ON SOME CRITERIA FOR A SET TO BE OF CLASS $N_{\mathfrak{B}}$

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1. Let D be a plane domain containing the point at infinity and E its complementary closed set. As to a sufficient condition for a compact set E to be of class $N_{\mathfrak{B}}$, Pfluger-Mori's criterion is well-known (Pfluger [10], Mori [6]). Various relations between the conditions of this type and the Hausdorff measure of the set E have been investigated recently by Kuroda and Ozawa (Kuroda [5], Ozawa and Kuroda [8], Ozawa [7]). For example they showed that Pfluger-Mori's condition implies that the set E is of one dimensional measure zero under some additional conditions (cf. [7], [8]). In the present paper we shall give an alternative proof of Pfluger-Mori's criterion and another criterion using analytic module and, further, prove some criteria for the set E to be of one dimensional measure zero.

2. We consider a set of doubly connected domains $R_n^{(k)}$ ($k = 1, 2, \ldots, \nu(n) < \infty$; $n = 1, 2, \ldots$) satisfying the following conditions;

(i) the closure of $R_n^{(k)}$ is contained in D,

(ii) the boundary of $R_n^{(k)}$ consists of two rectifiable closed Jordan curves $C_{1n}^{(k)}$ and $C_{2n}^{(k)}$,

(iii) $C_{1n}^{(k)}$ contains $C_{2n}^{(k)}$ in its interior and the point at infinity in its exterior $F_n^{(k)}$,

(iv) the interior $G_n^{(k)}$ of $C_{2n}^{(k)}$ contains at least one point of E and the set E is contained in $\bigcup_{k=1}^{\nu(n)} G_n^{(k)}$,

(v) $R_n^{(j)}$ lies in $F_n^{(k)}$ for any $k \neq j$,

(vi) each $R_{n+1}^{(k)}$ is contained in a certain $G_n^{(k)}$ and

(vii) $\{D_n\}$ is an exhaustion of D, where D_n is defined by $\bigcap_{k=1}^{\nu(n)} (F_n^{(k)} \cup R_n^{(k)})$.

Let $\log \mu_n^{(k)}$ be the modulus of the ring domain $R_n^{(k)}$ and $\mu_n = \min_{1 \leq k \leq \nu(n)} \mu_n^{(k)}$. Pfluger-Mori's criterion can be stated as follows.

THEOREM 1. *If there exists an exhaustion $\{D_n\}$ of D satisfying*

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$$(1) \qquad \limsup_{m\to\infty}\left(\sum_{n=1}^{m}\log\mu_n - \frac{1}{2}\log\nu(m)\right) = +\infty,$$

then the set E is of class $N_{\mathfrak{B}}$.

We give a proof of this theorem using the following

LEMMA (Golusin [4]). *Let R be a bounded ring domain whose outer boundary C_1 and inner boundary C_2 are both closed Jordan curves and let A_1 and A_2 be areas of domains bounded by C_1 and C_2 respectively. Then it holds*

$$\mu^2 \leqq \frac{A_1}{A_2},$$

where $\log\mu$ *is the modulus of* R.

Proof of Theorem 1. Let $E_{1m}^{(k)}$ be the complement $F_m^{(k)c}$ of $F_m^{(k)}$ and $E_{2m}^{(k)}$ be $G_m^{(k)}$ and put $E_{jm} = \bigcup_{k=1}^{\nu(m)} E_{jm}^{(k)}$ and $D_{jm} = E_{jm}^c$ $(j=1, 2)$. Consider a meromorphic function $f(z)$ which is univalent in D_{2m} and normalized at infinity:

$$f(z) = z + \text{terms in } z^{-1},$$

and which gives the maximal area of the complementary set of $f(D_{2m})$. The existence of such a function is well-known (cf. [2], [11]) and the value of the maximal area equals $\frac{\pi}{2}S(E_{2m})$, where $S(M)$ is the span of the component of M^c containing $z=\infty$ for a compact set M.

Let $A_{1m}^{(k)}$ and $A_{2m}^{(k)}$ be areas of the images $f(E_{1m}^{(k)})$ and $f(E_{2m}^{(k)})$ respectively. Then, by Lemma, we have

$$(\mu_m^{(k)})^2 \leqq \frac{A_{1m}^{(k)}}{A_{2m}^{(k)}}$$

for any k and m, because of the conformal invariance of $\mu_m^{(k)}$. Hence we get

$$(2) \qquad \mu_m^2 \leqq \frac{\sum_{k=1}^{\nu(m)} A_{1m}^{(k)}}{\sum_{k=1}^{\nu(m)} A_{2m}^{(k)}} \leqq \frac{S(E_{1m})}{S(E_{2m})},$$

since

$$\sum_{k=1}^{\nu(m)} A_{2m}^{(k)} = \frac{\pi}{2}S(E_{2m}) \text{ and } \sum_{k=1}^{\nu(m)} A_{1m}^{(k)} \leqq \frac{\pi}{2}S(E_{1m}).$$

Next we consider the family $\mathfrak{B}$ consisting of functions $g(z)$ regular in D_{2m} and normalized at infinity:

$$g(z) = \frac{a}{z} + \text{higher terms in } z^{-1},$$

and whose moduli are bounded by one. There exists a function $g_{0m}(z)$ which gives the maximum α_m of $|a|$ and maps D_{2m} onto the $\nu(m)$ sheeted unit disc (cf. [1], [2], [3]). Evidently

$$\iint_{D_{2m}} |g'_{0m}(z)|^2 dx\, dy = \nu(m)\pi.$$

On the other hand, in the family $\mathfrak{D}$ of functions $h(z)$ being regular in D_{2m} and satisfying

$$\iint_{D_{2m}} |h'(z)|^2 dx\, dy \leqq \pi.$$

The quantity $\underset{h\in\mathfrak{D}}{\text{Max}}(\lim_{z\to\infty}|z h(z)|)$ is equal to $\sqrt{\frac{1}{2} S(E_{2m})}$ (cf. [2], [11]). Since $g_{0m}(z)/\sqrt{\nu(m)}$ is in $\mathfrak{D}$, we have

$$\alpha_m \leqq \sqrt{\frac{1}{2}\nu(m) S(E_{2m})} \tag{3}$$

and, by (2),

$$\sqrt{2}\,\alpha_m \leqq \frac{\sqrt{\nu(m)}\ \sqrt{S(E_{1m})}}{\mu_m}.$$

Since E_{1m} is contained in $E_{2,m-1}$, it holds

$$S(E_{1m}) \leqq S(E_{2,m-1})$$

by the monotonicity of span and hence we obtain

$$\sqrt{2}\,\alpha_m \leqq \frac{\sqrt{\nu(m)}\ \sqrt{S(E_{1,m-1})}}{\mu_m \mu_{m-1}}.$$

We continue this procedure and finally get

$$\sqrt{2}\,\alpha_m \leqq \frac{\sqrt{\nu(m)}\ \sqrt{S(E_{11})}}{\prod_{n=1}^{m} \mu_n}.$$

Therefore, our assumption (1) implies that $\lim_{m\to\infty} \alpha_m = 0$, i.e., E belongs to the class $N_{\mathfrak{B}}$([1], [2], [3]).

Next we turn to a metrical test. Let $r_m^{(k)}$ be the outer mapping radius of $E_{2m}^{(k)}$ and $f_m^{(k)}(z)$ be a regular function which maps the domain $E_{2m}^{(k)c}$ univalently onto the unit disc under the normalization $\lim_{z\to\infty} z f_m^{(k)}(z) > 0$. At infinity, $f_m^{(k)}(z)$ has the expansion

$$f_m^{(k)}(z) = \frac{r_m^{(k)}}{z} + \text{higher terms in } z^{-1}.$$

By Minkowski's inequality we have

$$\left(\iint_{D_{2m}} \left|\sum_{k=1}^{\nu(m)} f_m^{(k)\prime}(z)\right|^2 dx\,dy\right)^{1/2} \leqq \sum_{k=1}^{\nu(m)} \left(\iint_{D_{2m}} |f_m^{(k)\prime}(z)|^2 dx\,dy\right)^{1/2}$$
$$\leqq \sum_{k=1}^{\nu(m)} \left(\iint_{E_{2m}^{(k)c}} |f_m^{(k)\prime}(z)|^2 dx\,dy\right)^{1/2} = \nu(m)\sqrt{\pi}\,.$$

Thus we see that $\sum_{k=1}^{\nu(m)} f_m^{(k)}(z)/\nu(m)$ is contained in $\mathfrak{D}$ and

$$\sqrt{2}\sum_{k=1}^{\nu(m)} r_m^{(k)} \leqq \nu(m)\sqrt{S(E_{2m})}\,.$$

The inequality (2) and the same procedure as in the proof of Theorem 1 yield

$$\sqrt{2}\sum_{k=1}^{\nu(m)} r_m^{(k)} \leqq \frac{\nu(m)\sqrt{S(E_{11})}}{\prod_{n=1}^{m} \mu_n}\,.$$

If $d_m^{(k)}$ is the diameter of $E_{2m}^{(k)}$, then $d_m^{(k)} \leqq 4 r_m^{(k)}$. Hence, if $\limsup_{m\to\infty} \left(\sum_{n=1}^{m} \log \mu_n - \log \nu(m)\right) = +\infty$, then $\varliminf_{m\to\infty} \sum_{k=1}^{\nu(m)} d_m^{(k)} = 0$. Thus we have the following

THEOREM 2. *If there exists an exhaustion of D such that*

$$\limsup_{m\leftarrow\infty} \left(\sum_{n=1}^{m} \log \mu_n - \log \nu(m)\right) = +\infty,$$

then E has one dimensional measure zero.

3. We consider now suitable domains conformally equivalent to members of the exhaustion $\{D_n\}$ in 2 satisfying the condition (1). Let $f_m(z)$ be a meromorphic function in D_{2m} which is normalized at $z = \infty$:

$$f_m(z) = z + \text{terms in } z^{-1}$$

and maps D_{2m} univalently onto a domain bounded by $\nu(m)$ circumferences. Denote by $\rho_m^{(k)}$ the diameter of $f_m(E_{2m}^{(k)})$ and by $A_{1m}^{(k)}$ the area of $f_m(E_{1m}^{(k)})$. Then, we get

$$\mu_m^{(k)} \leqq \frac{2\sqrt{A_{1m}^{(k)}}}{\sqrt{\pi}\ \rho_{1m}^{(k)}}.$$

Schwarz's inequality yields

$$\mu_m^{(k)} \leqq \frac{2\sum_{k=1}^{\nu(m)}\sqrt{A_{1m}^{(k)}}}{\sqrt{\pi}\sum_{k=1}^{\nu(m)}\rho_m^{(k)}} \leqq \frac{2\sqrt{\nu(m)}\ (\sum_{k=1}^{\nu(m)}A_{1m}^{(k)})^{1/2}}{\sqrt{\pi}\sum_{k=1}^{\nu(m)}\rho_m^{(k)}} \leqq \frac{\sqrt{2}\ \sqrt{\nu(m)}\ \sqrt{S(E_{1m})}}{\sum_{k=1}^{\nu(m)}\rho_m^{(k)}},$$

whence follows

$$\sum_{k=1}^{\nu(m)}\rho_m^{(k)} \leqq \frac{\sqrt{2}\ \sqrt{\nu(m)}\ \sqrt{S(E_{11})}}{\prod_{n=1}^{m}\mu_n}$$

by the same argument as in the proof of Theorem 1. Thus we have

THEOREM 3. *If there exists an exhaustion $\{D_n\}$ satisfying the condition* (1), *then we can select a sequence of mapping functions $\{f_{n_\nu}\}$ corresponding to a subsequence $\{D_{n_\nu}\}$ of $\{D_n\}$ and make one dimensional measure of the boundary of the image $f_{n_\nu}(D_{n_\nu})$ arbitrarily small with ν tending to infinity.*

4. We consider an exhaustion $\{D_n\}$ of D in the usual sense. The set $D_n - \overline{D_{n-1}}$ consists of a finite number of multiply connected domains $G_{n-1}^{(k)}$ $(k = 1, 2, \ldots, (n-1))$. We denote the outer boundary curve of $G_{n-1}^{(k)}$ by $C_{(k)}^{n-1}$ and inner boundary curves by $C_n^{(k)}$ respectively; both of them are oriented positively with respect to $G_{n-1}^{(k)}$. Then the analytic module $\sigma_n^{(k)}$ of $G_{n-1}^{(k)}$ is defined by

$$\sigma_n^{(k)} = \inf_f \left(\int_{C_{n-1}^{(k)}} f\,\overline{df} \Big/ \int_{C_n^{(k)}} f\,\overline{df}\right),$$

where $f(z)$ is analytic in $G_{n-1}^{(k)}$ and $\int_{C_{n-1}^{(k)}} f\,\overline{df} > 0$ (see [9]).

Put $D_m^c = E_m$ and $\sigma_m = \underset{1\leqq k\leqq \nu(m)}{\mathrm{Min}}\ \sigma_m^{(k)}$. Considering the same function meromorphic in D_m as in **1**, which is univalent and normalized at infinity and gives the maximal area of the complementary set of the image of D_m, we obtain an inequality

$$\sigma_m \leqq \frac{S(E_{m-1})}{S(E_m)}$$

from the definition of σ_m. Hence, we get

$$A_n^2 \leqq \frac{\nu(m) S(E_1)}{\prod_{n=1}^{m} \sigma_n}$$

by the inequality (3). From this follows

THEOREM 4. *If there exists an exhaustion $\{D_n\}$ of D such that*

$$\limsup_{m\to\infty} \left(\sum_{n=1}^{m} \log \sigma_n - \log \nu(m) \right) = +\infty,$$

then the set E is of class $N_{\mathfrak{B}}$.

We can also get the corresponding metrical criterion:

THEOREM 5. *If there exists an exhaustion $\{D_n\}$ of D satisfying the condition:*

$$\limsup_{m\to\infty} \left(\sum_{n=1}^{m} \log \sigma_n - 2\log \nu(m) \right) = +\infty,$$

then the set E is of one dimensional measure zero.

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