*

2.2. The key lemma

Lemma 2.6. *Let X be a topological vector space and let $k \in \mathbb{N}$, $n \in \mathbb{N}^k$ and $A \in L(X)^k$ be such that $A_j A_l = A_l A_j$ for $1 \leq l, j \leq k$. Then, for each x from $\mathcal{A}(n, A)$ defined in (2.1), there is a common finite-dimensional invariant subspace Y for A_1, \dots, A_k such that for any sequence $\{z_m\}_{m \in \mathbb{Z}_+}$ in \mathbb{K}^k satisfying $|z_m| \rightarrow \infty$, there exist sequences $\{x_m\}_{m \in \mathbb{Z}_+}$ and $\{y_m\}_{m \in \mathbb{Z}_+}$ in Y for which*

$$x_m \rightarrow 0, \quad e^{A_{z_m}} x_m \rightarrow x, \quad y_m \rightarrow x \quad \text{and} \quad e^{A_{z_m}} y_m \rightarrow 0, \tag{2.9}$$

where $A_s = (s_1 A_1 + \dots + s_k A_k)|_Y$.

Proof. Since $x \in \varkappa(n, T)$, there is $y \in X$ such that $x = A_1^{n_1} \cdots A_k^{n_k} y$ and $A_j^{2n_j} y = 0$ for $1 \leq j \leq k$. Let $N_j = \{1, \dots, 2n_j\}$ and $Q_j = \{1, \dots, n_j\}$ for $1 \leq j \leq k$. Define $M = N_1 \times \cdots \times N_k$ and $M_0 = Q_1 \times \cdots \times Q_k$. Let $h_l = A_1^{2n_1 - l_1} \cdots A_k^{2n_k - l_k} y$ for $l \in M$ and let $Y = \text{span}\{h_l : l \in M\}$. Clearly, Y is finite dimensional and $A_j h_l = 0$ if $l_j = 1$, $A_j h_l = h_{l'}$ if $l_j > 1$, where $l'_r = l_r$ for $r \neq j$ and $l'_j = l_j - 1$. Hence Y is invariant for each A_j . Consider $J \in L(\mathbb{K}^M, Y)$ defined by $J e_l = h_l$ for $l \in M$. Also, let $E = \text{span}\{e_l : l \in M_0\}$ and $T_j \in L(\mathbb{K}^M)$ be as in Corollary 2.5. Taking into account the definition of T_j and the action of A_j on h_l , we see that $A_j J = J T_j$ for $1 \leq j \leq k$. Clearly, $n \in M_0$ and therefore $e_n \in E$. Since $x = A_1^{n_1} \cdots A_k^{n_k} y$, we have $x = h_n$. By Corollary 2.5, there exist sequences $\{u_m\}_{m \in \mathbb{Z}_+}$ and $\{v_m\}_{m \in \mathbb{Z}_+}$ in \mathbb{K}^M such that $u_m \rightarrow e_n$, $e^{\langle z_m, T \rangle} u_m \rightarrow 0$, $v_m \rightarrow 0$ and $e^{\langle z_m, T \rangle} v_m \rightarrow e_n$. Now let $y_m = J u_m$ and $x_m = J v_m$ for $m \in \mathbb{Z}_+$. Then $\{x_m\}$ and $\{y_m\}$ are sequences in Y . From the relations $A_j J = J T_j$ and the fact that \mathbb{K}^M and Y are finite dimensional, it follows that $x_m \rightarrow 0$, $y_m \rightarrow J e_n = x$, $e^{A z_m} x_m \rightarrow J e_n = x$ and $e^{A z_m} y_m \rightarrow 0$. Thus (2.9) is satisfied. \square

From now on, if $A = (A_1, \dots, A_k)$ is a k -tuple of continuous linear operators on a topological vector space X and $z \in \mathbb{K}^k$, we write

$$\langle z, A \rangle = z_1 A_1 + \cdots + z_k A_k.$$

We also use the following convention. Let X be a topological vector space and let $S \in L(X)$. By saying that e^S is well defined, we mean that for each $x \in X$, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} S^n x$$

converges in X and defines a continuous linear operator denoted e^S .

Corollary 2.7. *Let X be a topological vector space, let $k \in \mathbb{N}$ and let $A \in L(X)^k$ be a k -tuple of pairwise commuting operators such that, for any $z \in \mathbb{K}^k$, $e^{\langle z, A \rangle}$ is well defined. Then, for each x and y from the space $\ker^\dagger(A)$ defined in (2.1) and any sequence $\{z_m\}_{m \in \mathbb{Z}_+}$ in \mathbb{K}^k satisfying $|z_m| \rightarrow \infty$, there is a sequence $\{u_m\}_{m \in \mathbb{Z}_+}$ in X such that $u_m \rightarrow x$ and $e^{\langle z_m, A \rangle} u_m \rightarrow y$.*

Proof. Fix a sequence $\{z_m\}_{m \in \mathbb{Z}_+}$ in \mathbb{K}^k satisfying $|z_m| \rightarrow \infty$. Let Σ be the set of $(x, y) \in X^2$ for which there exists a sequence $\{u_m\}_{m \in \mathbb{Z}_+}$ in X such that $u_m \rightarrow x$ and $e^{\langle z_m, A \rangle} u_m \rightarrow y$. By Lemma 2.6, $\varkappa(n, A) \times \{0\} \subseteq \Sigma$ and $\{0\} \times \varkappa(n, A) \subseteq \Sigma$ for any $n \in \mathbb{N}^k$, where $\varkappa(n, A)$ is defined in (2.1). On the other hand, Σ is a linear subspace of $X \times X$. Thus

$$\ker^\dagger(A) \times \ker^\dagger(A) = \text{span} \bigcup_{n \in \mathbb{N}^k} ((\varkappa(n, A) \times \{0\}) \cup (\{0\} \times \varkappa(n, A))) \subseteq \Sigma.$$

\square

2.3. Mixing semigroups and extended backward shifts

We start by proving Proposition 1.1. Proposition G below is Proposition 1 in [17], while Theorem U below can be found in [17, pp. 348–349].

Proposition G. *Let X be a topological space and let $\mathcal{F} = \{T_\alpha : \alpha \in A\}$ be a family of continuous maps from X to X such that $T_\alpha T_\beta = T_\beta T_\alpha$ and $T_\alpha(X)$ is dense in X for any $\alpha, \beta \in A$. The set of universal elements for \mathcal{F} is then either empty or dense in X .*

Theorem U. *Let X be a Baire topological space, let Y be a second countable topological space and let $\{T_a : a \in A\}$ be a family of continuous maps from X into Y . The set of universal elements for $\{T_a : a \in A\}$ is then dense in X if and only if $\{(x, T_a x) : x \in X, a \in A\}$ is dense in $X \times Y$.*

Proof of Proposition 1.1. Assume that $\{T_t\}_{t \in A}$ is hereditarily hypercyclic. That is, $\{T_{t_n} : n \in \mathbb{Z}_+\}$ is universal for any sequence $\{t_n\}_{n \in \mathbb{Z}_+}$ in A satisfying $|t_n| \rightarrow \infty$. Applying this to $t_n = nt$ with $t \in A, |t| > 0$, we see that T_t is hypercyclic. Since any hypercyclic operator has dense range [17], $T_t(X)$ is dense in X if $|t| > 0$. Assume that $\{T_t\}_{t \in A}$ is non-mixing. Then there are non-empty open subsets U and V of X and a sequence $\{t_n\}_{n \in \mathbb{Z}_+}$ in A such that $|t_n| \rightarrow \infty$ and $|t_n| > 0, T_{t_n}(U) \cap V = \emptyset$ for each $n \in \mathbb{Z}_+$. Since T_{t_n} have dense ranges and commute, Proposition G implies that the set W of universal elements for $\{T_{t_n} : n \in \mathbb{Z}_+\}$ is dense in X . Hence we can pick $x \in W \cap U$. Since x is universal for $\{T_{t_n} : n \in \mathbb{Z}_+\}$, there is an $n \in \mathbb{Z}_+$ for which $T_{t_n} x \in V$. Hence $T_{t_n} x \in T_{t_n}(U) \cap V = \emptyset$. This contradiction completes the proof of the first part of Proposition 1.1.

Next, assume that X is Baire separable and metrizable, $\{T_t\}_{t \in A}$ is mixing and $\{t_n\}_{n \in \mathbb{Z}_+}$ is a sequence in A such that $|t_n| \rightarrow \infty$. By the definition of mixing, for any non-empty open subsets U and V of $X, T_{t_n}(U) \cap V \neq \emptyset$ for all sufficiently large $n \in \mathbb{Z}_+$. Hence $\{(x, T_{t_n} x) : x \in X, n \in \mathbb{Z}_+\}$ is dense in $X \times X$. By Theorem U, $\{T_{t_n} : n \in \mathbb{Z}_+\}$ is universal. □

Proposition 2.8. *Let X be a topological vector space and let $A = (A_1, \dots, A_k) \in L(X)^k$ be an EBS_k-tuple such that $e^{(z,A)}$ is well defined for $z \in \mathbb{K}^k$ and $\{e^{(z,A)}\}_{z \in \mathbb{K}^k}$ is an operator group. Then $\{e^{(z,A)}\}_{z \in \mathbb{K}^k}$ is mixing.*

Proof. Assume the contrary. We can then find non-empty open subsets U and V of X and a sequence $\{z_m\}_{m \in \mathbb{Z}_+}$ in \mathbb{K}^k such that $|z_m| \rightarrow \infty$ and $e^{(z_m,A)}(U) \cap V = \emptyset$ for each $m \in \mathbb{Z}_+$. Let Σ be the set of $(x, y) \in X^2$ for which there is a sequence $\{x_m\}_{m \in \mathbb{Z}_+}$ in X such that $x_m \rightarrow x$ and $e^{(z_m,A)} x_m \rightarrow y$. By Corollary 2.7, $\ker^\dagger(A) \times \ker^\dagger(A) \subseteq \Sigma$. Since A is an EBS_k-tuple, $\ker^\dagger(A)$ is dense in X and therefore Σ is dense in $X \times X$. In particular, Σ meets $U \times V$, which is not possible since $e^{(z_m,A)}(U) \cap V = \emptyset$ for any $m \in \mathbb{Z}_+$. This contradiction shows that $\{e^{(z,A)}\}_{z \in \mathbb{K}^k}$ is mixing. □

Theorem 2.9. *Let X be a separable Banach space and let $(A_1, \dots, A_k) \in L(X)^k$ be an EBS_k-tuple. Then $\{e^{(z,A)}\}_{z \in \mathbb{K}^k}$ is a hereditarily hypercyclic uniformly continuous operator group on X .*

Proof. Since the A_j are pairwise commuting and X is a Banach space, $\{e^{\langle z, A \rangle}\}_{z \in \mathbb{K}^k}$ is a uniformly continuous operator group. By Proposition 1.1, it suffices to verify that $\{e^{\langle z, A \rangle}\}_{z \in \mathbb{K}^k}$ is mixing. It remains to apply Proposition 2.8. \square

We will extend the above theorem to more general topological vector spaces. Recall that a subset A of a vector space is called *balanced* if $zx \in A$ whenever $x \in A$, $z \in \mathbb{K}$ and $|z| \leq 1$. A subset D of a topological vector space X is called a *disc* if D is convex, balanced and bounded. For a disc D , the space $X_D = \text{span}(D)$ is endowed with the norm, being the Minkowskii functional [24] of D . Boundedness of D implies that the norm topology of X_D is stronger than the topology inherited from X . D is called a *Banach disc* if the normed space X_D is complete. It is well known [8] that a compact disc is a Banach disc.

Lemma 2.10. *Let X be a topological vector space, let p be a continuous seminorm on X , let $D \subset X$ be a Banach disc, let q be the norm of X_D , let $k \in \mathbb{N}$ and let $A \in L(X)^k$ be a k -tuple of pairwise commuting operators. Assume also that $A_j(X) \subseteq X_D$ for $1 \leq j \leq k$ and that there is a $a > 0$ such that $q(A_j x) \leq ap(x)$ for any $x \in X$ and $1 \leq j \leq k$. Then, for each $z \in \mathbb{K}^k$, $e^{\langle z, A \rangle}$ is well defined. Moreover, $\{e^{\langle z, A \rangle}\}_{z \in \mathbb{K}^k}$ is a uniformly continuous operator group and the map $z \mapsto f(e^{\langle z, A \rangle} x)$ from \mathbb{K}^k to \mathbb{K} is analytic for any $x \in X$ and $f \in X'$. Furthermore, if X_D is separable and dense in X and B is an EBS_k -tuple, then $\{e^{\langle z, A \rangle}\}_{z \in \mathbb{K}^k}$ is hereditarily hypercyclic, where $B_j \in L(X_D)$ are restrictions of A_j to X_D .*

Proof. Since D is bounded, there is a $c > 0$ such that $p(x) \leq cq(x)$ for each $x \in X_D$. Since $q(A_j x) \leq ap(x)$ for each $x \in X$, we have $q(A_j A_l x) \leq ap(A_l x) \leq caq(A_l x) \leq ca^2 p(x)$. Iterating this argument, we see that

$$q(A_1^{n_1} \cdots A_k^{n_k} x) \leq c^{|n|-1} a^{|n|} p(x) \quad \text{for any } x \in X \text{ and } n \in \mathbb{Z}_+^k, |n| > 0, \tag{2.10}$$

where $|n| = n_1 + \cdots + n_k$. By (2.10), for each $x \in X$ and $z \in \mathbb{K}^k$, the series

$$\sum_{n \in \mathbb{Z}_+^k, |n| > 0} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1! \cdots n_k!} A_1^{n_1} \cdots A_k^{n_k} x \tag{2.11}$$

converges absolutely in the Banach space X_D . Since the series

$$\sum_{m=1}^{\infty} \frac{1}{m!} \langle z, A \rangle^m x$$

can be obtained from (2.11) by an appropriate ‘bracketing’, it is also absolutely convergent in X_D . Hence the last series converges in X and therefore the formula

$$e^{\langle z, A \rangle} x = \sum_{m=0}^{\infty} \frac{1}{m!} \langle z, A \rangle^m x$$

defines a linear operator on X . Next, representing $e^{\langle z, A \rangle} x - x$ by the series (2.11) and using (2.10), we obtain

$$q(e^{\langle z, A \rangle} x - x) \leq \frac{p(x)}{c} \sum_{n \in \mathbb{Z}_+^k, |n| > 0} \frac{|z_1|^{n_1} \cdots |z_k|^{n_k}}{n_1! \cdots n_k!} (ac)^{|n|} = \frac{p(x)}{c} (e^{ac\|z\|} - 1),$$

where $\|z\| = |z_1| + \dots + |z_k|$. By the above inequality, each $e^{\langle z, A \rangle}$ is continuous and $\{e^{\langle z, A \rangle}\}_{z \in \mathbb{K}^k}$ is uniformly continuous. The semigroup property follows in a standard way from the fact that A_j are pairwise commuting. Applying $f \in X'$ to the series (2.11) and, using (2.10), one immediately obtains the power series expansion of the map $z \mapsto f(e^{\langle z, A \rangle} x)$. Hence each $z \mapsto f(e^{\langle z, A \rangle} x)$ is analytic.

Assume now that X_D is separable and dense in X , that $B_j \in L(X_D)$ are restrictions of A_j to X_D and that $B = (B_1, \dots, B_k)$ is an EBS $_k$ -tuple. By Theorem 2.9, $\{e^{\langle z, B \rangle}\}_{z \in \mathbb{K}^k}$ is hereditarily hypercyclic. Since each $e^{\langle z, B \rangle}$ is the restriction of $e^{\langle z, A \rangle}$ to X_D and X_D is dense in X and carries a topology stronger than the one inherited from X , $\{e^{\langle z, A \rangle}\}_{z \in \mathbb{K}^k}$ is also hereditarily hypercyclic. \square

3. ℓ_1 -sequences, equicontinuous sets and the class \mathfrak{M}

Definition 3.1. We say that a sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ in a topological vector space X is an ℓ_1 -sequence if the series $\sum_{n=0}^\infty a_n x_n$ converges in X for each $a \in \ell_1$ and, for any neighbourhood U of 0 in X , there is $n \in \mathbb{Z}_+$ such that $D_n \subseteq U$, where

$$D_n = \left\{ \sum_{k=0}^\infty a_k x_{n+k} : a \in \ell_1, \|a\| \leq 1 \right\}.$$

If X is a locally convex space, the latter condition is satisfied if and only if $x_n \rightarrow 0$.

Lemma 3.2. Let $\{x_n\}_{n \in \mathbb{Z}_+}$ be an ℓ_1 -sequence in a topological vector space X . Then the closed balanced convex hull D of $\{x_n : n \in \mathbb{Z}_+\}$ is compact and metrizable. Moreover, $D = D'$, where

$$D' = \left\{ \sum_{n=0}^\infty a_n x_n : a \in \ell_1, \|a\|_1 \leq 1 \right\},$$

X_D is separable and $E = \text{span}\{x_n : n \in \mathbb{Z}_+\}$ is dense in the Banach space X_D .

Proof. Let $Q = \{a \in \ell_1 : \|a\|_1 \leq 1\}$ be endowed with the coordinate-wise convergence topology. Then Q is metrizable and compact as a closed subspace of $\mathbb{D}^{\mathbb{Z}_+}$, where $\mathbb{D} = \{z \in \mathbb{K} : |z| \leq 1\}$. Obviously, the map $\Phi : Q \rightarrow D'$, $\Phi(a) = \sum_{n=0}^\infty a_n x_n$ is onto. Using the definition of an ℓ_1 -sequence, one can in a routine way verify that Φ is continuous. Hence D' is compact and metrizable as a continuous image of a compact metrizable space. Thus D' , being also balanced and convex, is a Banach disc. Let $u \in X_{D'}$ and $a \in \ell_1$ be such that $u = \Phi(a)$. One can easily see that $p_{D'}(u_n - u) \rightarrow 0$, where $u_n = \sum_{k=0}^n a_k x_k$. Hence $u_n \rightarrow u$ in X . Moreover, if $u \in D'$, then the u_n are in the balanced convex hull of $\{x_n\}_{n \in \mathbb{Z}_+}$. Thus D is dense and closed in D' and therefore $D = D'$. Hence $p_D(u_n - u) \rightarrow 0$ for each $u \in X_D$. Since $u_n \in E$, E is dense in X_D and X_D is separable. \square

Lemma 3.3. Let X be a topological vector space. Then the following are equivalent:

- (1) $X \in \mathfrak{M}_0$;
- (2) there exists a Banach disc D in X with dense linear span such that X_D is separable;
- (3) there exists an ℓ_1 -sequence in X with dense linear span.

Proof. Obviously, Lemma 3.3(2) implies Lemma 3.3(1). Lemma 3.2 ensures that Lemma 3.3(3) implies Lemma 3.3(2). It remains to verify that Lemma 3.3(1) implies Lemma 3.3(3). Assume that $X \in \mathfrak{M}_0$. Then there is a dense linear subspace Y of X carrying its own topology τ stronger than the topology inherited from X such that $Y = (Y\tau)$ is a separable \mathcal{F} -space. Clearly, any ℓ_1 -sequence in Y with dense linear span is also an ℓ_1 sequence in X with dense linear span. Thus it suffices to find an ℓ_1 -sequence with dense linear span in Y . To this end, we pick a dense subset $A = \{y_n : n \in \mathbb{Z}_+\}$ of Y and a base $\{U_n\}_{n \in \mathbb{Z}_+}$ of neighbourhoods of 0 in Y such that each U_n is balanced and $U_{n+1} + U_{n+1} \subseteq U_n$ for $n \in \mathbb{Z}_+$. Pick a sequence $\{c_n\}_{n \in \mathbb{Z}_+}$ of positive numbers such that $x_n = c_n y_n \in U_n$ for each $n \in \mathbb{Z}_+$. It is now easy to demonstrate that $\{x_n\}_{n \in \mathbb{Z}_+}$ is an ℓ_1 -sequence in Y with dense span. \square

Lemma 3.4. *Let X be a separable metrizable topological vector space and let $\{f_n\}_{n \in \mathbb{Z}_+}$ be a linearly independent sequence in X' . There then exist sequences $\{x_n\}_{n \in \mathbb{Z}_+}$ in X and $\{\alpha_{k,j}\}_{k,j \in \mathbb{Z}_+, j < k}$ in \mathbb{K} such that $\text{span}\{x_k : k \in \mathbb{Z}_+\}$ is dense in X , $g_n(x_k) = 0$ for $n \neq k$ and $g_n(x_n) \neq 0$ for $n \in \mathbb{Z}_+$, where $g_n = f_n + \sum_{j < n} \alpha_{n,j} f_j$.*

Proof. Let $\{U_n\}_{n \in \mathbb{Z}_+}$ be a base of topology of X . We inductively construct sequences $\{\alpha_{k,j}\}_{k,j \in \mathbb{Z}_+, j < k}$ in \mathbb{K} and $\{y_n\}_{n \in \mathbb{Z}_+}$ in X such that, for any $k \in \mathbb{Z}_+$,

$$y_k \in U_k, \quad g_k(y_k) \neq 0 \text{ and } g_k(y_m) = 0 \text{ if } m < k, \text{ where } g_k = f_k + \sum_{j < k} \alpha_{k,j} f_j. \quad (3.1)$$

Let $g_0 = f_0$. Since $f_0 \neq 0$, there is $y_0 \in U_0$ such that $f_0(y_0) = g_0(y_0) \neq 0$. This provides us with the base of induction. Assume now that $n \in \mathbb{N}$ and that y_k and $\alpha_{k,j}$ with $j < k < n$ satisfying (3.1) are already constructed. According to (3.1), we can find $\alpha_{n,0}, \dots, \alpha_{n,n-1} \in \mathbb{K}$ such that $g_n(y_m) = 0$ for $m < n$, where $g_n = f_n + \sum_{j < n} \alpha_{n,j} f_j$. Since the f_j are linearly independent, $g_n \neq 0$ and therefore there is $y_n \in U_n$ such that $g_n(y_n) \neq 0$. This concludes the inductive procedure.

Using (3.1), one can choose a sequence $\{\beta_{k,j}\}_{k,j \in \mathbb{Z}_+, j < k}$ in \mathbb{K} such that $g_n(x_n) \neq 0$ for $n \in \mathbb{Z}_+$ and $g_n(x_k) = 0$ for $k \neq n$, where $x_k = y_k + \sum_{j < k} \beta_{k,j} y_j$. Since $y_n \in U_n$, $\{y_n : n \in \mathbb{Z}_+\}$ is dense in X . Hence $\text{span}\{x_n : n \in \mathbb{Z}_+\} = \text{span}\{y_n : n \in \mathbb{Z}_+\}$ is dense in X . \square

Lemma 3.5. *Let $X \in \mathfrak{M}_1$. There then exists a linearly independent equicontinuous sequence $\{f_n : n \in \mathbb{Z}_+\}$ in X' such that $\varphi \subseteq \{\{f_n(x)\}_{n \in \mathbb{Z}_+} : x \in X\}$.*

Proof. Since $X \in \mathfrak{M}_1$, there is a continuous seminorm p on X for which the normed space $X_p = X/\ker p$ with the norm $\|x + \ker p\| = p(x)$ is infinite dimensional. Since every infinite-dimensional normed space admits a biorthogonal sequence, we can choose sequences $\{x_n\}_{n \in \mathbb{Z}_+}$ in X and $\{g_n\}_{n \in \mathbb{Z}_+}$ in X'_p such that $\|g_n\| \leq 1$ for each $n \in \mathbb{Z}_+$ and $g_n(x_k + \ker p) = \delta_{n,k}$ for $n, k \in \mathbb{Z}_+$, where $\delta_{n,k}$ is the Kronecker delta. Now let $f_n : X \rightarrow \mathbb{K}$, $f_n(x) = g_n(x + \ker p)$. The above properties of g_n can be rewritten in terms of f_n in the following way: $|f_n(x)| \leq p(x)$ and $f_n(x_k) = \delta_{n,k}$ for any $n, k \in \mathbb{Z}_+$ and $x \in Y$. Since $f_n(x_k) = \delta_{n,k}$, we have $\varphi \subseteq \{\{f_n(x)\}_{n \in \mathbb{Z}_+} : x \in X\}$. By the inequality $|f_n(x)| \leq p(x)$, $\{f_n : n \in \mathbb{Z}_+\}$ is equicontinuous. \square

Lemma 3.6. *Let $X \in \mathfrak{M}$. Then there exist an ℓ_1 -sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ in X with dense linear span and an equicontinuous sequence $\{f_k\}_{k \in \mathbb{Z}_+}$ in X' such that $f_k(x_n) = 0$ if $k \neq n$ and $f_k(x_k) \neq 0$ for each $k \in \mathbb{Z}_+$.*

Proof. According to Lemma 3.3, there is a Banach disc D in X such that X_D is separable and dense in X . By Lemma 3.5, there is a linearly independent equicontinuous sequence $\{g_n\}_{n \in \mathbb{N}}$ in X' . Since X_D is dense in X , the functionals $g_n|_{X_D}$ on X_D are linearly independent. Applying Lemma 3.4 to the sequence $\{g_n|_{X_D}\}$, we find sequences $\{y_n\}_{n \in \mathbb{Z}_+}$ in X_D and $\{\alpha_{k,j}\}_{k,j \in \mathbb{Z}_+, j < k}$ in \mathbb{K} such that $E = \text{span}\{y_k : k \in \mathbb{Z}_+\}$ is dense in X_D , $h_n(y_k) = 0$ for $n \neq k$ and $h_n(y_n) \neq 0$ for $n \in \mathbb{Z}_+$, where $h_n = g_n + \sum_{j < n} \alpha_{n,j} g_j$. Consider $f_n = c_n h_n$, where $c_n = (1 + \sum_{j < n} |\alpha_{n,j}|)^{-1}$. Since $\{g_n : n \in \mathbb{N}\}$ is equicontinuous, $\{f_n : n \in \mathbb{N}\}$ is also equicontinuous. Next, let $x_n = b_n y_n$, where $b_n = 2^{-n} q(x_n)^{-1}$ and q is the norm of the Banach space X_D . Since x_n converges to 0 in X_D , $\{x_n\}_{n \in \mathbb{N}}$ is an ℓ_1 -sequence in X_D . Since X_D is dense in X , $\text{span}\{x_n : n \in \mathbb{Z}_+\} = E$ is dense in X_D , and the topology of X_D is stronger than the one inherited from X , $\{x_n\}_{n \in \mathbb{N}}$ is an ℓ_1 -sequence in X with dense linear span. Finally, since $f_n(x_k) = c_n b_k h_n(y_k)$, we see that $f_n(x_k) = 0$ if $n \neq k$ and $f_n(x_n) \neq 0$ for any $n \in \mathbb{Z}_+$. Thus all required conditions are satisfied. \square

3.1. Proof of Proposition 1.5

Let X be a separable \mathcal{F} -space. We have to show that X belongs to \mathfrak{M} if and only if $\dim X' > \aleph_0$.

First, assume that $X \in \mathfrak{M}$. Then there is a continuous seminorm p on X such that $X_p = X/\ker p$ is infinite dimensional. We endow X_p with the norm $\|x + \ker p\| = p(x)$. The dual X'_p of the normed space X_p is naturally contained in X' . Since the algebraic dimension of the dual of any infinite-dimensional normed space is at least 2^{\aleph_0} [8], we have $\dim X' \geq \dim X'_p \geq 2^{\aleph_0} > \aleph_0$.

Assume now that $\dim X' > \aleph_0$ and let $\{U_n\}_{n \in \mathbb{Z}_+}$ be a base of neighbourhoods of 0 in X . Then X' is the union of subspaces $Y_n = \{f \in X' : |f| \text{ is bounded on } U_n\}$ for $n \in \mathbb{Z}_+$. Since $\dim X' > \aleph_0$, we can pick $n \in \mathbb{Z}_+$ such that Y_n is infinite dimensional. Now let p be the Minkowskii functional of U_n . Then the open unit ball of p is exactly the balanced convex hull W of U_n . Since $U_n \subseteq W$, p is a continuous seminorm on X . Since each $f \in Y_n$ is bounded on W and Y_n is infinite dimensional, $X/\ker p$ is also infinite dimensional. Hence $X \in \mathfrak{M}_1$. Since X , as a separable \mathcal{F} -space, belongs to \mathfrak{M}_0 , we see that $X \in \mathfrak{M}$. The proof is complete.

4. Proof of Theorem 1.4

Let $X \in \mathfrak{M}$. By Lemma 3.6, there exist an ℓ_1 -sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ in X and an equicontinuous sequence $\{f_k\}_{k \in \mathbb{Z}_+}$ in X' such that $E = \text{span}\{x_n : n \in \mathbb{Z}_+\}$ is dense in X , $f_k(x_n) = 0$ if $k \neq n$ and $f_k(x_k) \neq 0$ for each $k \in \mathbb{Z}_+$. Since $\{f_k\}$ is equicontinuous, there is a continuous seminorm p on X such that each $|f_k|$ is bounded by 1 on the unit ball of p . Since $\{x_n\}$ is an ℓ_1 -sequence in X , Lemma 3.2 implies that the balanced convex closed hull D of $\{x_n : n \in \mathbb{Z}_+\}$ is a Banach disc in X . Let q be the norm of the Banach space X_D . Then $q(x_n) \leq 1$ for each $n \in \mathbb{Z}_+$.

Lemma 4.1. Let $\alpha, \beta: \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ be any maps and let $a = \{a_n\}_{n \in \mathbb{Z}_+} \in \ell_1$. The formula

$$Tx = \sum_{n \in \mathbb{Z}_+} a_n f_{\alpha(n)}(x) x_{\beta(n)} \quad (4.1)$$

then defines a continuous linear operator on X . Moreover, $T(X) \subseteq X_D$ and $q(Tx) \leq \|a\|p(x)$ for each $x \in X$, where $\|a\|$ is the ℓ_1 -norm of a .

Proof. Since $\{f_k\}$ is equicontinuous, $\{f_{\alpha(n)}(x)\}_{n \in \mathbb{Z}_+}$ is bounded for any $x \in X$. Since $\{x_n\}$ is an ℓ_1 -sequence and $a \in \ell_1$, the series in (4.1) converges for any $x \in X$ and therefore defines a linear operator on X . Moreover, if $p(x) \leq 1$, then $|f_k(x)| \leq 1$ for each $k \in \mathbb{Z}_+$. Since $q(x_m) \leq 1$ for $m \in \mathbb{Z}_+$, (4.1) implies that $q(Tx) \leq \|a\|$ if $p(x) \leq 1$. Hence $q(Tx) \leq \|a\|p(x)$ for each $x \in X$. It follows that T is continuous and takes values in X_D . \square

Fix a bijection $\gamma: \mathbb{Z}_+^k \rightarrow \mathbb{Z}_+$. By e_j we denote the element of \mathbb{Z}_+^k defined by $(e_j)_l = \delta_{j,l}$. For $n \in \mathbb{Z}_+^k$, we write $|n| = n_1 + \dots + n_k$. Let

$$\varepsilon_m = \min\{|f_{\gamma(n)}(x_{\gamma(n)})|: n \in \mathbb{Z}_+^k, |n| = m + 1\} \quad \text{for } m \in \mathbb{Z}_+.$$

Since $f_j(x_j) \neq 0$, $\varepsilon_m > 0$ for $m \in \mathbb{Z}_+$. Pick any sequence $\{\alpha_m\}_{m \in \mathbb{Z}_+}$ of positive numbers satisfying

$$\alpha_{m+1} \geq 2^m \alpha_m \varepsilon_m^{-1} \quad \text{for any } m \in \mathbb{Z}_+ \quad (4.2)$$

and consider the operators $A_j: X \rightarrow X$ defined by the formula

$$A_j x = \sum_{n \in \mathbb{Z}_+^k} \frac{\alpha_{|n|} f_{\gamma(n+e_j)}(x)}{\alpha_{|n|+1} f_{\gamma(n+e_j)}(x_{\gamma(n+e_j)})} x_{\gamma(n)} \quad \text{for } 1 \leq j \leq k.$$

By (4.2), the series defining A_j can be written as

$$A_j x = \sum_{n \in \mathbb{Z}_+^k} c_{j,n} f_{\gamma(n+e_j)}(x) x_{\gamma(n)} \quad \text{with } 0 < |c_{j,n}| < 2^{-|n|}$$

and therefore

$$\sum_{n \in \mathbb{Z}_+^k} |c_{j,n}| \leq C = \sum_{n \in \mathbb{Z}_+^k} 2^{-|n|}.$$

Then each A_j has shape (4.1) with $\|a\| \leq C$. By Lemma 4.1, $A_j \in L(X)$, $A_j(X) \subseteq X_D$ and $q(Tx) \leq Cp(x)$ for any $x \in X$. Using the definition of A_j and the equalities $f_m(x_j) = 0$ for $m \neq j$, it is easy to verify that $A_j A_l x_n = A_l A_j x_n$ for any $1 \leq j < l \leq k$ and $n \in \mathbb{Z}_+$. Indeed, for any $n \in \mathbb{Z}_+$, there is a unique $m \in \mathbb{Z}_+^k$ such that $n = \gamma(m)$. If either $m_j = 0$ or $m_l = 0$, we have $A_j A_l x_n = A_l A_j x_n = 0$. If $m_j \geq 1$ and $m_l \geq 1$, then

$$A_j A_l x_n = A_l A_j x_n = \frac{\alpha_{|m|-2}}{\alpha_{|m|}} x_{\gamma(m-e_j-e_l)}.$$

Since E is dense in X , A_1, \dots, A_n are pairwise commuting. By Lemma 2.10, $e^{\langle z, A \rangle}$ are well defined for $z \in \mathbb{K}^k$, $\{e^{\langle z, A \rangle}\}_{z \in \mathbb{K}^k}$ is a uniformly continuous operator group and the map $z \mapsto f(e^{\langle z, A \rangle} x)$ from \mathbb{K}^k to \mathbb{K} is analytic for any $x \in X$ and $f \in X'$. It remains to show that $\{e^{\langle z, A \rangle}\}_{z \in \mathbb{K}^k}$ is hereditarily hypercyclic. By Lemma 3.2, X_D is separable. According to Lemma 2.10, it suffices to prove that $B \in L(X_D)^k$ is an EBS_k -tuple, where B_j are restrictions of A_j to X_D . Clearly, the B_j commute as restrictions of commuting operators. Using the relations $f_m(x_j) = 0$ for $m \neq j$ and $f_j(x_j) \neq 0$, it is easy to see that the set $\mathcal{N}(m, B)$, defined in (2.1), contains $E_m = \text{span}\{x_{\gamma(n)} : n \in \mathbb{Z}_+^k, n_j \leq m_j - 1, 1 \leq j \leq k\}$ for each $m \in \mathbb{N}^k$. Hence $\ker^\dagger B$, defined in (2.1), contains E , which is dense in X_D by Lemma 3.2. Thus B is an EBS_k -tuple. The proof of Theorem 1.4 is complete.

5. Spaces without supercyclic semigroups $\{T_t\}_{t \in \mathbb{R}_+}$

Lemma 5.1. *Let X be a finite-dimensional topological vector space of \mathbb{R} -dimension greater than 2. There is then no supercyclic strongly continuous operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ on X .*

Proof. As is well known, any strongly continuous operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ on \mathbb{K}^n has shape $\{e^{tA}\}_{t \in \mathbb{R}_+}$, where $A \in L(\mathbb{K}^n)$. Assume the contrary. There then exist $n \in \mathbb{N}$ and $A \in L(\mathbb{K}^n)$ such that $\{e^{tA}\}_{t \in \mathbb{R}_+}$ is supercyclic and $\dim_{\mathbb{R}} \mathbb{K}^n > 2$. Since the e^{tA} are invertible and commute with each other, Proposition G implies that the set W of universal elements for $\{ze^{tA} : z \in \mathbb{K}, t \in \mathbb{R}_+\}$ is dense in \mathbb{K}^n . On the other hand, for each $c > 0$ and any $x \in \mathbb{K}^n$, from the restriction on n it follows that the closed set $\{ze^{tA}x : z \in \mathbb{K}, 0 \leq t \leq c\}$ is nowhere dense in \mathbb{K}^n (the smoothness of the map $(z, t) \mapsto ze^{tA}x$ implies that the topological dimension of $\{ze^{tA}x : z \in \mathbb{K}, 0 \leq t \leq c\}$ is less than that of \mathbb{K}^n). Hence each $x \in W$ is universal for $\{ze^{tA} : z \in \mathbb{K}, t > c\}$ for any $c > 0$. Now, if (a, b) is a subinterval of $(0, \infty)$, it is easy to see that the family $\{ze^{tkA} : z \in \mathbb{K}, a < t < b, k \in \mathbb{Z}_+\}$ contains $\{ze^{tA} : z \in \mathbb{K}, t > c\}$ for a sufficiently large $c > 0$. Hence for each $x \in W$, the set $\{ze^{tkA}x : z \in \mathbb{K}, a < t < b, k \in \mathbb{Z}_+\}$ is dense in \mathbb{K}^n . Since (a, b) is arbitrary and W is dense in \mathbb{K}^n , $\{t, x, ze^{tkA}x : t \in \mathbb{R}_+, z \in \mathbb{K}, x \in \mathbb{K}^n, k \in \mathbb{Z}_+\}$ is dense in $\mathbb{R}_+ \times \mathbb{K}^n \times \mathbb{K}^n$. By Theorem U, the family $\{F_{z,k} : z \in \mathbb{K}, k \in \mathbb{Z}_+\}$ of maps $F_{z,k} : \mathbb{R}_+ \times \mathbb{K}^n \rightarrow \mathbb{K}^n, F_{z,k}(t, x) = ze^{tkA}x$ has a dense set $U_0 \subset \mathbb{R}_+ \times \mathbb{K}^n$ of universal elements. Hence the projection U of U_0 onto \mathbb{K}^n is dense in \mathbb{K}^n . On the other hand, U is exactly the set of $x \in \mathbb{K}^n$ that are supercyclic for e^{tA} for some $t \in \mathbb{R}_+$. In particular, there is $t \in \mathbb{R}_+$ such that e^{tA} is supercyclic. This contradicts the fact [26] that there are no supercyclic operators on finite-dimensional spaces of real dimension greater than 2. \square

Remark 5.2. In the proof of Lemma 5.1 we have shown that a strongly continuous supercyclic operator semigroup on a finite-dimensional space must contain supercyclic operators. It is worth mentioning that Conejero *et al.* [12] proved that every T_t with $t > 0$ is hypercyclic for any hypercyclic strongly continuous operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ on an \mathcal{F} -space. Bernal-González and Grosse-Erdmann [6] gave an example of a supercyclic strongly continuous operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ on a real Hilbert space such that T_t is not supercyclic for t from a dense subset of \mathbb{R}_+ . It seems to remain unknown whether

T_t with $t > 0$ must all be supercyclic for every supercyclic strongly continuous operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ on a complex \mathcal{F} -space.

The following (trivial under the Continuum Hypothesis) result is Lemma 2 in [25].

Lemma 5.3. *Let (M, d) be a separable complete metric space, let X be a topological vector space, let $f: M \rightarrow X$ be a continuous map and let $\tau = \dim \operatorname{span} f(M)$. Then either $\tau \leq \aleph_0$ or $\tau = 2^{\aleph_0}$.*

Lemma 5.4. *Let $\{T_t\}_{t \in \mathbb{R}_+}$ be a strongly continuous operator semigroup on a topological vector space X , let $x \in X$ and let $C(x) = \operatorname{span}\{T_t x: t \in \mathbb{R}_+\}$. Then either $\dim C(x) < \aleph_0$ or $\dim C(x) = 2^{\aleph_0}$.*

Proof. By Lemma 5.3, either $\dim C(x) \leq \aleph_0$ or $\dim C(x) = 2^{\aleph_0}$. It remains to rule out the case $\dim C(x) = \aleph_0$. Assume that $\dim C(x) = \aleph_0$. Restricting the T_t to the invariant subspace $C(x)$, we can without loss of generality assume that $C(x) = X$. Thus $\dim X = \aleph_0$ and therefore X is the union of an increasing sequence $\{X_n\}_{n \in \mathbb{Z}_+}$ of finite-dimensional subspaces. First, we shall show that for each $\varepsilon > 0$, the space $X_\varepsilon = \operatorname{span}\{T_t x: t \geq \varepsilon\}$ is finite dimensional.

Let $\varepsilon > 0$ and $0 < \alpha < \varepsilon$. Then $[\alpha, \varepsilon]$ is the union of closed sets $A_n = \{t \in [\alpha, \varepsilon]: T_t x \in X_n\}$ for $n \in \mathbb{Z}_+$. By the Baire Category Theorem, there is $n \in \mathbb{Z}_+$ such that A_n has non-empty interior in $[\alpha, \varepsilon]$. Hence we can pick $a, b \in \mathbb{R}$ such that $\alpha \leq a < b \leq \varepsilon$ and $T_t x \in X_n$ for any $t \in [a, b]$. We shall show that $T_t x \in X_n$ for $t \geq a$. Assume it is not the case. Then the number $c = \inf\{t \in [a, \infty): T_t x \notin X_n\}$ belongs to $[b, \infty)$. Since $\{t \in \mathbb{R}_+: T_t x \in X_n\}$ is closed, $T_c x \in X_n$. Since $[a, b]$ is uncountable and the span of $\{T_t: t \in [a, b]\}$ is finite dimensional, we can pick $a \leq t_0 < t_1 < \dots < t_n \leq b$ and $c_1, \dots, c_{n-1} \in \mathbb{K}$ such that $T_{t_n} x = c_1 T_{t_1} x + \dots + c_{n-1} T_{t_{n-1}} x$. Since $T_c x \in X_n$, by definition of c , there is $t \in (c, c + t_n - t_{n-1})$ such that $T_t x \notin X_n$. Since $t > c \geq t_n$, the equality $T_{t_n} x = c_1 T_{t_1} x + \dots + c_{n-1} T_{t_{n-1}} x$ implies that

$$T_t x = T_{t-t_n} T_{t_n} x = T_{t-t_n} \sum_{j=1}^{n-1} c_j T_{t_j} x = \sum_{j=1}^{n-1} c_j T_{t-t_n+t_j} x \in X_n$$

because $a \leq t - t_n + t_j \leq c$ for $1 \leq j \leq n - 1$. This contradiction proves that $T_t x \in X_n$ for each $t \geq a$. Hence $X_\varepsilon \subseteq X_n$ and therefore X_ε is finite dimensional for each $\varepsilon > 0$.

Since $T_t(X) = T_t(C(x)) \subseteq X_t$, T_t has finite rank for any $t > 0$. Let $t > 0$. Since T_t has finite rank, $F_t = \ker T_t$ is a closed subspace of X of finite codimension. Clearly, F_t is T_s -invariant for each $s \in \mathbb{R}_+$. Passing to quotient operators, $S_s \in L(X/F_t)$, $S_s(u + F_t) = T_s u + F_t$, we get a strongly continuous semigroup $\{S_s\}_{s \in \mathbb{R}_+}$ on the finite-dimensional space X/F_t . Hence there is $A \in L(X/F_t)$ such that $S_s = e^{sA}$ for $s \in \mathbb{R}_+$. Thus each S_s is invertible and is a quotient of T_s , we obtain $\operatorname{rk} T_s \geq \operatorname{rk} S_s = \dim X/F_t = \operatorname{rk} T_t$ for any $t > 0$ and $s \geq 0$. Thus all T_t for $t > 0$ have the same rank $k \in \mathbb{N}$. Passing to the limit as $t \rightarrow 0$, we see that the identity operator $I = T_0$ is the strong operator topology limit of a sequence of rank k operators. Hence $\operatorname{rk} I \leq k$. That is, X is finite dimensional. This contradiction completes the proof. \square

Lemma 5.5. *Let X be a topological vector space in which the linear span of each metrizable compact subset has dimension less than 2^{\aleph_0} . Then, for any strongly continuous operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ on X and any $x \in X$, the space $C(x) = \text{span}\{T_t x : t \in \mathbb{R}_+\}$ is finite dimensional.*

Proof. Let $\{T_t\}_{t \in \mathbb{R}_+}$ be a strongly continuous operator semigroup on X and let $x \in X$. By strong continuity, $K_n = \{T_t x : 0 \leq t \leq n\}$ is compact and metrizable for any $n \in \mathbb{N}$. Hence $\dim E_n < 2^{\aleph_0}$ for any $n \in \mathbb{N}$, where $E_n = \text{span}(K_n)$. Since the sum of countably many cardinals strictly less than 2^{\aleph_0} is strictly less than 2^{\aleph_0} , $\dim C(x) \leq \sum_{n=1}^{\infty} \dim E_n < 2^{\aleph_0}$. By Lemma 5.4, $C(x)$ is finite dimensional. \square

Applying Lemma 5.1 if X is finite dimensional and Lemma 5.5 otherwise, we get the following result.

Corollary 5.6. *Let X be a topological vector space such that $\dim_{\mathbb{R}} X > 2$ and the linear span of each metrizable compact subset of X has dimension less than 2^{\aleph_0} . Then there is no strongly continuous supercyclic operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ on X .*

Corollary 5.7. *Let X be an infinite-dimensional topological vector space such that $\dim_{\mathbb{R}} X' > 2$ and in X'_σ the span of any compact metrizable subset has dimension less than 2^{\aleph_0} . Then there is no strongly continuous supercyclic operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ on X .*

Proof. Assume that there exists a supercyclic strongly continuous operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ on X . It is straightforward to verify that $\{T'_t\}_{t \in \mathbb{R}_+}$ is a strongly continuous semigroup on X'_σ . Pick any finite-dimensional subspace L of X'_σ such that $\dim_{\mathbb{R}} L > 2$. By Lemma 5.5, $E = \text{span}\{T'_t f : t \in \mathbb{R}_+, f \in L\}$ is finite dimensional. Since $L \subseteq E$, $\dim_{\mathbb{R}} E > 2$. Since E is T'_t -invariant for any $t \in \mathbb{R}_+$, its annihilator $F = \{x \in X : f(x) = 0 \text{ for any } f \in E\}$ is T_t -invariant for each $t \in \mathbb{R}_+$. Thus we can consider the quotient operators $S_t \in L(X/F)$, $S_t(x + F) = T_t x + F$. Then $\{S_t\}_{t \in \mathbb{R}_+}$ is a strongly continuous operator semigroup on X/F . Moreover, $\{S_t\}_{t \in \mathbb{R}_+}$ is supercyclic since $\{T_t\}_{t \in \mathbb{R}_+}$ is. Now since $\dim E = \dim X/F$, $2 < \dim_{\mathbb{R}} X/F < \aleph_0$. By Lemma 5.1, there are no strongly continuous supercyclic operator semigroups on X/F . This contradiction completes the proof. \square

Proof of Theorem 1.6. Theorem 1.6 follows immediately from Corollaries 5.6 and 5.7. \square

6. Examples, remarks and questions

Note that if $(X, \tau) \in \mathfrak{M}$ is locally convex, then $(X, \theta) \in \mathfrak{M}$ for any locally convex topology θ on X such that $\theta \neq \sigma(X, X')$ and (X, θ) has the same dual X' as (X, τ) . This is an easy application of the Mackey–Arens Theorem [24]. Moreover, if $(X, \tau) \in \mathfrak{M}$ is locally convex, the hereditarily hypercyclic uniformly continuous group from Theorem 1.4 is strongly continuous and hereditarily hypercyclic on X equipped with the weak topology.

Unfortunately, the nature of the weak topology does not allow such a semigroup to be made uniformly continuous.

Assume now that X is an infinite-dimensional separable \mathcal{F} -space. If $\dim X' > \aleph_0$, Proposition 1.5 and Theorem 1.4 provide uniformly continuous hereditarily hypercyclic operator groups $\{T_t\}_{t \in \mathbb{K}^k}$ on X . If $2 < \dim_{\mathbb{R}} X' \leq \aleph_0$, Theorem 1.6 does not allow a supercyclic strongly continuous operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$ on X . Similarly, if $1 \leq \dim X' \leq \aleph_0$, there are no hypercyclic strongly continuous operator semigroups $\{T_t\}_{t \in \mathbb{R}_+}$ on X . It leaves unexplored the case $X' = \{0\}$.

Question 6.1. *Characterize infinite-dimensional separable \mathcal{F} -spaces X such that $X' = \{0\}$ and X admits a hypercyclic strongly continuous operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$. In particular, is it true that an \mathcal{F} -space X with $X' = \{0\}$ supporting a hypercyclic operator also supports a hypercyclic strongly continuous operator semigroup $\{T_t\}_{t \in \mathbb{R}_+}$?*

Recall that an infinite-dimensional topological vector space X is called *rigid* if $L(X)$ consists only of the operators of the form λI for $\lambda \in \mathbb{K}$. Since there exist rigid separable \mathcal{F} -spaces [19], there are separable infinite-dimensional \mathcal{F} -spaces which support no cyclic operators or cyclic strongly continuous operator semigroups $\{T_t\}_{t \in \mathbb{R}_+}$. Of course, $X' = \{0\}$ if X is rigid. We show that the equality $X' = \{0\}$ for an \mathcal{F} -space is not an obstacle for having uniformly continuous hereditarily hypercyclic operator groups. The spaces we consider are $L_p[0, 1]$ for $0 \leq p < 1$.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with μ being σ -finite. Recall that if $0 < p < 1$, then $L_p(\Omega, \mu)$ consists of (classes of equivalence up to being equal almost everywhere with respect to μ of) measurable functions $f: \Omega \rightarrow \mathbb{K}$ satisfying $q_p(f) = \int_{\Omega} |f(x)|^p \lambda(dx) < \infty$ with the topology defined by the metric $d_p(f, g) = q_p(f - g)$. The space $L_0(\Omega, \mu)$ consists of (equivalence classes of) all measurable functions $f: \Omega \rightarrow \mathbb{K}$ with the topology defined by the metric $d_0(f, g) = q_0(f - g)$, where

$$q_0(h) = \sum_{n=0}^{\infty} \frac{2^{-n}}{\mu(\Omega_n)} \int_{\Omega_n} \frac{|f(x)|}{1 + |f(x)|} \mu(dx)$$

and $\{\Omega_n\}_{n \in \mathbb{Z}_+}$ is a sequence of measurable subsets of Ω such that $\mu(\Omega_n) < \infty$ for each $n \in \mathbb{Z}_+$ and Ω is the union of Ω_n . Although d_0 depends on the choice of $\{\Omega_n\}$, the topology defined by this metric does not depend on this choice. If $\Omega = [0, 1]^k$ or $\Omega = \mathbb{R}^k$ and μ is the Lebesgue measure, we omit the notation for the underlying measure and σ -algebra and simply write $L_p([0, 1]^k)$ or $L_p(\mathbb{R}^k)$. We also replace $L_p([0, 1])$ by $L_p[0, 1]$. Note [19] that $X = L_p[0, 1]$ for $0 \leq p < 1$ is a separable infinite-dimensional \mathcal{F} -space satisfying $X' = \{0\}$. Moreover, for any $p \in [0, 1)$ and $k \in \mathbb{N}$, $L_p([0, 1]^k)$ is isomorphic to $L_p[0, 1]$ and $L_p(\mathbb{R}^k)$ is isomorphic to $L_p[0, 1]$.

Example 6.2. Let $0 < p < 1$, let $X = L_p([0, 1]^k)$ and let $T_j \in L(X)$ be defined by the formula

$$T_j f(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n) = f(x_1, \dots, x_{j-1}, \frac{1}{2}x_j, x_{j+1}, \dots, x_n), \quad 1 \leq j \leq k.$$

Then $\{e^{(t, T)}\}_{t \in \mathbb{K}^k}$ is a uniformly continuous and hereditarily hypercyclic operator group.

Proof. That the T_j are pairwise commuting, that $e^{\langle t, T \rangle}$ is well defined for each $t \in \mathbb{K}^k$ and that $\{e^{\langle t, T \rangle}\}_{t \in \mathbb{K}^k}$ is a uniformly continuous operator group are easily verified. Moreover, T is an EBS_k -tuple. Namely, $\ker^\dagger T$ consists of all $f \in X$ that vanish in a neighbourhood of $(0, \dots, 0)$, and therefore it is dense. By Corollary 2.8, $\{e^{\langle t, T \rangle}\}_{t \in \mathbb{K}^k}$ is mixing. By Proposition 1.1, $\{e^{\langle t, T \rangle}\}_{t \in \mathbb{K}^k}$ is hereditarily hypercyclic. \square

It is worth noting that the above example does not work for $X = L_0([0, 1]^k)$: $e^{\langle t, T \rangle}$ is not well defined for each non-zero $t \in \mathbb{K}^k$. Nevertheless, we can produce a strongly continuous hereditarily hypercyclic operator group $\{T_t\}_{t \in \mathbb{R}^k}$ on $L_0(\mathbb{R}^k)$.

Example 6.3. Let $k \in \mathbb{N}$, let $X = L_0(\mathbb{R}^k)$ and, for each $t \in \mathbb{R}^k$, let $T_t \in L(X)$ be defined by the formula $T_t f(x) = f(x - t)$. Then $\{T_t\}_{t \in \mathbb{R}^k}$ is a strongly continuous hereditarily hypercyclic operator group.

Proof. The fact that $\{T_t\}_{t \in \mathbb{R}^k}$ is a strongly continuous operator group is obvious. Pick a sequence $\{t_n\}_{n \in \mathbb{Z}_+}$ of vectors in \mathbb{R}^k such that $|t_n| \rightarrow \infty$ as $n \rightarrow \infty$. Clearly, the space E of functions from X with bounded support is dense in X . It is easy to see that $T_{t_n} f \rightarrow 0$ and $T_{t_n}^{-1} f = T_{-t_n} f \rightarrow 0$ for each $f \in E$. Hence $\{T_{t_n} : n \in \mathbb{Z}_+\}$ satisfies the universality criterion from [7]. Thus $\{T_{t_n} : n \in \mathbb{Z}_+\}$ is universal and therefore $\{T_t\}_{t \in \mathbb{R}^k}$ is hereditarily hypercyclic. \square

Since $L_p([0, 1]^k)$ and $L_p(\mathbb{R}^k)$ are isomorphic to $L_p[0, 1]$, we obtain the following corollary.

Corollary 6.4. *Let $k \in \mathbb{N}$ and $0 \leq p < 1$. Then there exists a hereditarily hypercyclic strongly continuous operator group $\{T_t\}_{t \in \mathbb{R}^k}$ on $L_p[0, 1]$.*

Ansari [1] asked whether the $L_p[0, 1]$ for $0 \leq p < 1$ support hypercyclic operators. This question was answered affirmatively by Grosse-Erdmann [17, Remark 4b]. Corollary 6.4 provides a ‘very strong’ affirmative answer to the same question. Finally, we would like to mention a class of topological vector spaces very different from the spaces in \mathfrak{M} in terms of operator semigroups. Recall that operator semigroups from Theorem 1.4 on spaces $X \in \mathfrak{M}$ depend analytically on the parameter: the map $t \mapsto f(T_t x)$ from \mathbb{K}^k to \mathbb{K} is analytic for any $x \in X$ and $f \in X'$.

Proposition 6.5. *Let a locally convex space X be the union of a sequence of its closed linear subspaces $\{X_n\}_{n \in \mathbb{Z}_+}$ such that $X_n \neq X$ for each $n \in \mathbb{Z}_+$. Assume also that $\{T_t\}_{t \in \mathbb{R}_+}$ is a strongly continuous operator semigroup such that the function $t \mapsto f(T_t x)$ from \mathbb{R}_+ to \mathbb{K} is real-analytic for any $x \in X$ and $f \in X'$. Then $\{T_t\}_{t \in \mathbb{R}_+}$ is non-cyclic.*

Proof. Let $x \in X$. Clearly, \mathbb{R}_+ is the union of closed sets $A_n = \{t \in \mathbb{R}_+ : T_t x \in X_n\}$ for $n \in \mathbb{Z}_+$. By the Baire Theorem, there is $n \in \mathbb{Z}_+$ such that A_n contains an interval (a, b) . Now let $f \in X'$ be such that $X_n \subseteq \ker f$. The function $t \mapsto f(T_t x)$ then vanishes on (a, b) . Since this function is analytic, it is identically 0. That is, $f(T_t x) = 0$ for any $t \in \mathbb{R}_+$ and any $f \in X'$ vanishing on X_n . By the Hahn–Banach Theorem, $T_t x \in X_n$ for each $t \in \mathbb{R}_+$. Hence x is not cyclic for $\{T_t\}_{t \in \mathbb{R}_+}$. Since x is arbitrary, $\{T_t\}_{t \in \mathbb{R}_+}$ is non-cyclic. \square

Note that a countable locally convex direct sum of infinite-dimensional Banach spaces may admit a hypercyclic operator [10]. This observation together with the above proposition make the following question more intriguing.

Question 6.6. *Let X be the locally convex direct sum of a sequence of separable infinite-dimensional Banach spaces. Does X admit a hypercyclic strongly continuous semigroup $\{T_t\}_{t \in \mathbb{R}_+}$?*

6.1. A question by Bermúdez *et al.*

Using [2, Theorem 2.2] and Theorem 2.9, one can easily see that if T is an extended backward shift on a separable infinite-dimensional Banach space X , then both $I + T$ and e^T are hereditarily hypercyclic. Clearly, an extended backward shift T has dense range and dense generalized kernel $\ker^* T = \bigcup_{n=1}^{\infty} \ker T^n$. The converse is not true in general. This leads to the following question.

Question 6.7. *Let T be a continuous linear operator (having dense range and dense generalized kernel) on a separable Banach space. Is it true that $I + T$ and/or e^T are mixing or at least hypercyclic?*

This reminds us of the following question [3] posed by Bermúdez *et al.*

Question B²CP. *Let X be a complex Banach space and let $T \in L(X)$ be such that its spectrum $\sigma(T)$ is connected and contains 0. Does hypercyclicity of $I + T$ imply hypercyclicity of e^T ? Does hypercyclicity of e^T imply hypercyclicity of $I + T$?*

We shall show that the answer to both parts of the above question is negative. Before doing this, we would like to raise a similar question, which remains open.

Question 6.8. *Let X be a Banach space and let $T \in L(X)$ be quasinilpotent. Is hypercyclicity of $I + T$ equivalent to hypercyclicity of e^T ?*

If the answer is affirmative, then the following interesting question naturally arises.

Question 6.9. *Let T be a quasinilpotent bounded linear operator on a complex Banach space X and let f be an entire function of one variable such that $f(0) = f'(0) = 1$. Is it true that hypercyclicity of $f(T)$ is equivalent to hypercyclicity of $I + T$?*

We introduce some notation. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, let $\mathcal{H}^2(\mathbb{D})$ be the Hardy–Hilbert space on the unit disc and let $\mathcal{H}^\infty(\mathbb{D})$ be the space of bounded holomorphic functions $f: \mathbb{D} \rightarrow \mathbb{C}$. It is well known that, for $\alpha \in \mathcal{H}^\infty(\mathbb{D})$, the multiplication operator $M_\alpha f(z) = \alpha(z)f(z)$ is a bounded linear operator on $\mathcal{H}^2(\mathbb{D})$. It is also clear that $\sigma(M_\alpha) = \alpha(\mathbb{D})$.

Godefroy and Shapiro [15, Theorem 4.9] proved that if $\alpha \in \mathcal{H}^\infty(\mathbb{D})$ is not a constant function, then the Hilbert space adjoint M_α^* is hypercyclic if and only if $\alpha(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$. Moreover, they proved hypercyclicity by means of applying the *Kitai Criterion* [14, 20], which automatically [16] provides hereditary hypercyclicity. Thus their result can be stated in the following form.

Proposition 6.10. *Let $\alpha \in \mathcal{H}^\infty(\mathbb{D})$ be non-constant. Then M_α^* is hereditarily hypercyclic if $\alpha(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$, and M_α^* is non-hypercyclic if $\alpha(\mathbb{D}) \cap \mathbb{T} = \emptyset$.*

We show that the answer to both parts of Question B²CP is negative. Consider $U \subset \mathbb{C}$, being the interior of the triangle with vertices $-1, i$ and $-i$. That is, $U = \{a + bi : a, b \in \mathbb{R}, a < 0, b - a < 1, b + a > -1\}$. Next, let $V = \{a + bi : a, b \in \mathbb{R}, 0 < b < 1, |a| < 1 - \sqrt{1 - b^2}\}$. The boundary of V consists of the interval $[-1 + i, 1 + i]$ and two circle arcs. Clearly, U and V are bounded, open, connected and simply connected. Moreover, $(1 + U) \cap \mathbb{T} \neq \emptyset$, where $1 + U = \{1 + z : z \in U\}$, and $e^U \subseteq \mathbb{D}$, where $e^U = \{e^z : z \in U\}$. Similarly, $(1 + V) \cap \mathbb{D} = \emptyset$ and $e^V \cap \mathbb{T} \neq \emptyset$. By the Riemann Theorem [21], there exist holomorphic homeomorphisms $\alpha : \mathbb{D} \rightarrow U$ and $\beta : \mathbb{D} \rightarrow V$. Obviously, $\alpha, \beta \in \mathcal{H}^\infty(\mathbb{D})$ and are non-constant. Since $I + M_\alpha^* = M_{1+\alpha}^*$, $e^{M_\beta^*} = M_{e^\beta}^*$ and both $(1 + \alpha)(\mathbb{D}) = 1 + U$ and $e^\beta(\mathbb{D}) = e^V$ intersect \mathbb{T} , Proposition 6.10 implies that $I + M_\alpha^*$ and $e^{M_\beta^*}$ are hereditarily hypercyclic. Since $I + M_\beta^* = M_{1+\beta}^*$, $e^{M_\alpha^*} = M_{e^\alpha}^*$, $e^\alpha(\mathbb{D}) = e^U$ is contained in \mathbb{D} and $(1 + \beta)(\mathbb{D}) = 1 + V$ does not meet \mathbb{D} , Proposition 6.10 implies that $e^{M_\alpha^*}$ and $I + M_\beta^*$ are non-hypercyclic. Finally, $\sigma(M_\alpha^*) = \bar{U}$ and $\sigma(M_\beta^*) = -\bar{V}$. Hence the spectra of M_α^* and M_β^* are connected and contain 0. Thus we have arrived at the following result, which answers Question B²CP in the negative.

Proposition 6.11. *There exist bounded linear operators A and B on a separable infinite-dimensional complex Hilbert space such that $\sigma(A)$ and $\sigma(B)$ are connected and contain 0, $I + A$ and e^B are hereditarily hypercyclic, while e^A and $I + B$ are non-hypercyclic.*

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