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A SECOND ORDER SUPERLINEAR OSCILLATION CRITERION

BY

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ABSTRACT. A new oscillation criterion is given for general superlinear ordinary differential equations of second order of the form $x''(t)+a(t)f[x(t)]=0$, where $a\in C([t_0,\infty))$, $f\in C(R)$ with $yf(y)>0$ for $y\neq 0$ and $\int_{\pm 1}^{\pm\infty}[1/f(y)]\,dy<\infty$, and f is continously differentiable on $R-\{0\}$ with $f'(y)\geq 0$ for all $y\neq 0$. In the special case of the differential equation $x''(t)+a(t)\,|x(t)|^{\gamma}\operatorname{sgn} x(t)=0$ $(\gamma>1)$ this criterion leads to an oscillation result due to Wong [9].

1. **Introduction.** This paper deals with the problem of oscillation of second order superlinear ordinary differential equations of the form

$$x''(t)+a(t)f[x(t)]=0, \tag{E}$$

where *a is a continuous real-valued function on an interval* $[t_0,\infty)$ *without any restriction on its sign and f is a continuous real-valued function on the real line* R *with the sign property*

$$yf(y)>0 \quad \textit{for all} \quad y\neq 0.$$

It will be supposed that *f is continuously differentiable on* R-$\{0\}$ *with*

$$f'(y)\geq 0 \quad \textit{for all} \quad y\neq 0$$

and that *f is strongly superlinear in the sense that*

$$\int^{\infty}\frac{dy}{f(y)}<\infty \quad \textit{and} \quad \int^{-\infty}\frac{dy}{f(y)}<\infty.$$

Only such solutions x of the differential equation (E) which exist on some interval $[t_x,\infty)$, $t_x\geqq t_0$, are considered. A solution of (E) is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise it is said to be *nonoscillatory*. Equation (E) is called oscillatory if all its solutions are oscillatory.

For the special case where $f(y)=|y|^{\gamma}\operatorname{sgn} y$, $y\in R$ $(\gamma>1)$, i.e. for the

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differential equation

$$(E_0) \qquad x''(t) + a(t)\,|x(t)|^\gamma \operatorname{sgn} x(t) = 0 \quad (\gamma > 1),$$

Wong [9] proved the following oscillation criterion.

THEOREM A. *Equation* (E_0) *is oscillatory if*

$$(A_1) \qquad \liminf_{t\to\infty} \int_0^t a(s)\,ds > -\infty$$

and

$$(A_2) \qquad \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^t \int_{t_0}^s a(\tau)\,d\tau\,ds = \infty.$$

Also, Onose [5] showed Theorem B below concerning the oscillation of the differential equation (E).

THEOREM B. *Equation* (E) *is oscillatory if* (A_1) *and* (A_2) *hold and:*
(F_0) *there exists a positive constant* k *such that*

$$f'(y) \geq k \quad \textit{for all} \quad y \neq 0.$$

It is obvious that Theorem B cannot be applied to the special case of the differential equation (E_0) and hence Theorem A is not covered by Theorem B. Our purpose here is to give a new oscillation criterion for the differential equation (E), which has Theorem A as a particular case. Our result can be applied in some cases in which Theorems A and B are not applicable.

It is noteworthy that the conditions (A_1) and (A_2) are also sufficient for the oscillation of the differential equation (E) with $f(y) = y$, $y \in \boldsymbol{R}$. The validity of this result for the linear case follows from a result of Hartman [2], see also Coles [1] and Macki and Wong [4]. Also, the well-known criterion of Wintner [7] states that the condition

$$(A_2') \qquad \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^t \int_{t_0}^s a(\tau)\,d\tau\,ds = \infty$$

suffices for the oscillation in the linear case. Moreover, it is remarkable that, if the differential equation (E) is strongly sublinear in the sense that

$$\int_{+0} \frac{dy}{f(y)} < \infty \quad \text{and} \quad \int_{-0} \frac{dy}{f(y)} < \infty,$$

then the condition (A_2) implies oscillation even in the case where (A_1) fails (cf. Kamenev [3]).

For a general discussion on second order nonlinear oscillation problems we refer the reader to the paper of Wong [8]. We refer also to the survey article by Ševelo [6] where a detailed account on second order nonlinear oscillation and its physical motivations is presented.

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2. **Main result.** We state and prove here our oscillation criterion.

THEOREM. *Equation* (E) *is oscillatory if* (A_1) *and* (A_2) *hold and:*
(F) f *is such that*

$$(*)\qquad \int^{\infty} \frac{\sqrt{f'(y)}}{f(y)}\,dy<\infty \quad \textit{and} \quad \int^{-\infty} \frac{\sqrt{f'(y)}}{f(y)}\,dy<\infty$$

and

$$\min\left\{\inf_{y>0} \frac{\left[\int_y^{\infty} \frac{\sqrt{f'(z)}}{f(z)}\,dz\right]^2}{\int_y^{\infty} \frac{dz}{f(z)}},\quad \inf_{y<0} \frac{\left[\int_y^{-\infty} \frac{\sqrt{f'(z)}}{f(z)}\,dz\right]^2}{\int_y^{-\infty} \frac{dz}{f(z)}}\right\}>0.$$

Proof. The substitution $v=-x$ transforms the differential equation (E) into the equation $v''(t)+a(t)\hat{f}[v(t)]=0$, where $\hat{f}(y)=-f(-y)$, $y \in R$. The function $\hat{f}$ is subject to the conditions posed on f. Thus, with respect to the nonoscillatory solutions of (E) we can restrict our attention only to the positive ones.

Let x be a positive solution on an interval $[T, \infty)$, $T \geq \max\{1, t_0\}$, of the differential equation (E). We put

$$w(t)=x'(t)/f[x(t)], \quad t \geq T.$$

Then for $t \geq T$ we obtain

$$w'(t)=-a(t)-w^2(t)f'[x(t)]$$

and consequently we have

$$(1)\qquad \int_T^t a(s)\,ds=w(T)-w(t)-\int_T^t \{\sqrt{(f'[x(s)])}w(s)\}^2\,ds, \qquad t \geq T.$$

As in [9] (cf. also [5]), we distinguish three cases of the behavior of x':

Case 1: x' *is oscillatory*. Then there exists a sequence $(t_\nu)_{\nu=1,2,\ldots}$ in $[T, \infty)$ with $\lim_{\nu\to\infty} t_\nu=\infty$ and such that $x'(t_\nu)=0$, $\nu=1,2,\ldots$. Thus, by the condition (A_1), (1) gives

$$(2)\qquad \int_T^{\infty} \{\sqrt{(f'[x(t)])}w(t)\}^2\,dt<\infty$$

and so there exists a positive constant N such that

$$(3)\qquad \int_T^t \{\sqrt{(f'[x(s)])}w(s)\}^2\,ds \leq N \quad \text{for every} \quad t \geq T.$$

Furthermore, by using the Schwarz inequality, for $t \geq T$ we get

$$\left|\int_T^t \sqrt{(f'[x(s)])}w(s)\,ds\right|^2 \leq \left(\int_T^t ds\right)\int_T^t \{\sqrt{(f'[x(s)])}w(s)\}^2\,ds$$
$$=(t-T)\int_T^t \{\sqrt{(f'[x(s)])}w(s)\}^2\,ds$$

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and consequently, in view of (3),

$$\left|\int_T^t \sqrt{(f'[x(s)])}w(s)\,ds\right|^2 \leq Nt \quad \text{for all} \quad t \geqq T. \tag{4}$$

Condition (F) says that for some positive constant M we have

$$\int_{x(t)}^{\infty} \frac{dy}{f(y)} \leq M\left[\int_{x(t)}^{\infty} \frac{\sqrt{f'(y)}}{f(y)}\,dy\right]^2, \qquad t \geq T. \tag{5}$$

Now, by setting

$$K_1 = \int_{x(T)}^{\infty} [1/f(y)]\,dy > 0 \quad \text{and} \quad K_2 = \int_{x(T)}^{\infty} [\sqrt{(f'(y))}/f(y)]\,dy > 0$$

and taking into account (5) and (4), for any $t \geq T$ we obtain

$$\begin{aligned}
\left|\int_T^t w(s)\,ds\right| &= \left|\int_{x(T)}^{x(t)} \frac{dy}{f(y)}\right| = \left|K_1 - \int_{x(t)}^{\infty} \frac{dy}{f(y)}\right| \leq K_1 + \int_{x(t)}^{\infty} \frac{dy}{f(y)} \\
&\leq K_1 + M\left[\int_{x(t)}^{\infty} \frac{\sqrt{f'(y)}}{f(y)}\,dy\right]^2 = K_1 + M\left[K_2 - \int_{x(T)}^{x(t)} \frac{\sqrt{f'(y)}}{f(y)}\,dy\right]^2 \\
&\leq K_1 + M\left[K_2 + \left|\int_{x(T)}^{x(t)} \frac{\sqrt{f'(y)}}{f(y)}\,dy\right|\right]^2 \\
&= K_1 + M\left[K_2 + \left|\int_T^t \sqrt{(f'[x(s)])}w(s)\,ds\right|\right]^2 \\
&\leq K_1 + M(K_2 + \sqrt{Nt})^2 = K_1 + MK_2^2 + 2MK_2\sqrt{N}\sqrt{t} + MNt
\end{aligned}$$

and therefore

$$\left|\int_T^t w(s)\,ds\right| \leq c_1 + c_2 t \quad \text{for every} \quad t \geq T, \tag{6}$$

where $c_1 = K_1 + MK_2^2$, $c_2 = M(2K_2\sqrt{N} + N)$. Next, from (1) we derive

$$\int_T^t \int_T^s a(\tau)\,d\tau\,ds \leq (t-T)w(T) - \int_T^t w(s)\,ds, \qquad t \geq T$$

and hence, by (6), we have

$$\frac{1}{t}\int_T^t \int_T^s a(\tau)\,d\tau\,ds \leq c_2 + w(T) + \frac{c_1 - Tw(T)}{t} \quad \text{for all} \quad t \geq T,$$

which contradicts condition (A_2).

Case 2: $x' > 0$ *on* $[T^*, \infty)$ *for some* $T^* \geq T$. Then (1) gives

$$\int_T^t a(s)\,ds \leq w(T) \quad \text{for} \quad t \geq T^*$$

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and hence

$$\frac{1}{t}\int_{T^*}^{t}\int_{T}^{s} a(\tau)\,d\tau\,ds \le w(T)\left(1-\frac{T^*}{t}\right) \quad \text{for every} \quad t \ge T^*,$$

which again contradicts (A_2).

Case 3: $x'<0$ *on* $[T^*,\infty)$ *for some* $T^* \ge T$. By condition (A_1), from (1) it follows that for some constant C

$$-w(t) \ge C + \int_{T}^{t} \{\sqrt{(f'[x(s)])}w(s)\}^2\,ds \quad \text{for all} \quad t \ge T^*. \tag{7}$$

If (2) holds, then we arrive at a contradiction by the procedure of Case 1. So, we suppose that (2) fails and we consider a $\hat{T} \ge T^*$ so that

$$D \equiv C + \int_{T}^{T} \{\sqrt{(f'[x(s)])}w(s)\}^2\,ds > 0.$$

Then, by multiplying (7) through by

$$f'[x(t)]w(t)\Big/\left[C+\int_{T}^{t}\{\sqrt{(f'[x(s)])}w(s)\}^2\,ds\right]$$

and next integrating over $[\hat{T}, t]$, we obtain

$$\log\frac{C+\int_{T}^{t}\{\sqrt{(f'[x(s)])}w(s)\}^2\,ds}{D} \ge \log\frac{f[x(\hat{T})]}{f[x(t)]} \quad \text{for} \quad t \ge \hat{T}.$$

Hence, for every $t \ge \hat{T}$ we have

$$C+\int_{T}^{t}\{\sqrt{(f'[x(s)])}w(s)\}^2\,ds \ge Df[x(\hat{T})]/f[x(t)]$$

and consequently, by (7),

$$x'(t) \le -Df[x(\hat{T})] < 0 \quad \text{for all} \quad t \ge \hat{T}.$$

the last inequality leads to the contradiction $\lim_{t\to\infty} x(t) = -\infty$.

REMARK 1. For the special case where $f(y) = |y|^\gamma \operatorname{sgn} y$, $y \in R$ $(\gamma > 1)$, it is easy to verify that

$$\int_{y}^{(\operatorname{sgn} y)\infty} \frac{dz}{f(z)} = \frac{|y|^{1-\gamma}}{\gamma-1} \quad \text{and} \quad \int_{y}^{(\operatorname{sgn} y)\infty} \frac{\sqrt{f'(z)}}{f(z)}\,dz = \frac{2\gamma^{1/2}|y|^{(1-\gamma)/2}}{\gamma-1} \quad \text{for} \quad y \ne 0.$$

This means that (F) is satisfied. Hence, by applying our theorem to the particular case of the differential equation (E_0), we arrive at Theorem A.

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Now, we introduce the following conditions:

(F_1) (*) *holds and*

$$\min\left\{\inf_{y>0}\sqrt{f'(y)}\int_y^{\infty}\frac{\sqrt{f'(z)}}{f(z)}\,dz,\ \inf_{y<0}\sqrt{f'(y)}\int_y^{-\infty}\frac{\sqrt{f'(z)}}{f(z)}\,dz\right\}>0.$$

(F_2) (*) *holds, f' is increasing on* $(0,\infty)$ *and decreasing on* $(-\infty,0)$, *and*

$$\min\left\{\inf_{y>0}f'(y)\int_y^{\infty}\frac{dz}{f(z)},\ \inf_{y<0}f'(y)\int_y^{-\infty}\frac{dz}{f(z)}\right\}>0.$$

(F_3) (*) *holds, f' is positive and continuously differentiable on R-{0} with $yf''(y)\geq 0$ for all $y\neq 0$, $ff''/(f')^2$ is bounded on R-{0}.*

For these conditions we have

$$(F_3)\Rightarrow(F_2)\Rightarrow(F_1)\Rightarrow(F).$$

Indeed: (i) $(F_3)\Rightarrow(F_2)$. Suppose that (F_3) holds and consider a positive constant c so that

$$\frac{f(y)f''(y)}{[f'(y)]^2}\leq c \quad \text{for all} \quad y\neq 0.$$

Then for every $y\neq 0$ we obtain

$$\frac{f''(y)}{[f'(y)]^2}\operatorname{sgn} y\leq\frac{c}{f(y)}\operatorname{sgn} y$$

and consequently

$$\int_y^{(\operatorname{sgn} y)\infty}\frac{f''(z)}{[f'(z)]^2}\,dz\leq c\int_y^{(\operatorname{sgn} y)\infty}\frac{dz}{f(z)}.$$

Hence,

$$\frac{1}{f'(y)}-\lim_{|z|\to\infty}\frac{1}{f'(z)}\leq c\int_y^{(\operatorname{sgn} y)\infty}\frac{dz}{f(z)} \quad \text{for} \quad y\neq 0. \tag{8}$$

But, since f is increasing on R-{0}, we get

$$2\int_{y/2}^{y}\frac{dz}{f(z)}\geq\frac{y}{f(y)} \quad \text{for all} \quad y\neq 0,$$

which, because of the fact that f is strongly superlinear, gives

$$\lim_{|y|\to\infty}\frac{f(y)}{y}=\infty.$$

This means that

$$\lim_{|y|\to\infty}f'(y)=\infty.$$

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By this fact, (8) leads to

$$f'(y)\int_y^{(\operatorname{sgn} y)\infty}\frac{dz}{f(z)}\geq\frac{1}{c}\quad\text{for every}\quad y\neq 0$$

and so (F_2) is satisfied.

(ii) $(F_2)\Rightarrow(F_1)$. This implication is obvious.

(iii) $(F_1)\Rightarrow(F)$. We assume that (F_1) is fulfilled. Then there exists a positive constant d such that

$$\sqrt{f'(y)}\int_y^{(\operatorname{sgn} y)\infty}\frac{\sqrt{f'(z)}}{f(z)}\,dz\geq d\quad\text{for all}\quad y\neq 0.$$

Therefore, for $y\neq 0$ we derive

$$\left[\int_y^{(\operatorname{sgn} y)\infty}\frac{\sqrt{f'(z)}}{f(z)}\,dz\right]\frac{\sqrt{f'(y)}}{f(y)}\operatorname{sgn} y\geq\frac{d}{f(y)}\operatorname{sgn} y$$

and so

$$\int_y^{(\operatorname{sgn} y)\infty}\left[\int_u^{(\operatorname{sgn} y)\infty}\frac{\sqrt{f'(z)}}{f(z)}\,dz\right]\frac{\sqrt{f'(u)}}{f(u)}\,du\geq d\int_y^{(\operatorname{sgn} y)\infty}\frac{dz}{f(z)}.$$

Thus,

$$\frac{1}{2}\left[\int_y^{(\operatorname{sgn} y)\infty}\frac{\sqrt{f'(z)}}{f(z)}\,dz\right]^2\geq d\int_y^{(\operatorname{sgn} y)\infty}\frac{dz}{f(z)}\quad\text{for every}\quad y\neq 0,$$

which means that (F) holds.

3. **Examples and remarks.** Here, we give three examples of differential equations of the form (E) for which all assumptions of our theorem on the function f are satisfied. We also provide an example of an equation of the form (E) where the condition (F) fails while all other assumptions on f hold. These examples will be used to demonstrate the fact that the class of the differential equations of the form (E), for which the conditions on f of our theorem are valid, is sufficiently wide, and to compare our criterion with Theorem B.

EXAMPLE 1. Consider the case where

$$f(y)=|y|^{\gamma}\{\lambda+\sin[\log(1+|y|)]\}\operatorname{sgn} y,\qquad y\in\boldsymbol{R},$$

where $\gamma>1$ and $\lambda>1+1/\gamma$. In this case the function f is continuous on $\boldsymbol{R}$ and has the sign property $yf(y)>0$ for all $y\neq 0$. Also, f is strongly superlinear, since

$$f(y)\operatorname{sgn} y\geq(\lambda-1)\,|y|^{\gamma}\quad\text{for every}\quad y\neq 0.$$

Moreover, f is continuously differentiable on $\boldsymbol{R}$-$\{0\}$ with

$$f'(y)=\gamma\,|y|^{\gamma-1}\{\lambda+\sin[\log(1+|y|)]\}+\frac{|y|^{\gamma}}{1+|y|}\cos[\log(1+|y|)],\qquad y\neq 0.$$

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Hence,

$$0<\gamma[\lambda-(1+1/\gamma)]\,|y|^{\gamma-1}\le f'(y)\le\gamma[\lambda+(1+1/\gamma)]\,|y|^{\gamma-1}\quad\text{for}\quad y\neq 0$$

and consequently

$$\frac{f(y)}{\sqrt{f'(y)}}\operatorname{sgn} y\ge\frac{(\lambda-1)\,|y|^{\gamma}}{\sqrt{\gamma[\lambda+(1+1/\gamma)]}\,|y|^{(\gamma-1)/2}}=\frac{\lambda-1}{\sqrt{\gamma[\lambda+(1+1/\gamma)]}}\,|y|^{(\gamma+1)/2}$$

for all $y\neq 0$. So, $f/\sqrt{f'}$ is strongly superlinear (i.e. (*) holds), since $(\gamma+1)/2>1$. Furthermore, for any $y\neq 0$ we get

$$\begin{aligned}\int_y^{(\operatorname{sgn} y)\infty}\frac{\sqrt{f'(z)}}{f(z)}\,dz&=\int_{|y|}^{\infty}\frac{\sqrt{f'(z)}}{f(z)}\,dz\ge\frac{\sqrt{\gamma[\lambda-(1+1/\gamma)]}}{\lambda+1}\int_{|y|}^{\infty}\frac{dz}{z^{(\gamma+1)/2}}\\&=\frac{2\sqrt{\gamma[\lambda-(1+1/\gamma)]}}{(\lambda+1)(\gamma-1)}\,|y|^{(1-\gamma)/2}.\end{aligned}$$

Therefore, for all $y\neq 0$ we have

$$\sqrt{f'(y)}\int_y^{(\operatorname{sgn} y)\infty}\frac{\sqrt{f'(z)}}{f(z)}\,dz\ge\frac{2\gamma[\lambda-(1+1/\gamma)]}{(\lambda+1)(\gamma-1)}>0,$$

which means that (F_1) holds. Thus, condition (F) is fulfilled. Hence, we derive the following result: *Under the conditions* (A_1) *and* (A_2), *the differential equation*

$$x''(t)+a(t)\,|x(t)|^{\gamma}\,\{\lambda+\sin[\log(1+|x(t)|)]\}\operatorname{sgn} x(t)=0\qquad(\gamma>1,\ \lambda>1+1/\gamma)$$

is oscillatory. Note that this result cannot be obtained from Theorem B, since here (F_0) fails.

EXAMPLE 2. Put

$$f(y)=\frac{|y|^{2\gamma}\operatorname{sgn} y}{1+|y|^{\gamma}},\qquad y\in\boldsymbol{R},$$

where $\gamma>1$. Then f is a continuous function on $\boldsymbol{R}$ such that $yf(y)>0$ for every $y\neq 0$. The function f is continuously differentiable on $\boldsymbol{R}$-{0} and

$$f'(y)=\frac{\gamma\,|y|^{2\gamma-1}\,(2+|y|^{\gamma})}{(1+|y|^{\gamma})^2}>0,\qquad y\neq 0.$$

Moreover, f is strongly superlinear and for all $y\neq 0$

$$\int_y^{(\operatorname{sgn} y)\infty}\frac{dz}{f(z)}=\int_{|y|}^{\infty}\frac{dz}{f(z)}=\int_{|y|}^{\infty}\frac{dz}{z^{2\gamma}}+\int_{|y|}^{\infty}\frac{dz}{z^{\gamma}}=\frac{|y|^{1-2\gamma}}{2\gamma-1}+\frac{|y|^{1-\gamma}}{\gamma-1}.$$

Furthermore, we obtain

$$\frac{\sqrt{f'(y)}}{f(y)\operatorname{sgn} y}=\frac{\sqrt{\gamma}\,|y|^{(2\gamma-1)/2}\sqrt{2+|y|^{\gamma}}}{|y|^{2\gamma}}\le\frac{\sqrt{\gamma}\cdot\sqrt{|y|^{\gamma}+|y|^{\gamma}}}{|y|^{\gamma+1/2}}=\frac{\sqrt{2\gamma}}{|y|^{(\gamma+1)/2}}$$

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for all y with $|y| \geq 2^{1/\gamma}$. This, because of the fact that $(\gamma+1)/2 > 1$, ensures the validity of (*). Now, we have

$$\left[\int_y^{(\operatorname{sgn} y)\infty} \frac{\sqrt{f'(z)}}{f(z)}\, dz\right]^2 \Big/ \int_y^{(\operatorname{sgn} y)\infty} \frac{dz}{f(z)} = \left[\sqrt{\gamma}\int_{|y|}^{\infty} \frac{\sqrt{2+z^\gamma}}{z^{\gamma+1/2}}\, dz\right]^2 \Big/ \left(\frac{|y|^{1-2\gamma}}{2\gamma-1} + \frac{|y|^{1-\gamma}}{\gamma-1}\right)$$

for all $y \neq 0$. But, it is a matter of elementary calculations to verify that

$$\lim_{u\to 0+0} \left\{\left[\sqrt{\gamma}\int_u^{\infty} \frac{\sqrt{2+z^\gamma}}{z^{\gamma+1/2}}\, dz\right]^2 \Big/ \left(\frac{u^{1-2\gamma}}{2\gamma-1} + \frac{u^{1-\gamma}}{\gamma-1}\right)\right\} = \frac{8\gamma}{2\gamma-1} > 0$$

and

$$\lim_{u\to\infty} \left\{\left[\sqrt{\gamma}\int_u^{\infty} \frac{\sqrt{2+z^\gamma}}{z^{\gamma+1/2}}\, dz\right]^2 \Big/ \left(\frac{u^{1-2\gamma}}{2\gamma-1} + \frac{u^{1-\gamma}}{\gamma-1}\right)\right\} = \frac{4\gamma}{\gamma-1} > 0.$$

This means that the condition (F) is satisfied and so we have the result: *The differential equation*

$$x''(t) + a(t)\frac{|x(t)|^{2\gamma}}{1+|x(t)|^\gamma} \operatorname{sgn} x(t) = 0 \quad (\gamma > 1)$$

is oscillatory if (A_1) *and* (A_2) *are satisfied.* It is remarkable that we cannot derive this result from Theorem B.

EXAMPLE 3. Let us consider the continuous function f defined by

$$f(y) = |y|^\gamma \operatorname{sgn} y + y, \qquad y \in \boldsymbol{R},$$

where $\gamma > 1$. Obviously, $yf(y) > 0$ for $y \neq 0$. Also, we observe that

$$f(y)\operatorname{sgn} y \geq |y|^\gamma \quad \text{for all} \quad y \neq 0$$

and consequently f is strongly superlinear. Furthermore, the function f is twice continuously differentiable on $\boldsymbol{R}$-{0} with

$$f'(y) = \gamma\,|y|^{\gamma-1} + 1 \quad \text{and} \quad f''(y) = \gamma(\gamma-1)\,|y|^{\gamma-2} \operatorname{sgn} y \quad \text{for} \quad y \neq 0.$$

Thus, $f'(y) > 0$ and $yf''(y) > 0$ for every $y \neq 0$. Moreover, $f/\sqrt{f'}$ is strongly superlinear, since for $|y| \geq 1$ we have

$$\frac{f(y)}{\sqrt{f'(y)}}\operatorname{sgn} y = \frac{|y|^\gamma + |y|}{\sqrt{\gamma\,|y|^{\gamma-1}+1}} \geq \frac{|y|^\gamma}{\sqrt{\gamma\,|y|^{\gamma-1} + |y|^{\gamma-1}}} = \frac{1}{\sqrt{\gamma+1}}\,|y|^{(\gamma+1)/2},$$

where $(\gamma+1)/2 > 1$. Now, for every $y \neq 0$ we obtain

$$\frac{f(y)f''(y)}{[f'(y)]^2} = \gamma(\gamma-1)\frac{|y|^{\gamma-1}\,(|y|^{\gamma-1}+1)}{(\gamma\,|y|^{\gamma-1}+1)^2} < \gamma(\gamma-1)\frac{|y|^{\gamma-1}}{|y|^{\gamma-1}+1} < \gamma(\gamma-1).$$

Namely, $ff''/(f')^2$ is bounded on $\boldsymbol{R}$-{0}. Hence, *the conditions* (A_1) *and* (A_2) *are sufficient for the oscillation of the differential equation*

$$x''(t) + a(t)[|x(t)|^\gamma \operatorname{sgn} x(t) + x(t)] = 0 \quad (\gamma > 1).$$

We notice that this result follows also from Theorem B.

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EXAMPLE 4. The function f defined as follows

$$f(y) = y \log^2(\mu + |y|), \qquad y \in \boldsymbol{R},$$

where $\mu \geq 1$, is continuous on $\boldsymbol{R}$ and such that $yf(y) > 0$ for all $y \neq 0$. Also, f is continuously differentiable on $\boldsymbol{R}$-{0} and

$$f'(y) = \log^2(\mu + |y|) + \frac{2\,|y|}{\mu + |y|} \log(\mu + |y|) > 0 \quad \text{for every} \quad y \neq 0.$$

Moreover, we have

$$\int_{\pm e}^{\pm\infty} \frac{dy}{f(y)} = \int_{e}^{\infty} \frac{dy}{y \log^2(\mu + y)} \leq \int_{e}^{\infty} \frac{dy}{y \log^2 y} = \int_{1}^{\infty} \frac{dz}{z^2} < \infty$$

and consequently the function f is strongly superlinear. But, (*) is not here satisfied. Indeed,

$$\int_{\pm 1}^{\pm\infty} \frac{\sqrt{f'(y)}}{f(y)}\, dy = \int_{1}^{\infty} \frac{\sqrt{f'(y)}}{f(y)}\, dy \geq \int_{1}^{\infty} \frac{\sqrt{\log^2(\mu + y)}}{y \log^2(\mu + y)}\, dy = \int_{1}^{\infty} \frac{dy}{y \log(\mu + y)}$$

$$\geq \int_{1}^{\infty} \frac{dy}{(\mu + y)\log(\mu + y)} = \int_{\log(\mu+1)}^{\infty} \frac{dz}{z} = \infty.$$

So, condition (F) fails and hence *our theorem cannot be applied for the differential equation*

$$x''(t) + a(t)x(t)\log^2(\mu + |x(t)|) = 0 \quad (\mu \geq 1).$$

In the case of this equation, *Theorem* B *is also not applicable if* $\mu = 1$, *while for* $\mu > 1$ *Theorem* B *can be applied* to conclude that (A_1) and (A_2) suffice for the oscillation.

REMARK 2. Condition (F) holds by itself in a numerous special cases of differential equations of the form (E), as for example the cases of the differential equation (E_0) and of the equations of Examples 1, 2 and 3. Thus, we can say that the assumption (F) on the function f is not very restrictive.

REMARK 3. The example of the differential equation (E_0) and Examples 1 and 2 show that our theorem can be applied in some cases in which Theorem B cannot be applied. On the other hand, by Example 4 (with $\mu > 1$), there exist cases in which Theorem B can be applied while our criterion is not applicable. Moreover, Example 3 demonstrates that it is possible to have cases where our theorem as well as Theorem B are applicable, while Example 4 (with $\mu = 1$) ensures that in some cases both these criteria cannot be applied.

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