ON ANNIHILATOR IDEALS OF SKEW MONOID RINGS

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Abstract. A ring $R$ is called a left APP-ring if the left annihilator $l_R(Ra)$ is pure as a left ideal of $R$ for every $a \in R$; $R$ is called (left principally) quasi-Baer if the left annihilator of every (principal) left ideal of $R$ is generated by an idempotent. Let $R$ be a ring and $M$ an ordered monoid. Assume that there is a monoid homomorphism $\phi : M \rightarrow \text{Aut}(R)$. We give a necessary and sufficient condition for the skew monoid ring $R \ast M$ (induced by $\phi$) to be left APP (left principally quasi-Baer, quasi-Baer, respectively).

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1. Introduction. Throughout this paper, $R$ denotes a ring with unity. Recall that $R$ is a right PP-ring if the right annihilator of an element of $R$ is generated by an idempotent. Armendariz showed that polynomial rings over right PP-rings need not be right PP in the example in [1]. Also the concept of right PP-rings is not a Morita invariant property because $\mathbb{Z}[x]$ is PP but the $2 \times 2$ full matrix ring over $\mathbb{Z}[x]$ is not a right PP-ring [1]. In order to consider the natural question of how much of the right PP condition transfers to polynomial rings or matrix rings, a concept of left APP-rings was introduced and considered in [15]. By [15], Proposition 2.3, right PP-rings are left APP. It was shown in [15], Theorem 3.8 and Corollary 3.12, that the left APP condition is a Morita invariant property and transfers to a variety of polynomial extensions.

On the other hand, a ring $R$ is (quasi-)Baer if the left annihilator of every non-empty subset (every left ideal) of $R$ is generated by an idempotent of $R$. Clark defined quasi-Baer rings in [7] and used them to characterise when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semi-group algebra. As a generalisation of quasi-Baer rings, Birkenmeier, Kim and Park in [5] introduced the concept of left principally quasi-Baer rings. A ring $R$ is called left principally quasi-Baer if the left annihilator of a principal left ideal of $R$ is generated by an idempotent. Observe that biregular rings and quasi-Baer rings are left principally quasi-Baer. Clearly the concept of left APP-rings is a common generalisation of left principally quasi-Baer rings and right PP-rings. For more details and examples of quasi-Baer rings, left principally quasi-Baer rings and left APP-rings, see [2–6, 12, 13 and 15].

In this paper we consider the left APP property, left principal quasi-Baerness and quasi-Baerness of the skew monoid ring $R \ast M$. Some necessary and sufficient

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conditions for the skew monoid ring \( R \ast M \) to be left APP (left principally quasi-Baer, quasi-Baer) are obtained.

Let \( R \) be a ring, and let \( M \) be an ordered monoid. Assume that there exists a monoid homomorphism \( \phi : M \rightarrow Aut(R) \). For any \( g \in M \) and any \( r \in R \), we denote by \( r^g \) the image of \( r \) under \( \phi(g) \). Then we can form a skew monoid ring \( R \ast M \) (induced by the monoid homomorphism \( \phi \)) by taking its elements to be finite formal combinations \( \sum_{g \in M} a_g g \), with multiplication induced by

\[
(a_g)(b_h) = (a_g b_h^g)(gh).
\]

If \( \phi \) is weakly rigid (that is to say \( ab = 0 \) implies \( a^g b = ab^g = 0 \) for any \( a, b \in R \) and any \( g \in M \)), then it has been proved in \([14]\) that the skew monoid ring \( R \ast M \) is quasi-Baer if and only if \( R \) is quasi-Baer. If \( R \) is \( \phi \)-compatible, then it has been proved in \([8]\) that \( R \ast M \) is (left principally) quasi-Baer if and only if \( R \) is (left principally) quasi-Baer. It was shown in \([15]\), Theorem 3.10, that if \( R \) is a left APP-ring, \( M \) is a u.p.-monoid, and the monoid homomorphism \( \phi : M \rightarrow Aut(R) \) satisfies the condition that for every \( a \in R \), the left ideal \( \sum_{g \in M} Ra^g \) is finitely generated, then the skew monoid ring \( R \ast M \) is a left APP-ring. When \( M \) is a group or \( Im(\phi) \) is a group, a necessary and sufficient condition for the skew monoid ring \( R \ast M \) to be (left principally) quasi-Baer was given in \([9]\). In this paper we will show that for an ordered monoid \( M \) and a monoid homomorphism \( \phi : M \rightarrow Aut(R) \), the skew monoid ring \( R \ast M \) is a left APP-ring (a left principally quasi-Baer ring, a quasi-Baer ring, respectively) if and only if the left annihilator \( l_R(\sum_{g \in M} Ra^g) \) is right s-unital for every \( a \in R \). (The left annihilator of \( \sum_{g \in M} Ra^g \) is generated by an idempotent for every \( a \in R \), and the left annihilator of left ideal \( \sum_{h \in S} \sum_{g \in M} Rb^g \) of \( R \) is generated by an idempotent for every subset \( S \) of \( R \), respectively.)

For a non-empty subset \( Y \) of \( R \), \( l_R(Y) \) and \( r_R(Y) \) denote the left and right annihilators of \( Y \) in \( R \), respectively. A monoid \( M \) is said to be ordered if the elements of \( M \) are linearly ordered with respect to the relation \( < \) and if for all \( x, y, z \in M \), \( x < y \) implies \( zx < zy \) and \( xz < yz \). It is well known that any submonoid of a free group or a torsion-free nilpotent group is an ordered monoid. We denote by \( \eta \) the identity of a monoid \( M \).

2. Left APP-rings. An ideal \( I \) of \( R \) is said to be right s-unital if for each \( a \in I \) there exists an element \( x \in I \) such that \( ax = a \). Note that if \( I \) and \( J \) are right s-unital ideals, then so is \( I \cap J \). If \( a \in I \cap J \), then \( a \in aIJ \subseteq \eta(I \cap J) \). It follows from \([17]\), Theorem 1, that \( I \) is right s-unital if and only if for any finitely many elements \( a_1, a_2, \ldots, a_n \in I \) there exists an element \( x \in I \) such that \( a_i = a_i x \), \( i = 1, 2, \ldots, n \). A submodule \( N \) of a left \( R \)-module \( M \) is called a pure submodule if \( L \otimes_R N \rightarrow L \otimes_R M \) is a monomorphism for every right \( R \)-module \( L \). By \([16]\), Proposition 11.3.13, an ideal \( I \) is right s-unital if and only if \( R/I \) is flat as a left \( R \)-module if and only if \( I \) is pure as a left ideal of \( R \).

By \([15]\), a ring \( R \) is called a left APP-ring if the left annihilator \( l_R(Ra) \) is right s-unital as an ideal of \( R \) for any element \( a \in R \).

Right APP-rings may be defined analogously. Clearly every left principally quasi-Baer ring is a left APP-ring. (Thus the class of left APP-rings includes all biregular rings and all quasi-Baer rings.) From \([15]\), it follows that right PP-rings are left APP and left APP-rings are quasi-Armendariz in the sense that
whenever \( f(x) = a_0 + a_1x + \cdots + a_nx^n, \) \( g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x] \) satisfy \( f(x)R[x]g(x) = 0 \), we have \( a_iRb_j = 0 \) for each \( i \) and \( j \) (see for example [10]).

Let \( M \) be a monoid and \( \phi : M \rightarrow Aut(R) \) a monoid homomorphism. The ring \( R \) is called left \( M \)-APP if the left annihilator \( l_R(\sum_{g \in M} Rb^g) \) is right s-unital for every \( b \in R \). Clearly if \( \phi(g) = 1 \) for every \( g \in M \), then \( R \) is left \( M \)-APP if and only if \( R \) is left APP. It is easy to see that if \( R \) is a left Noetherian and left APP-ring, then \( R \) is left \( M \)-APP for any monoid \( M \) (in fact, there exists a maximal element \( \sum_{g \in N} Rb^g \) in the set \( \{\sum_{g \in N} Rb^g | N \subseteq M, |N| < \infty\} \), which is unique, and so \( l_R(\sum_{g \in N} Rb^g) = l_R(\sum_{g \in N_0} Rb^g) = \cap_{g \in N} l_R(Rb^g) \) is right s-unital).

**Remark 1.** (1) It follows from [15], Theorem 3.10, that if \( R \) is a left APP-ring and \( M \) an ordered monoid and if the monoid homomorphism \( \phi : M \rightarrow Aut(R) \) satisfies the condition that for every \( a \in R \), the left ideal \( \sum_{g \in M} Rb^g \) is finitely generated, then the skew monoid ring \( R \ast M \) is a left APP-ring. Thus, by Theorem 2, \( R \) is a left \( M \)-APP-ring. Remark 3.11 of [15] gave some examples of left \( M \)-APP-rings.

(2) For a given left APP-ring \( T \), let

\[
R = \left\{ (a_n)_{n \in \mathbb{Z}} \in \prod T | a_n \text{ is eventually constant} \right\},
\]

which is a subring of the countably infinite direct product \( \prod T \). Define an automorphism \( \sigma \) of \( R \) by \( \sigma((a_n)) = (a_{n+1}) \). Let \( M = \mathbb{N} \cup \{0\} \). Define \( \phi : M \rightarrow Aut(R) \) via \( \phi(0) = 1 \) and \( \phi(n) = \sigma^n \) for every \( n \in \mathbb{N} \). Suppose that \( w = (\ldots, a, a, a, a, \ldots) \in l_R(\sum_{g \in M} Rb^g) \), where \( b = (b_n)_{n \in \mathbb{Z}} \in R \). Then \( wRb^g = 0 \) for each \( g \in M \). Thus for any \( s \leq n \leq t, a_nTb_n = 0 \), \( a_nTb_{n-1} = 0 \), \( a_nTb_{n-2} = 0 \), \( \ldots \). Since \( (b_n)_{n \in \mathbb{Z}} \) is eventually constant, the left ideal \( Tb_n + Tb_{n-1} + \cdots \) is finitely generated. By Proposition 2.6 of [15], \( l_T(Tb_n + Tb_{n-1} + \cdots) \) is right s-unital. Thus \( a_n = a_n a_n' \) for some \( a_n' \in T_T(Tb_n + Tb_{n-1} + \cdots) \). Similarly \( a' = a a' \) for some \( a' \in T_T(Tb_n + Tb_{n-1} + \cdots) \). Now it is easy to see that \( w = w w' \), where \( w' = (\ldots, a', a', a', a', \ldots) \in l_R(\sum_{g \in M} Rb^g) \). Therefore \( R \) is left \( M \)-APP.

If we take \( T = S[[x]] \), where

\[
S = \left( \prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z} \right) / \left( \bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z} \right),
\]

then, by Example 2.5 of [15], \( T \) is an APP-ring, but \( T \) is neither PP nor principally quasi-Baer. Thus \( R \) is neither PP nor principally quasi-Baer.

(3) The following example (see [9]) shows that left \( M \)-APP rings need not be left APP. Let \( F \) be a field; let \( A = F[s, t] \) be a commutative polynomial ring; and consider the ring \( R = A/(st) \). Let \( \tilde{s} = s + (st) \) and \( \tilde{t} = t + (st) \) in \( R \). Define an automorphism \( \sigma \) of \( R \) by \( \sigma(\tilde{s}) = \tilde{t} \) and \( \sigma(\tilde{t}) = \tilde{s} \). Then \( l_R(R\tilde{s}) = R\tilde{t} \). Clearly this ideal is not right s-unital. Thus \( R \) is not a left APP-ring. By Example 2 of [9], any non-zero ideal \( I \) of \( R \) with \( \sigma(I) = I \) is essential in \( R \), and so \( l_R(I) = 0 \). (Note that \( R \) is a reduced ring.) Let \( M = \mathbb{N} \cup \{0\} \). Define \( \phi : M \rightarrow Aut(R) \) via \( \phi(0) = 1 \) and \( \phi(n) = \sigma^n \) for every \( n \in \mathbb{N} \). Therefore \( R \) is \( M \)-APP.

The following is our main result which gives a necessary and sufficient condition for the skew monoid ring \( R \ast M \) to be a left APP-ring.
**Theorem 2.** Let $R$ be a ring, $M$ an ordered monoid and $\phi : M \to \text{Aut}(R)$ a monoid homomorphism. Then the following are equivalent:

1. The skew monoid ring $R \rtimes M$ is a left APP-ring.
2. $R$ is a left $M$-APP-ring.

**Proof.** $(2) \implies (1)$. Let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n$, and let $\beta = b_1h_1 + b_2h_2 + \cdots + b_nh_m \in R \rtimes M$ satisfy $\alpha(R \rtimes M)\beta = 0$. Without loss of generality, we assume that $g_i < g_j$ and $h_i < h_j$ if $i < j$. Suppose that $c_1, c_2, \ldots, c_n \in R$ are such that $a_i = c_i^g$ for $i = 1, 2, \ldots, n$. We will show that $c_i \in l_R(\sum_{g \in M} Rb_g^{\phi})$ for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$ by induction on $n$.

For any $c \in R$ and any $g \in M$, from $\alpha(R \rtimes M)\beta = 0$ it follows that

$$(a_1g_1 + a_2g_2 + \cdots + a_ng_n)(cg)(b_1h_1 + b_2h_2 + \cdots + b_nh_m) = 0.$$ 

Considering the coefficient of the largest element $g_ngh_m$ in above, we obtain $a_n\epsilon_n b_n^{g_n} = 0$. This implies that $(c_n\epsilon_n b_n^{g_n})^g = 0$. Thus $c_n\epsilon_n b_n^{g_n} = 0$, since $(-)^g$ is an automorphism of $R$. So $c_n\epsilon_n b_n^{g_n} = 0$ for all $g \in M$, which implies that $c_n(\sum_{g \in M} Rb_g^{\phi}) = 0$. That is to say $c_n \in l_R(\sum_{g \in M} Rb_g^{\phi})$. By $(2)$, $l_R(\sum_{g \in M} Rb_g^{\phi})$ is right s-unital. Thus there exists $e \in l_R(\sum_{g \in M} Rb_g^{\phi})$ such that $c_n = c_ne$. Now for every $c \in R$ and every $g \in M$, we have

$$0 = (a_1g_1 + a_2g_2 + \cdots + a_ng_n)(c_1eg_1h_1 + c_1eg_1h_2 + \cdots + c_1eg_1h_m)$$
$$= a_1e^g_1 \epsilon_1^g b_1^{g_1g} g_1gh_1 + \cdots + a_n \epsilon_{n-1}^g e_{n-1}^g b_{n-1}^{g_{n-1}g} g_{n-1}gh_m$$
$$+ a_ne^g_n \epsilon_n^g b_n^{g_n} g_ngh_{m-1} + a_ne^g_n \epsilon_n^g b_n^{g_n} g_ngh_m.$$

Since $a_ne^g_n \epsilon_n^g b_n^{g_n} g_m = a_ne^g_n \epsilon_n^g b_n^{g_n} g_m = 0$, $a_n \epsilon_{n-1}^g e_{n-1}^g b_{n-1}^{g_{n-1}g} g_{n-1}gh_m = 0$, considering the coefficient of the largest element $g_ngh_m$ in above, we have $a_n \epsilon_{n-1}^g e_{n-1}^g b_{n-1}^{g_{n-1}g} g_{n-1}gh_m = 0$. Thus

$$\left(c_n\epsilon_n b_n^{g_n}\right)^g = \left(c_n\epsilon_n b_n^{g_n}\right)^g = a_n \epsilon_{n-1}^g e_{n-1}^g b_{n-1}^{g_{n-1}g} g_{n-1}gh_m = a_n \epsilon_{n-1}^g e_{n-1}^g b_{n-1}^{g_{n-1}g} g_{n-1}gh_m = 0,$$

which implies that $c_n Rb_{n-1}^{g_n} = 0$ for every $g \in M$. Hence $c_n \in l_R(\sum_{g \in M} Rb_g^{\phi}) \cap l_R(\sum_{g \in M} Rb_{g-1}^{g_n})$. Noting that $l_R(\sum_{g \in M} Rb_g^{\phi})$ and $l_R(\sum_{g \in M} Rb_{g-1}^{g_n})$ are right s-unital ideals of $R$, so is $l_R(\sum_{g \in M} Rb_g^{\phi}) \cap l_R(\sum_{g \in M} Rb_{g-1}^{g_n})$. Thus there exists $f \in l_R(\sum_{g \in M} Rb_g^{\phi}) \cap l_R(\sum_{g \in M} Rb_{g-1}^{g_n})$ such that $c_n = c_nf$. Now for any $g \in M$ and any $c \in R$ we have $(a_1g_1 + a_2g_2 + \cdots + a_ng_n)(f cg)(b_1h_1 + b_2h_2 + \cdots + b_nh_m) = 0$.

Continuing this process, we obtain

$$c_n \in \bigcap_{j=1}^{m} l_R(\sum_{g \in M} Rb_g^{\phi}).$$

Thus for any $c \in R$ and any $g \in M$, $(a_n \epsilon_n b_n^{g_n} g_ngh_1 + \cdots + a_n \epsilon_n b_n^{g_n} g_ngh_m) = (c_n \epsilon_n b_n^{g_n} g_ngh_1 + \cdots + (c_n \epsilon_n b_n^{g_n} g_ngh_m) = a_n \epsilon_n b_n^{g_n} g_ngh_1 + \cdots + a_n \epsilon_n b_n^{g_n} g_ngh_m = (c_n \epsilon_n b_n^{g_n} g_ngh_1 + \cdots + (c_n \epsilon_n b_n^{g_n} g_ngh_m).$

So we have

$$(a_1g_1 + a_2g_2 + \cdots + a_{n-1}g_{n-1})(R \rtimes M)(b_1h_1 + b_2h_2 + \cdots + b_nh_m) = 0.$$

Now using induction on $n$, we obtain that $c_i \in \bigcap_{j=1}^{m} l_R(\sum_{g \in M} Rb_g^{\phi})$ for all $i$. Since $l_R(\sum_{g \in M} Rb_1^{\phi}), \ldots, l_R(\sum_{g \in M} Rb_m^{\phi})$ are right s-unital ideals, it is clear that
For every $\phi$ monoids via $R$ if $M$ quasi-Baer, then so are (respectively left principally quasi-Baer) ring if and only if left principally quasi-Baer. C. Y. Hong, N. K. Kim and T. K. Kwak showed in [3].

If $R$ that $\sum_{i=1}^{n} c_i e_i g_i = 0$. This shows that $a = ac_0$. Note that $M$ is cancellative. For any $r \in R$ and $g \in M$, from $(c_0 \eta + c_1 g_1 + \cdots + c_n g_n)(rg)(b\eta) = 0$ it follows that $c_0rb^g = 0$, which implies that $c_0rb^g = 0$. Thus $c_0(\sum_{g \in M} Rb^g) = 0$. This shows that $R$ is a left $M$-APP-ring.

**Corollary 3.** Let $R$ be a ring and $M$ an ordered monoid. Then the monoid ring $R[M]$ is left APP if and only if $R$ is left APP.

**Corollary 4.** Let $R$ be a ring and $\sigma$ a ring automorphism of $R$. Then the ring $R[x; \sigma]$ (respectively $R[x, x^{-1}; \sigma]$) is left APP if and only if the left annihilator of $\sum_{i=0}^{\infty} R\sigma^i(b)$ (respectively $\sum_{i=-\infty}^{\infty} R\sigma^i(b)$) is right s-unital for every $b \in R$.

**Proof.** Define a homomorphism $\phi : \mathbb{N} \cup \{0\} \rightarrow Aut(R) (\phi : \mathbb{Z} \rightarrow Aut(R))$ of monoids via $\phi(i) = \sigma^i$. Then the result follows from Theorem 2.

**3. Left principally quasi-Baer rings and quasi-Baer rings.** Let $R$ be a ring, $M$ a monoid and $\phi : M \rightarrow Aut(R)$ a monoid homomorphism; $R$ is called a left $M$-principally quasi-Baer ring if for any $a \in R$, the left annihilator of $\sum_{g \in M} Rg^\phi$ is generated by an idempotent. For the condition that $M$ is a group, left $M$-principally quasi-Baer rings were considered by Y. Hirano in [9]. Note that by Remark 1(3), left $M$-principally quasi-Baer rings need not be left principally quasi-Baer.

There are a lot of results concerning quasi-Baerness and left principal quasi-Baerness of polynomial extensions of a ring. G. F. Birkenmeier, J. Y. Kim and J. K. Park showed in [4], Theorem 1.8, that $R$ is quasi-Baer if and only if $R[[x]]$ is quasi-Baer if and only if $R[[x; \sigma]]$ is quasi-Baer if and only if $R[[x, x^{-1}]]$ is quasi-Baer, where $X$ is an arbitrary non-empty set of not necessarily commuting indeterminates. Furthermore, it was shown in [4], Theorem 1.2, that if $R$ is quasi-Baer, then so are $R[x; \sigma], R[[x; \sigma]], R[x, x^{-1}; \sigma]$ and $R[[x, x^{-1}; \sigma]]$. It was proved in [3], Theorem 2.1, that a ring $R$ is left principally quasi-Baer if and only if $R[x]$ is left principally quasi-Baer. C. Y. Hong, N. K. Kim and T. K. Kwak showed in [11], Corollaries 12, 15 and 22, that if $\sigma$ is a rigid endomorphism of $R$, then $R$ is a quasi-Baer (respectively left principally quasi-Baer) ring if and only if $R[x; \sigma, \delta]$ is a quasi-Baer (respectively left principally quasi-Baer) ring if and only if $R[[x; \sigma]]$ is a quasi-Baer ring. If $R$ is a ring and $(S, \leq)$ a strictly totally ordered monoid which satisfies the condition that $0 \leq s$ for every $s \in S$, then it is shown in [13] that $R$ is a quasi-Baer ring if and only if the ring $[[R^S \leq]]$ of generalised power series over $R$ is a quasi-Baer ring. If $M$
is an ordered monoid, then it is proved in [9], Theorem 1, that \( R[M] \) is quasi-Baer if and only if \( R \) is quasi-Baer. This result has been generalised by G. F. Birkenmeier and J. K. Park in [6], Theorem 1.2, by showing that if \( M \) is a u.p.-monoid, then \( R[M] \) is quasi-Baer (respectively left principally quasi-Baer) if and only if \( R \) is quasi-Baer (respectively left principally quasi-Baer). For skew monoid rings it was proved in [9], Theorem 2, that if \( R \) is a ring and \( M \) an ordered group acting on \( R \), then \( R * M \) is a left principally quasi-Baer ring if and only if \( R \) is a left \( M \)-principally quasi-Baer ring. It was also noted in [9], Remark, that if \( M \) is an ordered monoid and if there exists a monoid homomorphism \( \phi: M \rightarrow Aut(R) \) such that \( I_m(\phi) \) is a group, then the skew monoid ring \( R * M \) is a left principally quasi-Baer ring if and only if \( R \) is a left \( I_m(\phi) \)-principally quasi-Baer ring. Here we have the following result.

Theorem 5. Let \( R \) be a ring, \( M \) an ordered monoid and \( \phi: M \rightarrow Aut(R) \) a monoid homomorphism. Then the following statements are equivalent:

1. The skew monoid ring \( R * M \) is a left principally quasi-Baer ring.
2. \( R \) is a \( M \)-principally quasi-Baer ring.

Proof. \((2)\Rightarrow(1)\). Suppose that \( b_1h_1 + b_2h_2 + \cdots + b_nh_n \) belongs to \( R * M \), and consider the principal left ideal \( I = (R * M)(b_1h_1 + b_2h_2 + \cdots + b_nh_n) \) of \( R * M \). Without loss of generality, we assume that \( h_1 < h_2 < \cdots < h_n \). Let \( J \) denote the set of all coefficients of elements of \( I \). Then it is easy to see that

\[
J = \sum_{g \in M} Rb_1g + \sum_{g \in M} Rb_2g + \cdots + \sum_{g \in M} Rb_ng,
\]

By point (2), there exists an idempotent \( e_j \in R \) such that \( l_{R}(\sum_{g \in M} Rb_jg) = Re_j \), \( j = 1, 2, \ldots, m \). Let \( e = e_1e_2 \cdots e_m \). Then \( l_{R}(\sum_{g \in M} Rb_1g) + \sum_{g \in M} Rb_2g + \cdots + \sum_{g \in M} Rb_ng =\sum_{g \in M} Rb_1g = e \sum_{g \in M} Rb_1g \in l_{R}(\sum_{g \in M} Rb_jg) = Re \). Clearly \( e \in I_{R,M}(I) \). Suppose \( a_1g_1 + a_2g_2 + \cdots + a_ng_n \in I_{R,M}(I) \). Then \( (a_1g_1 + a_2g_2 + \cdots + a_ng_n)(R * M)(b_1h_1 + b_2h_2 + \cdots + b_nh_n) = 0 \). Without loss of generality, we assume that \( g_1 < g_2 < \cdots < g_n \). Suppose that \( c_1, c_2, \ldots, c_n \in R \) are such that \( a_i = c_i \) for \( i = 1, 2, \ldots, n \). Then, by analogy with the proof of Theorem 2, we have \( c_i \in l_{R}(\sum_{g \in M} Rb^\eta_jg) \) for \( i = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \). Thus \( c_i \in l_R(J) \), and so \( c_i = ce_i \), \( i = 1, 2, \ldots, n \). Now

\[
\left( \sum_{i=1}^{n} a_ig_i \right)(e \eta) = \sum_{i=1}^{n} a_i(e^g_i g_i) = \sum_{i=1}^{n} (c_i e^g_i g_i) = \sum_{i=1}^{n} c_i g_i = \sum_{i=1}^{n} a_ig_i,
\]

which implies \( \sum_{i=1}^{n} a_ig_i \in (R * M)(e \eta) \). Thus \( l_{R,M}(I) \leq (R * M)(e \eta) \). Therefore \( l_{R,M}(I) = (R * M)(e \eta) \), and so \( R * M \) is a principally quasi-Baer ring.

\((1)\Rightarrow(2)\). Suppose that the skew monoid ring \( R * M \) is a left principally quasi-Baer ring. Let \( b \in R \). We consider the left annihilator \( l_{R}(\sum_{g \in M} Rb^\eta) \). By Hypothesis (1), there exists an idempotent \( \alpha \in R * M \) such that \( l_{R,M}(I_{R,M}(b \eta)) = (R * M)\alpha \). We may write \( \alpha = e_0e_1g_1 + \cdots + e_ng_n \in R * M \), where \( e_i \in R \) and \( g_1, g_2, \ldots, g_n \) are distinct elements of \( M \). Note that the monoid \( M \) is cancellative. For any \( r \in R \) and any \( g \in M \), from \( 0 = (e_0e_1g_1 + \cdots + e_ng_n)(rg)(b \eta) = e_0rb^\eta g + e_1rb^\eta b^\eta g_1g + \cdots + e_nrb^\eta b^nrb^\eta g_n^\eta b^\eta g \) it follows \( e_0rb^\eta g = 0 \), and so \( e_0rb^\eta = 0 \). Since \( g \) is an arbitrary element of \( M \), we have \( e_0(\sum_{g \in M} Rb^\eta) = 0 \). Thus \( R e_0 \subseteq l_{R}(\sum_{g \in M} Rb^\eta) \). To prove the converse inclusion, let \( a \in l_{R}(\sum_{g \in M} Rb^\eta) \). Then for any \( r \in R \) and any \( g \in M \), \( (a \eta)(rg)(b \eta) = arb^\eta g = 0 \). Thus \( (a \eta)(R * M)(b \eta) = 0 \), and so \( (a \eta) = (a \eta)\alpha = (a \eta)(e_0e_1g_1 + \cdots + e_ng_n) = ae_0e_1g_1 + \cdots + ae_ng_n \). Considering the coefficient of \( \eta \) we obtain \( a = ae_0 \).
Hence \( l_R(\sum_{g \in M} Rb^g) \subseteq Re_0 \). In particular \( e_0 \) is an idempotent of \( R \). Hence \( R \) is a left \( M \)-principally quasi-Baer ring.

Let \( M \) be an ordered monoid and \( \phi : M \to Aut(R) \) a monoid homomorphism. \( R \) is called a left \( M \)-quasi-Baer ring if for any subset \( S \) of \( R \), the left annihilator of left ideal \( \sum_{b \in S} \sum_{g \in M} Rb^g \) of \( R \) is generated by an idempotent of \( R \). For the condition that \( G \) is a group, left \( G \)-quasi-Baer rings was considered by Y. Hirano in [9]. Note that by Remark 1(3), left \( M \)-quasi-Baer rings need not be left quasi-Baer. By analogy with the proof of Theorem 5 we have the following result.

**Theorem 6.** Let \( R \) be a ring, \( M \) an ordered monoid and \( \phi : M \to Aut(R) \) a monoid homomorphism. Then the following are equivalent:

1. The skew monoid ring \( R \ast M \) is a quasi-Baer ring.
2. \( R \) is a left \( M \)-quasi-Baer ring.

**Proof.** Let \( I \) be a left ideal of \( R \ast M \). Denote by \( I_0 \) the set of all coefficients of elements of \( I \). Let

\[
J = \sum_{b \in I_0} \sum_{g \in M} Rb^g.
\]

If \( R \) is left \( M \)-quasi-Baer, then there exists an idempotent \( e \in R \) such that \( l_R(J) = Re \). Now by analogy with the proof of Theorem 5 we can complete the proof.

**Corollary 7.** Let \( R \) be a ring and \( \sigma \) a ring automorphism of \( R \). Then

(i) the ring \( R[x; \sigma] \) (respectively \( R[x, x^{-1}; \sigma] \)) is left principally quasi-Baer if and only if the left annihilator of \( \sum_{i=0}^{\infty} R\sigma^i(b) \) (respectively \( \sum_{i=-\infty}^{\infty} R\sigma^i(b) \)) is generated by an idempotent for every \( b \in R \);

(ii) the ring \( R[x; \sigma] \) (respectively \( R[x, x^{-1}; \sigma] \)) is quasi-Baer if and only if the left annihilator of \( \sum_{b \in S} \sum_{i=0}^{\infty} R\sigma^i(b) \) (respectively \( \sum_{b \in S} \sum_{i=-\infty}^{\infty} R\sigma^i(b) \)) is generated by an idempotent for any subset \( S \) of \( R \).

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**References**