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A CHARACTERIZATION OF IDENTITIES IMPLYING CONGRUENCE MODULARITY I

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0. Introduction. In his thesis and [**24**], J. B. Nation showed the existence of certain lattice identities, strictly weaker than the modular law, such that if all the congruence lattices of a variety of algebras $\mathscr{K}$ satisfy one of these identities, then all the congruence lattices were even modular. Moreover Freese and Jónsson showed in [**10**] that from this "congruence modularity" of a variety of algebras one can even deduce the (stronger) Arguesian identity.

These and similar results [**3**; **5**; **9**; **12**; **18**; **21**] induced Jónsson in [**17**; **18**] to introduce the following notions. For a variety of algebras $\mathscr{K}$, $\mathbf{Con}(\mathscr{K}) = \mathbf{HSP}\Theta(\mathscr{K})$ is the (congruence) variety of lattices generated by the class $\Theta(\mathscr{K})$ of all congruence lattices $\Theta(A)$, $A \in \mathscr{K}$. Secondly if ϵ is a lattice identity, and Σ is a set of such, $\Sigma \models_c \epsilon$ holds if for any variety $\mathscr{K}$, $\mathbf{Con}(\mathscr{K}) \models \Sigma$ implies $\mathbf{Con}(\mathscr{K}) \models \epsilon$.

In [**2**] and [**16**] characterizations of $\mathbf{Con}(\mathscr{K}) \models$ mod and $\mathbf{Con}(\mathscr{K}) \models$ dist were found (mod (dist) is the modular (resp. distributive) law). These statements express the so-called congruence modularity or congruence distributivity of a variety $\mathscr{K}$. Furthermore in [**11**] it was shown that for a variety of semigroups $\mathscr{K}$, $\mathbf{Con}(\mathscr{K}) \models \epsilon$ where ϵ is any non-trivial lattice identity implies $\mathscr{K}$ is congruence modular.

The aforementioned results led to a conjecture that there existed no proper non-modular congruence varieties but this conjecture was shattered by a recent result of Polin [**25**], where a variety of algebras $\mathscr{P}$ is produced that is not congruence modular and which has $\mathbf{Con}(\mathscr{P}) \neq \mathscr{L}$. A detailed analysis of this variety $\mathscr{P}$ (and $\mathbf{Con}(\mathscr{P})$) has allowed us to produce several complete characterizations of congruence modularity and to answer some related questions about congruence varieties and the congruence satisfaction relation $\models_c$.

The main result (6.1) states that $\mathbf{Con}(\mathscr{P})$ is the smallest non-modular congruence variety (of lattices). The proof of this fact involves showing that the lattices, $\Theta(F_{\mathscr{P}}(n))$ ($n < \omega$), are in fact splitting lattices with conjugate splitting equations ζ_n. These results allow us to prove a very strong compactness result that $\Sigma \models_c$ mod if and only if $\delta \models_c$ mod for some $\delta \in \Sigma$. The splitting equations allow us to characterize "$\delta \models_c$ mod" in

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terms of the usual lattice satisfaction relation, viz: $\delta \vDash_c$ mod if and only if $\delta \vDash \zeta_n$ for some $n < \omega$. In Part II, the second author sharpens this last result to a recursive statement: if and only if $\delta \vDash \zeta_{h(\delta)}$ where h is a suitable function from the set of lattice equations into ω.

1. Preliminaries. We need two main results; one refers to generators of congruence varieties and the other to McKenzie's splitting lattices. The first result appears explicitly in Nation's thesis and is also a consequence of Wille's work on Mal'cev conditions for lattice identities, [**27**]. See also [**15**].

(1.1) Proposition. *Let $\mathscr{K}$ be a class of algebras closed under the formation of subalgebras. Then a lattice identity ϵ holds in* **Con**$(\mathscr{K})$ *if and only if it holds in* $\{\Theta(A): A \in \mathscr{K}$ *and A is finitely generated*$\}$. *Moreover if $\mathscr{K}$ is also closed under products,* **Con**$(\mathscr{K}) \vDash \epsilon$ *if and only if ϵ holds in* $\{\Theta(F_{\mathscr{K}}(n)): n < \omega\}$.

In [**22**], McKenzie developed the notions of a bounded homomorphism and a splitting lattice. These notions, and their subsequent development have had a profound effect on lattice theory. We recall the relevant definitions.

A subdirectly irreducible lattice L is called a *splitting lattice* if there exists a lattice equation ϵ (called the conjugate or splitting equation of L) such that for any variety $\mathscr{V}$ of lattices either $L \in \mathscr{V}$ or $\mathscr{V} \vDash \epsilon$ but not both. From Dean's result [**7**] that the variety of all lattices is generated by its finite members, one can easily show that all splitting lattices are finite.

An epimorphism $f: M \twoheadrightarrow L$ is *upper bounded* if there exists a function $\alpha: L \to M$ with $f \circ \alpha = 1_L$ and $1_M \leqq \alpha \circ f$. A finite lattice L is called an *upper-bounded-lattice* if there is an upper bounded epimorphism from some free lattice onto L. Lower bounded epimorphisms and lower-bounded-lattices are defined dually. A finite lattice L is called *bounded* if it is both upper and lower bounded.

Let L be a finite lattice and take $a \in L$. A finite non-empty subset $U \subseteq L$ is a *cover* of a if $a \leqq \bigvee U$. U is a *non-trivial cover* of a if in addition $a \nleqq u$ for all $u \in U$. Define $V \ll U$ to mean for all $v \in V$, $v \leqq u$ for some $u \in U$. A (non-trivial) cover U of a is called a *minimal cover* of a if, whenever V is a cover of a with $V \ll U$, then $U \subseteq V$. Since L is finite, minimal covers exists and are easily seen to consist of join-irreducible members of L.

Finally let

$$D_0(L) = \{a \in L: a \text{ has no non-trivial covers}\},\ D_{k+1}(L)$$
$$= \{a \in L: \text{every non-trivial minimal cover of } a \text{ is a subset of } D_k(L)\}$$

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and

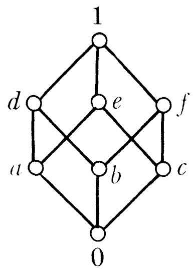
$$D(L) = \bigcup_{k<\omega} D_k(L).$$

$D_k'(L)$ and $D'(L)$ are defined dually.

(1.2) THEOREM ([**22**] and [**13**]). *For a finite subdirectly irreducible lattice L, the following are equivalent:*
(1) *L is a splitting lattice*
(2) *L is a bounded lattice*
(3) $D(L) = L = D'(L)$.

We refer the reader to [**19**] for historical notes on and the proof of this result. We note here that a splitting equation for L can be determined in the following way: Let $p \prec q$ be a (prime) critical quotient in L (i.e., one that generates the least non-trivial congruence on L), and f: $FL(X) \twoheadrightarrow L$ be a "suitable" epimorphism. Using $D(L) = L(D'(L) = L)$ one can construct the lower-bound (resp. upper bound) function β: $L \rightarrow FL(X)$ (resp. α: $L \rightarrow FL(X)$). This construction will depend on the join (resp. meet) irreducible elements and their minimal covers (resp. minimal co covers). A splitting equation for L is then given by $\beta(q) \leqq \alpha(p)$.

2. Polin's variety, $\mathscr{P}$. Polin created his variety by using the variety of Boolean algebras, $\mathscr{B}$, in two ways, externally and internally. Intuitively he considered an "external" or skeletal Boolean algebra, A, e.g.



and replaced: (i) each element $a \in A$, with another Boolean algebra, $\mathbf{S}(a)$,

(ii) every order relation $a \geqq b$, with a homomorphism $\xi_b{}^a$: $\mathbf{S}(a) \rightarrow \mathbf{S}(b)$,

(iii) assumed that the homomorphisms were "compatible" with the order relation, i.e.,

(a) $a \geqq b \geqq c$ imply $\xi_c{}^b \circ \xi_b{}^a = \xi_c{}^a$

(b) $\xi_a{}^a = \mathrm{id}_{\mathbf{S}(a)}$

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e.g. the commutative diagram of Boolean algebras:

$$\begin{tikzcd}
& \mathbf{S}(1) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbf{S}(d) \arrow[d] \arrow[dr] & \mathbf{S}(e) \arrow[dl] \arrow[dr] & \mathbf{S}(f) \arrow[dl] \arrow[d] \\
\mathbf{S}(a) \arrow[dr] & \mathbf{S}(b) \arrow[d] & \mathbf{S}(c) \arrow[dl] \\
& \mathbf{S}(0) &
\end{tikzcd}$$

Category theoretists would recognize such entities as functors $\mathbf{S}\colon (A, \geqq) \to \mathscr{B}$; we will need the set-theoretical description:

$$P = P(\mathbf{S}, A) = \cup_{a \in A} \{a\} \times \mathbf{S}(a).$$

P becomes an algebra of type $(2, 0, 1, 1)$ via:

$$(a, s) \cdot (b, t) = (a \cdot b, \xi_{ab}{}^a(s) \cdot \xi_{ab}{}^b(t))$$
$$1 = (1, 1)$$
$$(a, s)' = (a, s') \text{ (internal complement)}$$
$$(a, s)^+ = (a', 1) \text{ (external complement)}$$

where in both co-ordinates $x \cdot y$ is the meet of x and y.

Easy calculations show that $(P, \cdot, 1)$ is a meet-semilattice with unit $(1, 1)$ in which $(a, s) \geqq (b, t)$ if and only if $a \geqq b$ and $\xi_b{}^a(s) \geqq t$.

Polin showed that the (abstract) class of algebras having such a representation is equationally definable (in terms of $(\cdot, 1, ', {}^+)$) and in fact is a finitely based variety. His result is:

(2.1) THEOREM (Polin). **Con**$(\mathscr{P})$ *is a proper but non-modular variety of lattices.*

Since we will require a detailed analysis of congruence lattices of algebras in $\mathscr{P}$, we need a full description of congruence relations on members of $\mathscr{P}$.

(2.2) *Definition.* For $P = P(\mathbf{S}, A) \in \mathscr{P}$, and $\theta \in \Theta(P)$, define

(i) $\theta_* = \{(a, b) \in A^2\colon (a, 1)\theta(b, 1)\}$
(ii) $\theta_a = \{(s, t) \in \mathbf{S}(a)^2\colon (a, s)\theta(a, t)\}, (a \in A)$.

It follows easily that the above are congruence relations on their respective Boolean algebras.

(2.3) LEMMA. $(a, s)\theta(b, t)$ *if and only if* $a\theta_* b$ *and* $\xi_{ab}{}^a(s)\theta_{ab}\xi_{ab}{}^b(t)$.

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Proof. If $(a, s)\theta(b, t)$ then clearly $(a, 1) = (a, s)^{++}\theta(b, t)^{++} = (b, 1)$ and furthermore

$$(ab, \xi_{ab}{}^a(s)) = (a, s)\cdot(b, t)^{++}\theta(b, t)\cdot(a, s)^{++} = (ab, \xi_{ab}{}^b(t)).$$

Conversely if $a\theta_* b$ and $\xi_{ab}{}^a(s)\theta_{ab}\xi_{ab}{}^b(t)$ we have

$$(a, s) = (a, s)(a, 1)\theta(a, s)(b, 1) = (ab, \xi_{ab}{}^a(s))$$

and

$$(b, t) = (b, t)(b, 1)\theta(b, t)(a, 1) = (ab, \xi_{ab}{}^b(t))$$

and therefore $(a, s)\theta(b, t)$ by transitivity.

(2.4) LEMMA. *Assume $a \geqq b$. If $s\theta_a t$ then $\xi_b{}^a(s)\theta_b\xi_b{}^a(t)$. Moreover if $a\theta_* b$ holds, the reverse implication is also true.*

Proof. If $(a, s)\theta(b, t)$ then meeting with $(b, 1)$ and using the fact that $ab = b$ proves the required implication.

Now if $a\theta_* b$ and $\xi_b{}^a(s)\theta_b\xi_b{}^a(t)$ then $s\theta_a t$ follows from $b = ab$ and the previous lemma.

(2.5) *Definition.* Associated with each homomorphism $\xi_b{}^a: \mathbf{S}(a) \to \mathbf{S}(b)$, is a function $\kappa_b{}^a: \Theta(\mathbf{S}(b)) \to \Theta(\mathbf{S}(a))$ defined by:

$$\begin{aligned}\kappa_b{}^a(\psi) &= (\xi_b{}^a \times \xi_b{}^a)^{-1}[\psi]\\ &= \{(s, t) \in \mathbf{S}(a)^2: \xi_b{}^a(s)\psi\xi_b{}^a(t)\}.\end{aligned}$$

Clearly $\kappa_b{}^a$ preserves arbitrary intersections (= meets) and hence also preserves order. Moreover for $a \geqq b \geqq c$, $\kappa_b{}^a \circ \kappa_c{}^b = \kappa_c{}^a$.

(2.6) *Definition.* For $P = P(\mathbf{S}, A) \in \mathscr{P}$, let $\mathrm{Rep}(P)$ be the set of all $(\theta_*; (\theta_a)_{a\in A}) \in \Theta(A) \times \prod_{a\in A} \Theta(\mathbf{S}(a))$ satisfying:

($R1$) $a \geqq b$ implies $\theta_a \leqq \kappa_b{}^a(\theta_b)$

($R2$) $a \geqq b$ and $a\theta_* b$ imply $\theta_a = \kappa_b{}^a(\theta_b)$.

The previous lemmata provide us with the following result:

(2.7) THEOREM. *For $P = P(\mathbf{S}, A) \in \mathscr{P}$, $(\Theta(P), \leqq)$ and $(\mathrm{Rep}(P), \leqq)$ are isomorphic lattices where $\leqq$ on $\mathrm{Rep}(P)$ is the product order and meets in $\mathrm{Rep}(P)$ are computed component-wise.*

Subsequently we will identify congruences on $P(\mathbf{S}, A)$ with their representations. One might note at this time that the Polin algebra, P:

$$\begin{array}{c}\mathbf{2}\\ \downarrow\\ \mathbf{2}\end{array}$$

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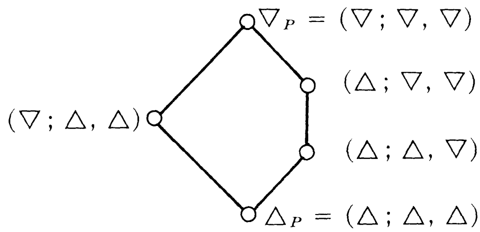
has as its congruence (= representation) lattice



where $(\alpha; \beta, \gamma) = (\theta_*; \theta_1, \theta_0)$.

(2.8) COROLLARY. *The map* $\theta \mapsto \theta_*$ *is a lattice homomorphism from* $\Theta(P)$ *to* $\Theta(A)$.

Proof. By (2.7), the map preserves (arbitrary) meets. For $\theta, \psi \in \Theta(P)$, $(a, 1)\theta \vee \psi(b, 1)$ if and only if $\exists\, n \in \mathbf{N}$ and $(a, 1) = (c_0, s_0), (c_1, s_1), \ldots (c_n, s_n) = (b, 1)$ such that for $i < n$,

$$(c_i, s_i)\theta(c_{i+1}, s_{i+1}) \quad (i \text{ even})$$

and

$$(c_i, s_i)\psi(c_{i+1}, s_{i+1}) \quad (i \text{ odd}).$$

Using $(\ )^{++}$ this is equivalent to: $\exists\, n \in \mathbf{N}$ and $a = c_0, c_1, \ldots, c_n = b$ such that for $i < n$

$$(c_i, 1)\theta(c_{i+1}, 1) \quad (i \text{ even})$$

$$(c_i, 1)\psi(c_{i+1}, 1) \quad (i \text{ odd}).$$

But this is equivalent to $a\theta_* \vee \psi_* b$.

The general formula for joins in $\Theta(P)$ is messy and not of much use. If however the "external" Boolean algebra, A, is finite, a reasonable and very useful method exists.

Firstly if A is finite, then every θ_* is of the form $\operatorname{con}(b, 1) = \operatorname{con}(0, b')$ for some $b \in A$. Secondly $A/\theta_* \cong [0, b]$ and by $(R2)$, the set $\{\theta_x : x \leqq b\}$ determines all other θ_a by

$$\theta_a = \kappa_{ab}{}^a(\theta_{ab}).$$

This is the main content of the following two results.

(2.9) LEMMA. *For* A *finite,* $\theta, \psi \in \Theta(P)$ *with* $\theta_* = \operatorname{con}(b, 1)$ *and* $\psi_* = \operatorname{con}(c, 1)$ *then* $\theta \leqq \psi$ *if and only if* $c \leqq b$ *and* $\theta_x \leqq \psi_x$ *for all* $x \leqq c$.

(2.10) LEMMA. *If* A *finite, and* $\theta = (\operatorname{con}(b, 1); (\theta_a)_{a \in A}), \psi = (\operatorname{con}(c, 1);$

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$(\psi_a)_{a \in A}) \in \Theta(P)$, *then* $\theta \vee \psi$ *is given by:*

(i) $(\theta \vee \psi)_* = \mathrm{con}(bc, 1)$
(ii) $(\theta \vee \psi)_a = \kappa_{abc}{}^a(\theta_{abc} \vee \psi_{abc})$.

This representation of congruences also allows us to describe all subdirectly irreducible members of $\mathscr{P}$.

(2.11) THEOREM. *The subdirectly irreducible members of* $\mathscr{P}$ *are (up to isomorphism) the following list:*

(1) $A = \mathbf{2}, \mathbf{S}(0) = \mathbf{S}(1) = \mathbf{1}$
(2) $A = \mathbf{1}, \mathbf{S}(0) = \mathbf{2}(0 = 1$ *in* $A)$
(3) A *arbitrary*, $\mathbf{S}(1) = \mathbf{2}$ *and* $\mathbf{S}(a) = \mathbf{1}, a < 1$.

(Note that the homomorphisms for all $\mathscr{P}$ algebras in the above list are uniquely defined and therefore need not be mentioned.)

Proof. It is easily seen that all algebras in the given list are indeed subdirectly irreducible members of $\mathscr{P}$. Conversely let $P = P(\mathbf{S}, A)$ be subdirectly irreducible.

If $A = \mathbf{1}$, then $(\ )^+$ is a constant unary operation and therefore $\mathbf{S}(0 = 1)$ must be the unique subdirectly irreducible Boolean algebra, $\mathbf{2}$.

If $A \neq \mathbf{1}$ then for any $b < 1$ we define two congruence relations on P by:

$$\theta_* = \Delta; \theta_a = \begin{cases} \nabla & a \leqq b \\ \Delta, & a \nleqq b \end{cases}$$
$$\psi_* = \mathrm{con}(b, 1), \psi_a = \Delta, a \leqq b.$$

Since $\psi \neq \Delta_P$ and $\theta \wedge \psi = \Delta_P$ we must have $\theta = \Delta_P$. But this forces for all $a \leqq b$, $\nabla = \Delta$ and therefore $\mathbf{S}(a) = \mathbf{1}$. Since $b < 1$ was arbitrary, $\mathbf{S}(b) = 1$ for all $b < 1$.

If $\mathbf{S}(1) = \mathbf{1}$, then $(\ )'$ is the identity function and our algebra is isomorphic to $(A, \cdot, 1, \mathrm{id}, ')$. Therefore $A = \mathbf{2}$.

If $\mathbf{S}(1) \neq \mathbf{1}$, then easy calculations show that it must be a subdirectly irreducible Boolean algebra. Therefore $\mathbf{S}(1) = \mathbf{2}$.

Note. Case (2) in our list is contained (vacuously) in Case (3).

(2.12) COROLLARY 1. $\mathscr{P}$ *is a locally finite variety. (Equivalently, finitely generated free* $\mathscr{P}$*-algebras are finite.)*

3. Congruence lattices of finite members of $\mathscr{P}$. By (1.1), $\mathbf{Con}(\mathscr{K}) = \mathbf{HSP}\{\Theta(F_{\mathscr{K}}(n)) : n \in \mathbf{N}\}$ for any variety of algebras $\mathscr{K}$. Therefore in order to determine $\mathbf{Con}(\mathscr{P})$, we need to know $\Theta(F_{\mathscr{P}}(n))$ for every $n \in \mathbf{N}$. In the next section these will be described by means of a special representation for free $\mathscr{P}$-algebras. Most of the details however can be seen more clearly by examining arbitrary finite algebras.

Throughout this section, $P = P(\mathbf{S}, A)$ will be a finite algebra in $\mathscr{P}$. Therefore A and all $\mathbf{S}(a)$, $a \in A$, are finite.

(3.1) *Definition*. For $b \in A$ and $u \in \mathbf{S}(b)$ and $v \in \mathbf{S}(0)$ define congruences on P by:

$$\phi_b: (\phi_b)_* = \operatorname{con}(b, 1);\ (\phi_b)_a = \nabla$$

$$\theta_{b,u}: (\theta_{b,u})_* = \operatorname{con}(b, 1);\ (\theta_{b,u})_a = \begin{cases} \operatorname{con}(u, 1), & a = b \\ \nabla \quad , & a < b \end{cases}$$

$$\psi_v = \theta_{0,v}.$$

The characterization of the subdirectly-irreducibles in (2.11) provides the following result.

(3.2) LEMMA. *The meet-irreducible congruences are precisely the following:*

(1) ϕ_p, $0 \prec p \in A$

(2) $\theta_{b,q}$, $0 \prec q \in \mathbf{S}(b)$

(3) ψ_s, $0 \prec s \in \mathbf{S}(0)$

with their respective unique covers given by:

(1) ∇

(2) $\phi_b, b \in A$

(3) $\nabla = \phi_0$.

(3.3) LEMMA. *The only order relations between the meet-irreducibles congruences is given by:*

$$\theta_{b,q} \leqq \phi_p \text{ if and only if } p \leqq b.$$

Proof. A $\theta_{b,q}$ produces a factor algebra with either a trivial "external" Boolean algebra (if $b = 0$) or one with only trivial "internal" Boolean algebras except at $\mathbf{S}(1)$. Therefore no such two can be comparable. Since $|P/\phi_p| = 2$, all ϕ_p's are maximal, and therefore the only comparabilities can be of the form $\theta_{b,q} \leqq \phi_p$. Now

$$(\theta_{b,q})_* = \operatorname{con}(b, 1) \text{ and } (\phi_p)_* = \operatorname{con}(p, 1).$$

Since $(\phi_p)_a = \nabla$ for all $a \in A$

$$\begin{aligned} \theta_{b,q} \leqq \phi_p &\Leftrightarrow (\theta_{b,q})_* \leqq (\phi_p)_* \\ &\Leftrightarrow \operatorname{con}(b, 1) \subseteq \operatorname{con}(p, 1) \\ &\Leftrightarrow p \leqq b. \end{aligned}$$

(3.4) LEMMA. *For all* $0 \prec p \in A$, ϕ_p *is meet-prime.*

Proof. Define τ_p by

$$(\tau_p)_* = \operatorname{con}(p', 1) = \operatorname{con}(0, p)$$
$$(\tau_p)_x = \Delta \ (x \leqq p').$$

Now for $q \succ 0$ in A, $q \neq p$, $q \leqq p'$ and $(\tau_p)_* \leqq (\phi_q)_*$. For $b \in A$ and $0 \prec q \in \mathbf{S}(b)$ $\theta_{b,q} \nleqq \phi_p \Leftrightarrow p \nleqq b \Leftrightarrow b \leqq p' \Leftrightarrow \tau_p \leqq \theta_{b,q}$.

Since $\Theta(P)$ is generated by its meet-irreducibles,

$$\Theta(P) = [\Delta, \phi_p] \cup [\tau_p, \nabla]$$

and ϕ_p is meet-prime.

(3.5) LEMMA. *For $b \in A$ and $0 \prec q \in \mathbf{S}(b)$, $\theta_{b,q}$ is meet-prime if and only if for all $c < b$, $\xi_c{}^b(q) = 0$.*

Proof. Define $\rho_{b,q} \in \Theta(P)$ by:

$$(\rho_{b,q})_* = \Delta, \quad (\rho_{b,q})_a = \begin{cases} \text{con}\,(\xi_a{}^b(q'), 1), & b \geqq a \\ \Delta, & b \ngeqq a. \end{cases}$$

If $\xi_c{}^b(q) = 0$ for all $c < b$ then only $(\rho_{b,q})_b$ is non-trivial.

For meet-irreducible $\theta_{b,s}$, $0 \prec s \in \mathbf{S}(b)$, $s \neq q$, $s \leqq q'$ implies $\text{con}(q', 1) \subseteq \text{con}(s, 1)$ on $\mathbf{S}(b)$ and $\rho_{b,q} \leqq \theta_{b,s}$.

For meet-irreducible $\theta_{c,r}$, $0 \prec r \in \mathbf{S}(c)$ and $b \neq c$,

$$(\theta_{c,r})_b = \begin{cases} \kappa_c{}^b(\text{con}(r, 1)), & b \geqq c \\ \nabla, & b \ngeqq c. \end{cases}$$

But $b > c$ implies $\xi_c{}^b(q') = 1$ hence

$$(\theta_{c,r})_b \geqq (\rho_{b,q})_b$$

and therefore $\rho_{b,q} \leqq \theta_{c,r}$.

For meet-irreducible ϕ_p, clearly $\phi_p \geqq \rho_{b,q}$. Therefore

$$\Theta(P) = [\Delta, \theta_{b,q}] \cup [\rho_{b,q}, \nabla].$$

Now suppose for some $c < b$, $\xi_c{}^b(q) \geqq r \succ 0$. Let

$$M_{c,r} = \{\theta_{c,r}\} \cup \{\phi_p : p \leqq b \wedge c'\}.$$
$$(\bigwedge M_{c,r})_* = \text{con}(c, 1) \wedge \text{con}(bc', 1) = \text{con}(b, 1) = (\theta_{b,q})_*$$
$$(\bigwedge M_{c,r})_b = \kappa_c{}^b(\text{con}(r, 1)) \wedge \nabla = \kappa_c{}^b(\text{con}(r, 1)) \leqq \text{con}(q, 1) = (\theta_{b,q})_b.$$

Therefore $\bigwedge M_{c,r} \leqq \theta_{b,q}$ and $M_{c,r}$ is clearly a non-trivial co-cover.

The following result is straightforward.

(3.6) LEMMA. *If $\theta_{b,q}$ is not meet-prime, its minimal co-covers are given by $M_{c,r} = \{\theta_{c,r}\} \cup \{\phi_p : p \leqq bc'\}$ where $c < b$ with $0 \prec r \leqq \xi_c{}^b(q)$.*

(3.7) THEOREM. $D'(\Theta(P)) = \Theta(P)$.

Proof. We need only show that every meet-irreducible belongs to $D'(\Theta(P))$. But we have easily by induction that if $\theta_{b,q}$ is not meet-prime, then

$$\theta_{b,q} \in D'_{|b|}(\Theta(P))$$

where $|b|$ is the number of atoms in A less than or equal to b.

In § 7, we will show that all $\Theta(P)$ satisfy $(SD_\wedge)$ and $(SD_\vee)$. This implies (c.f. [**6**]) that there is a bijective correspondence between the

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join-irreducibles and the meet irreducibles of $\Theta(P)$. The correspondence is given by:

$$\phi_p \leftrightarrow \tau_p,\ 0 \prec p \in A$$
$$\theta_{b,q} \leftrightarrow \rho_{b,q},\ 0 \prec q \in S(b),\ b \in A.$$

With this the reader can prove the following results.

(3.8) Lemma. *τ_p is join-prime for all atoms $p \in A$.*

(3.9) Lemma. *$\rho_{b,q}$ is join-prime if and only if for all $c < b$, $\xi_c{}^b(q) = 0$. If $\rho_{b,q}$ is not join-prime, its minimal covers are given by*

$$J_{c,r} = \{\rho_{c,r}\} \cup \{\tau_p\colon p \leqq bc'\}$$

where $c < b$ and $0 \prec r \leqq \xi_c{}^b(q)$.

(3.10) Theorem. $D(\Theta(P)) = \Theta(P)$.

(3.11) Theorem. *The congruence lattice of any finite algebra in $\mathscr{P}$ is a bounded lattice.*

Since the splitting lattices are precisely the subdirectly irreducible bounded lattices, we are interested in what finite algebras have such as their congruence lattices.

(3.12) Theorem. *If $\mathbf{2} = \mathbf{S}(1) \leqq \mathbf{S}(0)$, then $\Theta(P)$ is subdirectly irreducible with critical quotient $\theta_{1,1} \prec \rho_{1,1}$.*

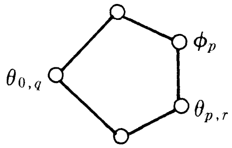
Proof. Since $(1, 0) \prec (1, 1)$ in P, we get $\theta_{1,1}$ as the largest congruence not identifying $(1, 0)$ and $(1, 1)$ and

$$\rho_{1,1} = \bigwedge\{\phi_p\colon p \text{ atom in } A\} = \operatorname{con}_P((1, 0), (1, 1)).$$

If we collapse any meet-irreducible of the form $\theta_{b,q}$ with its unique upper cover ϕ_b then by meeting with $\rho_{1,1}$ the interval $[\theta_{b,q} \wedge \rho_{1,1}, \rho_{1,1}]$ must also be collapsed, and

$$\theta_{b,q} \wedge \rho_{1,1} \leqq \theta_{1,1} < \rho_{1,1}.$$

If we collapse a meet irreducible ϕ_p with its unique cover, ∇, then by considering the pentagon



for $r > 0$ in $\mathbf{S}(p)$ with $\xi_0{}^p(r) = q$ we must collapse $\theta_{p,r}$ with ϕ_p which again collapses $\theta_{1,1}$ with $\rho_{1,1}$. Since $\mathbf{2} \leqq \mathbf{S}(0)$, such a $\theta_{0,q}$ exists. Therefore $\Theta(P)$ is subdirectly irreducible.

4. The congruence lattices of free algebras and their splitting equations. In this section, we will describe the algebras $F_{\mathscr{P}}(n)$, show that their respective congruence lattices $L_n = \Theta(F_{\mathscr{P}}(n))$ are subdirectly irreducible, and determine the respective splitting equations, ζ_n.

Since every algebra in $\mathscr{P}$ has a representation $P = P(\mathbf{S}, A)$, we assume that $F_{\mathscr{P}}(n)$ has this form and is generated by $(x_1, r_1), \ldots, (x_n, r_n)$. By using $(x_i, r_i) \mapsto (x_i, r_i)^{++} = (x_i, 1)$ we see that the external Boolean algebra will be n-generated and therefore should be $F_{\mathscr{B}}(n)$. Because the morphisms go downwards ($a \geqq b$ gives $\xi_b{}^a$: $\mathbf{S}(a) \to \mathbf{S}(b)$), no new elements will be added to $\mathbf{S}(x_i)$, that we do not get from $\{r_i\}$. This gives $\mathbf{S}(x_i) = F_{\mathscr{B}}(\{r_i\})$. Continuing in this manner we see that for any $a \in A = F_{\mathscr{B}}(\{x_1, \ldots, x_n\})$, $\mathbf{S}(a)$ contains $\{\xi_a{}^{x_i}(r_i)\colon x_i \geqq a\}$ and should be freely generated by that set. This provides us with a complete description of $F_{\mathscr{P}}(n) = P(\mathbf{S}, A)$, namely:

(1) $A = F_{\mathscr{B}}(\{x_1, \ldots, x_n\})$.

(2) For $a \in A$, $\mathbf{S}(a) = F_{\mathscr{B}}(\{r_i\colon x_i \geqq a\})$.

(3) For $a \geqq b$ in A, $\xi_b{}^a$ is the embedding monomorphism given by the embedding on the generators.

The proof of this fact is left to the reader.

We require a reasonable representation of this algebra.

Consider the free Boolean algebra, $\mathbf{2}^{2^n}$, on free generators $e_1, \ldots, e_n$. Let U be the set of all maps from $\{1, \ldots, n\}$ into $\{1, -1\}$. For $T \subseteq U$ let

$$\sigma(T) = \{i\colon \epsilon(i) = 1 \text{ for all } \epsilon \in T\}.$$

In particular $\sigma(\emptyset) = \{1, \ldots, n\}$. For each $\epsilon \in U$,

$$\prod_{i=1}^{n} e_i{}^{\epsilon(i)} \in \mathbf{2}^{2^n}$$

is an atom, where $e_i{}^1 = e_i$ and $e_i{}^{-1}$ is the complement of e_i. Thus the elements of $\mathbf{2}^{2^n}$ are in one-to-one correspondence with the subsets of U. Notice that $\{e_i\colon i \in \sigma(T)\}$ is the set of generators which lie above the element

$$\sum_{\epsilon \in T} \prod_{j=1}^{n} e_j{}^{\epsilon(j)},$$

which corresponds to T.

Consider the algebra $P(\mathbf{S}, A) \in \mathscr{P}$ with $A = \mathbf{2}^{2^n}$ and $\mathbf{S}(T)$ the free Boolean algebra with free generating set $\{r_i{}^T\colon i \in \sigma(T)\}$ and if $T_1 \supseteq T_2$

$$\xi_{T_2}{}^{T_1}(r_i{}^{T_1}) = r_i{}^{T_2}$$

(note that $T_1 \supseteq T_2$ implies $\sigma(T_1) \subseteq \sigma(T_2)$). Subsequently we will drop the superscript and let $r_i^T = r_i$ and consider $\mathbf{S}(T_1)$ to be embedded in $\mathbf{S}(T_2)$ if $T_1 \supseteq T_2$.

Notice that e_{i_0} corresponds to the set $\{\epsilon \in U: \epsilon(i_0) = 1\}$ and

$$\sigma(\{\epsilon \in U: \epsilon(i_0) = 1\}) = \{i_0\}.$$

(4.1) LEMMA. *$P(\mathbf{S}, A)$ as described above is isomorphic to $F_{\mathscr{P}}(n)$ with free generators (e_i, r_i), $i = 1, \ldots, n$.*

We shall now describe the meet irreducible elements of L_n. For $\gamma \in L_n$ recall $\gamma_* = \{(T_1, T_2) \in (2^U)^2: (T_1, 1)\gamma(T_2, 1)\}$. We let the *critical coordinates* associated with $\gamma \in L_n$ be those $T \subseteq U$ which are least in their γ_* equivalence classes. By (2.9) $\gamma \in L_n$ is determined by γ_* and its values at its critical coordinates.

Recall that $\sigma(\epsilon) = \{i: \epsilon(i) = 1\}$. Let $\epsilon, \mu \in U$ and $\omega: \sigma(\epsilon) \to \{\pm 1\}$. Following the results of § 3, we define ψ_μ, ϕ_ϵ, and $\theta_{\epsilon,\omega} \in L_n$ by

$$(\psi_\mu)_* = \operatorname{con}(1, 0) = \operatorname{con}(U, \emptyset) = \nabla$$

$$(\psi_\mu)_T = \operatorname{con}\left(1, \prod_{i\epsilon\sigma(T)} r_i^{\mu(i)}\right), \quad T \subseteq U.$$

$$(\phi_\epsilon)_* = \operatorname{con}(U, \{\epsilon\})$$

$$(\phi_\epsilon)_T = \operatorname{con}(1, 0) = \nabla, T \subseteq U$$

$$(\theta_{\epsilon,\omega})_* = \operatorname{con}(U, \{\epsilon\})$$

$$(\theta_{\epsilon,\omega})_T = \begin{cases} \operatorname{con}\left(1, \prod_{i\epsilon\sigma(T)} r_i^{\omega(i)}\right) & \text{if } \epsilon \in T \\ \operatorname{con}(1, 0) = \nabla & \text{if } \epsilon \notin T \end{cases}$$

More generally, for $T \subseteq U$ and $\eta: \sigma(T) \to \{\pm 1\}$ we define ϕ_T and $\theta_{T,\eta}$ by

$$\phi_T = \bigwedge\nolimits_{\epsilon \in T} \phi_\epsilon$$

so that

$$(\phi_T)_* = \operatorname{con}(U, T)$$

$$(\phi_T)_S = \operatorname{con}(1, 0) = \nabla, S \subseteq U$$

$$(\theta_{T,\eta})_* = \operatorname{con}(U, T)$$

$$(\theta_{T,\eta})_S = \begin{cases} \operatorname{con}\left(1, \prod_{i\epsilon\sigma(S)} r_i^{\eta(i)}\right) & \text{if } S \supseteq T \\ \nabla \quad \text{if } S \not\supseteq T. \end{cases}$$

We define $\pi_{\epsilon\mu}$ to be $\phi_\epsilon \wedge \psi_\mu$. Note again that $\psi_\mu = \theta_{\emptyset,\mu}$.

(4.2) LEMMA. *The meet irreducible elements of L_n are precisely the $\theta_{T,\eta}$, $T \subseteq U$, $\eta: \sigma(T) \to \{\pm 1\}$ and the ϕ_ϵ, $\epsilon \in U$. Moreover each ϕ_ϵ is* *uniquely covered by the greatest element of L_n and $\theta_{T,\eta}$ is uniquely covered by ϕ_T.*

(4.3) LEMMA. *The ϕ_ϵ and $\theta_{\emptyset,\mu}$ are meet-prime. The non-trivial dual minimal covers of $\theta_{T,\eta}$ are the sets*

$$\{\theta_{S,\omega}\} \cup \{\phi_\epsilon : \epsilon \in T - S\}$$

where $S \subset T$ and $\omega \supseteq \eta$ (i.e. $\omega|_{\sigma(T)} = \eta$).

(4.4) COROLLARY. *For each n, $D'(L_n) = L_n$. In fact $\theta_{T,\eta} \in D_k'(L_n)$ where $k = |T|$, and $\phi_\epsilon \in D_0'(L_n)$.*

(4.5) LEMMA. *If $T \neq \emptyset$ then*

$$\theta_{T,\eta} = \bigvee_{\epsilon \in T} \bigvee_{\omega \supseteq \eta} \phi_T \wedge \theta_{\epsilon\omega}.$$

Proof. First note that the *-coordinates of both $\theta_{T,\eta}$ and $\phi_T \wedge \theta_{\epsilon\omega}$ are the same. Hence both sides of the equation have the same critical coordinates and it suffices to show that equality holds at each of these coordinates. Notice that the critical coordinates are precisely those S with $S \subseteq T$. Since $\omega \supseteq \eta$, it follows from the definitions that

$$(\theta_{T,\eta})_T = (\phi_T \wedge \theta_{\epsilon\omega})_T.$$

Since joins at critical coordinates are computed component-wise, the equation holds at T. Let $S \subset T$. Choose $\epsilon \in T - S$. Then $(\theta_{\epsilon\omega})_S = 1$, for any $\omega \geqq \eta$. Thus

$$(\phi_T \wedge \theta_{\epsilon\omega})_S = 1,$$

from which the equation follows.

Now we have from § 3 the join-irreducible elements of L_n. For $T \subseteq U$ and $\omega: \sigma(T) \to \{\pm 1\}$ let τ_T and $\rho_{T,\omega} \in L_n$ be defined by

$$(\tau_T)_* = \operatorname{con}(\emptyset, T)$$

$$(\tau_T)_S = \operatorname{con}(0, 0) = \triangle \; S \subseteq U$$

and

$$(\rho_{T,\omega})_* = \operatorname{con}(\emptyset, \emptyset) = \triangle$$

$$(\rho_{T,\omega})_S = \begin{cases} \operatorname{con}\left(0, \prod_{i \in \sigma(T)} r_1^{\omega(i)}\right) & \text{if } S \subseteq T \\ \triangle & \text{if } S \not\subseteq T. \end{cases}$$

(4.6) LEMMA. *The join-irreducible elements of L_n are precisely the τ_ϵ, $\epsilon \in U$, and $\rho_{T,\omega}$, $T \subseteq U$ and $\omega: \sigma(T) \to \{\pm 1\}$.*

(4.7) LEMMA. *For $\epsilon \in U$, τ_ϵ is join-prime. Let $T \subseteq U$ and $\eta: \sigma(T) \to \{\pm 1\}$. The non-trivial minimal covers of $\rho_{T,\eta}$ are the sets*

$$\{\tau_\epsilon\} \cup \{\rho_{S,\omega} : \omega : \sigma(S) \to \{\pm 1\}, \omega \supseteq \eta\} \text{ for all } S \subset T \text{ and } \epsilon \in T - S.$$

Recall that $\pi_{\delta,\mu} = \phi_\delta \wedge \psi_\mu$.

(4.8) LEMMA.

(1) $\tau_\epsilon = \bigwedge_{\substack{\delta \in U \\ \delta \neq \epsilon}} \bigwedge_{\mu \in U} \pi_{\delta,\mu}.$

(2) $\rho_{T,\eta} = \bigwedge_{\epsilon \in U} \bigwedge_{\substack{\mu \in U \\ \mu \not\supseteq \eta}} \pi_{\epsilon,\mu} \wedge \bigwedge_{\epsilon \in U-T} \bigwedge_{\omega : \sigma(\epsilon) \to \{\pm 1\}} \theta_{\epsilon,\omega}.$

Proof. Let γ be the right hand side of (2). Then

$$\gamma_* \leqq \left(\bigwedge_{\epsilon \in U} \bigwedge_{\mu \not\supseteq \eta} \pi_{\epsilon,\mu}\right)_* \leqq \left(\bigwedge_{\epsilon \in U} \phi_\epsilon\right)_* = \triangle.$$

Also

$$(\pi_{\epsilon,\mu})_S = \operatorname{con}\left(1, \prod_{i \epsilon \sigma(S)} r_i^{\mu(i)}\right).$$

Hence

$$\left(\bigwedge_{\epsilon \in U} \bigwedge_{\mu \not\supseteq \eta} \pi_{\epsilon,\mu}\right)_S \leqq \bigwedge_{\mu \not\supseteq \eta} \operatorname{con}\left(1, \prod_{i \in \sigma(S)} r_i^{\mu(i)}\right) = \operatorname{con}\left(1, \sum_{\mu \not\supseteq \eta} \prod_{i \in \sigma(S)} r_i^{\mu(i)}\right).$$

Now $(\rho_{T,\eta})_S = \operatorname{con}(0, \prod_{i \in \sigma(T)} r_i^{\eta(i)})$ if $S \subseteq T$ and $\triangle$ otherwise. If $S \subseteq T$ then $\sigma(S) \supseteq \sigma(T)$ and it is easy to see that

$$\sum_{\mu \not\supseteq \eta} \prod_{i \in \sigma(S)} r_i^{\mu(i)}$$

is just the sum of the atoms of $\Theta(\mathbf{S}(S))$ which are not below

$$\prod_{i \in \sigma(T)} r_i^{\eta(i)};$$

i.e., $\sum_{\mu \not\supseteq \eta} \prod_{\sigma(S)} r_i^{\mu(i)}$ is the complement of $\prod_{i \in \sigma(T)} r_i^{\eta(i)}$. Hence

$$\operatorname{con}\left(0, \prod_{\sigma(T)} r_i^{\eta(i)}\right) = \operatorname{con}\left(1, \sum_{\mu \subset \eta} \prod_{\sigma(S)} r_1^{\mu(i)}\right) \geqq \gamma_S.$$

If $S \not\subseteq T$ then there is a $\delta \in S - T$. Hence

$$\left(\bigwedge_{\omega : \sigma(\delta) \to \{\pm 1\}} \theta_{\delta,\omega}\right)_S = \bigwedge_\omega \operatorname{con}\left(1, \prod_{\sigma(S)} r_i^{\omega(i)}\right) = \operatorname{con}\left(1, \sum_\omega \prod_{\sigma(S)} r_i^{\omega(i)}\right).$$

Since $\delta \in S$, $\sigma(S) \not\subseteq \sigma(\delta)$ and ω is defined on all of $\sigma(S)$. Thus

$$\sum_\omega \prod_{\sigma(S)} r_i^{\omega(i)}$$

is just the sum of all the atoms, and hence 1. It follows that $\gamma_S = \triangle = (\rho_{T,\eta})_S$. We have shown that $\gamma \leqq \rho_{T,\eta}$. Since $\rho_{T,\eta} \leqq \pi_{\epsilon,\mu}$ for all $\epsilon \in U$ and $\mu \not\supseteq \eta$ and $\rho_{T,\eta} \leqq \theta_{\epsilon,\omega}$ for all $\epsilon \in U - T$ and all ω, we have $\gamma = \rho_{T,\eta}$ proving (2). The proof of (1) is similar.

(4.9) THEOREM. *For each $n < \omega$, L_n is a splitting lattice. L_n is generated by ϕ_ϵ, $\theta_{\epsilon,\omega}$, ψ_μ, ϵ, $\mu \in U$ and $\omega: \sigma(\epsilon) \to \{\pm 1\}$. The prime quotient $\phi_U = \rho_{U,\emptyset} > \theta_{U,\emptyset}$ is collapsed by every non-trivial lattice congruence on L_n. Moreover the equivalence relation generated by this prime quotient is the same as the congruence relation generated by it.*

Now let

$$X = \{x_\mu: \mu \in U\} \cup \{y_\epsilon: \epsilon \in U\} \cup \{z_{\epsilon,\omega}: \epsilon \in U, \omega: \sigma(\epsilon) \to \{\pm 1\}\}$$

be a set of variables. Let f be the homomorphism from $FL(X)$ onto L_n extending $f(x_\mu) = \psi_\mu$, $f(y_\epsilon) = \phi_\epsilon$, and $f(z_{\epsilon,\omega}) = \theta_{\epsilon,\omega}$.

(4.10) *Definition.* (1) Define maps α_0 and α from the meet-irreducibles of L_n into $FL(X)$ as follows:

$$\alpha_0(\psi_\mu) = x_\mu$$
$$\alpha_0(\theta_{\epsilon,\omega}) = z_{\epsilon,\omega}$$
$$\alpha_0(\phi_\epsilon) = y_\epsilon \vee \bigvee_{\omega:\sigma(\epsilon)\to\{\pm 1\}} z_{\epsilon,\omega}$$
$$\alpha_0(\theta_{T,\eta}) = \bigvee_{\epsilon\in T} \bigvee_{\omega\supseteq\eta} [(\bigwedge_{\delta\in T} Y_\delta) \wedge z_{\epsilon,\omega}] \; |T| \geqq 2.$$

Since the only order relations among the meet-irreducibles are $\theta_{T,\eta} < \phi_\epsilon$ if and only if $\epsilon \in T$, one can check that α_0 preserves order. By (4.5), $f\alpha_0(\gamma) = \gamma$ for all meet-irreducible γ.

(2) Now let

$$\alpha(\phi_\epsilon) = \alpha_0(\phi_\epsilon)$$
$$\alpha(\theta_{T,\eta}) = \alpha_0(\theta_{T,\eta}) \vee \bigvee_{S\subset T} \bigvee_{\omega\geqq\eta} \alpha(\phi_{T-S}) \wedge \alpha(\theta_{S,\omega})$$

where

$$\alpha(\phi_{T-S}) = \bigwedge_{\epsilon\in T-S} \alpha(\phi_\epsilon) = \bigwedge_{\epsilon\in T-S} \alpha_0(\phi_\epsilon).$$

Now define β_0 and β from the join-irreducible into $FL(X)$ by:

(3) $$\beta_0(\tau_\epsilon) = \bigwedge_{\substack{\delta\in U\\ \delta\neq\epsilon}} \bigwedge_{\mu\in U} x_\mu \wedge y_\delta$$

$$\beta_0(\rho_{T,\eta}) = \bigwedge_{\epsilon\in U} \bigwedge_{\substack{\mu\in U\\ \mu\supset\eta}} x_\mu \wedge y_\epsilon \wedge \bigwedge_{\epsilon\in T} \bigwedge_{\omega:\sigma(\epsilon)\to\{\pm 1\}} z_{\epsilon,\omega}.$$

Since the only comparabilities among the join-irreducibles are $\rho_{S,\omega} \leqq \rho_{T,\eta}$ if $S \subseteq T$ and $\omega \supseteq \eta$, β_0 preserves order. By Lemma 4.8 $f\beta_0(\gamma) = \gamma$ for all join-irreducibles γ. Now let

(4) $$\beta(\tau_\epsilon) = \beta_0(\tau_\epsilon)$$

$$\beta(\rho_{T,\eta}) = \beta_0(\rho_{T,\eta}) \wedge \bigwedge_{S\subset T} [\bigvee_{\epsilon\in T-S} \beta(\tau_\epsilon) \vee \bigvee_{\omega\not\supseteq\eta} \beta(\rho_{S,\omega})].$$

It follows from the remark at the end of Section 1 that the splitting equation, ζ_n, of L_n is given by:

(ζ_n) $\beta(\rho_{U,\emptyset}) \leqq \alpha(\theta_{U,\emptyset})$.

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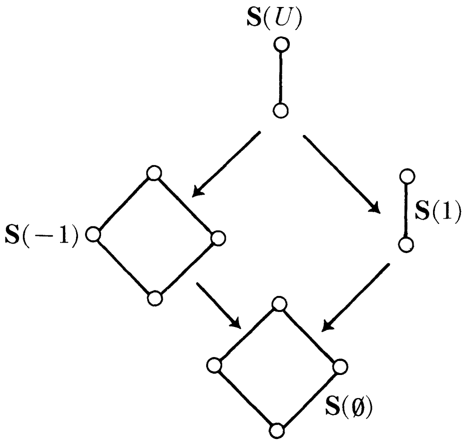
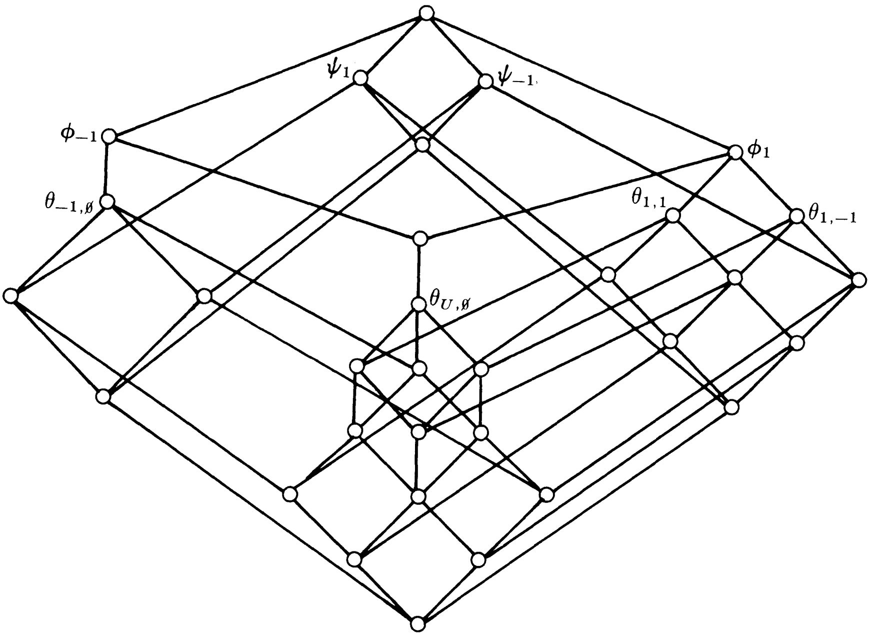
For $n = 0$, it is easy to see that $F_{\mathscr{P}}(0)$ is given by the algebra $\mathbf{2} \rightarrow \mathbf{2}$ in Section 2 which produces $L_0 = N_5$. For $n = 1$, $F_{\mathscr{P}}(1)$ and L_1 are given in the following diagrams where $U = \{\pm 1\}^1$ is identified with $\{\pm 1\}$ and $\emptyset$ also denotes the empty function.



$F_{\mathscr{P}}(1)$



$L_1 = \Theta(F_{\mathscr{P}}(1))$

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5. Main computations. In this section we show that any variety of algebras $\mathscr{K}$ such that $\Theta(\mathscr{K})$ satisfies ζ_n has modular congruence lattices. Let $\mathscr{K}$ be a variety of algebras with nonmodular congruence lattices. By [**2**, **4**], $\Theta(F_{\mathscr{K}}(a, b, c, d))$ contains one of the following sublattices.

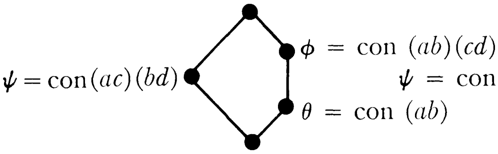

FIGURE 5.1

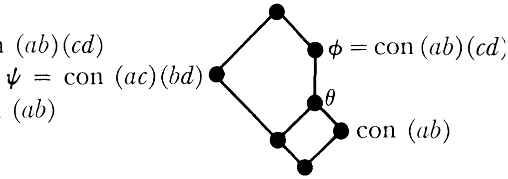

FIGURE 5.2

By considering $A = F_{\mathscr{K}}(a, b, c, d)/\psi \wedge \phi$ we may assume that the situation of Figure 5.1 applies and that $\psi \wedge \phi = 0$. Our goal is to find $B \in \mathscr{K}$ with $\Theta(B)$ failing ζ_n. If $\epsilon, \mu \in U$ and $\omega: \sigma(\epsilon) \rightarrow \{\pm 1\}$ then ϕ_ϵ, ψ_μ, and $\theta_{\epsilon,\omega}$ generate a copy of N_5, the five element nonmodular lattice. Thus L_n has several copies of N_5 "near the top". The desired B is obtained by taking a subdirect power of A in such a way that the N_5's of $\Theta(A)$ connect together in a manner similar to the way they connect in L_n.

Let $A \in \mathscr{K}$ be the algebra described above with congruence ϕ, θ, ψ as in Figure 5.1. That is A is an algebra in $\mathscr{K}$. A contains elements a, b, c, d and has congruence ϕ, θ, ψ such that $\phi = \mathrm{con}(ab)(cd)$, $\psi = \mathrm{con}(ac)(bd)$, $(a, b) \in \theta$ and $(c, d) \notin \theta$, and ϕ, θ, ψ generate N_5. We shall also assume that $\psi \wedge \phi = \triangle$. Thus whenever $(x, y) \in \psi \wedge \phi$ we shall conclude that $x = y$.

(5.1) *Definition.* Consider $A^{2^{2n}} = A^{(2^n)^2} = \{(a_{\epsilon\chi}): \epsilon, \chi \in U\}$.

(1) Let B be the subalgebra of $A^{2^{2n}}$ whose elements satisfy

$$(a_{\epsilon\chi}, a_{\delta\chi}) \in \psi \quad \epsilon, \delta, \chi \in U$$
$$(a_{\epsilon\chi}, a_{\epsilon\nu}) \in \phi \quad \epsilon, \chi, \nu \in U$$
$$(a_{\epsilon\chi}, a_{\epsilon\nu}) \in \theta \text{ if } \chi|_{\sigma(\epsilon)} = \nu|_{\sigma(\epsilon)}.$$

(2) Let $\mathbf{a} = (a_{\epsilon\psi})$, $\mathbf{b} = (b_{\epsilon\chi}) \in B$ and define congruences $\bar{\psi}_\mu$, $\bar{\phi}_\delta$, $\bar{\theta}_{\delta,\omega}$, $\bar{\pi}_{\delta,\mu} \in \Theta B$ for $\mu, \delta \in U$, $\omega: \sigma(\delta) \rightarrow \{\pm 1\}$, by

$$(\mathbf{a}, \mathbf{b}) \in \bar{\psi}_\mu \text{ if for all } \epsilon \in U \ (a_{\epsilon\mu}, b_{\epsilon\mu}) \in \psi$$
$$(\mathbf{a}, \mathbf{b}) \in \bar{\phi}_\delta \text{ if for all } \chi \in U \ (a_{\delta\chi}, b_{\delta\chi}) \in \phi$$
$$(\mathbf{a}, \mathbf{b}) \in \bar{\theta}_{\delta,\omega} \text{ if for all } \chi \text{ with } \chi \supseteq \omega \ (a_{\delta\chi}, b_{\delta\chi}) \in \theta$$
$$\bar{\pi}_{\delta,\mu} = \bar{\phi}_\delta \wedge \bar{\psi}_\mu.$$

Notice that $(\mathbf{a}, \mathbf{b}) \in \bar{\psi}_\mu$ if there exists an $\epsilon \in U$ such that $(a_{\epsilon\mu}, b_{\epsilon\mu}) \in \psi$. Similarly, the "for all" part of the definition of $\bar{\phi}_\delta$ and $\bar{\theta}_{\delta,\omega}$ may be replaced by "there exists". These facts follow from the definition of B.

Recall that

$$X = \{x_\mu: \mu \in U\} \cup \{y_\epsilon: \epsilon \in U\} \cup \{z_{\epsilon,\omega}: \epsilon \in U, \omega: \sigma(\epsilon) \rightarrow \{\pm 1\}\}.$$

Let g be the homomorphism from $FL(X)$ into $\Theta(B)$ which extends the map $g(x_\mu) = \bar{\psi}_\mu$, $g(y_\epsilon) = \bar{\phi}_\epsilon$, and $g(z_{\epsilon,\omega}) = \bar{\theta}_{\epsilon,\omega}$. We shall eventually show that

$$g(\beta(\rho_{U,\emptyset})) \nleq g(\alpha(\theta_{U,\emptyset})),$$

proving that ζ_n fails in $\Theta(B)$.

(5.2) LEMMA. *Fix $T \subseteq U$. Let $\mathbf{a} \in B$ such that $a_{\epsilon,\chi} = a_{\delta,\chi}$ for all $\epsilon, \delta \in T$ and all $\chi \in U$. Then $(a_{\epsilon,\chi}, a_{\delta,\mu}) \in \theta$ for all $\epsilon, \delta \in T$ and all χ and μ such that $\chi|_{\sigma(T)} = \mu|_{\sigma(T)}$.*

Proof. Suppose $\epsilon, \delta \in T$ and $\chi|_{\sigma(T)} = \mu|_{\sigma(T)}$. We induct on $|\{k: \chi(k) \neq \mu(k)\}|$. If this is zero the lemma holds. Suppose $\chi(k) \neq \mu(k)$ for some k. Then

$$k \notin \sigma(T) = \{i: \epsilon(i) = 1 \text{ for all } \epsilon \in T\}.$$

Hence there is a $\gamma \in T$ with $\gamma(k) = -1$. Let χ_1 be defined by $\chi_1(i) = \chi(i)$, $i \neq k$ and $\chi_1(k) = -\chi(k) = \mu(k)$. By (5.1), $\mathbf{a} \in B$ implies

$$(a_{\gamma,\chi}, a_{\gamma,\chi_1}) \in \theta.$$

Thus

$$a_{\epsilon,\chi} = a_{\gamma,\chi} \theta a_{\gamma,\chi_1}.$$

Since χ_1 and μ differ in one less place, the proof is complete.

(5.3) LEMMA. *Fix $\emptyset \neq T \subseteq U$. Let $\mathbf{a}, \mathbf{b} \in B$ be such that $a_{\epsilon,\chi} = a_{\delta,\chi}$ and $b_{\epsilon,\chi} = b_{\delta,\chi}$ for all $\epsilon, \delta \in T$ and all $\chi \in U$. Suppose that $(\mathbf{a}, \mathbf{b}) \in g(\alpha(\theta_{T,\eta}))$. Then $(a_{\epsilon,\chi}, b_{\epsilon,\chi}) \in \theta$ for all $\epsilon \in T$ and for all χ with $\chi \supseteq \eta$.*

Proof. Induct on $|T|$. If $T = \{\epsilon\}$ then by Lemma 4.3 the only nontrivial dual minimal covers of $\theta_{\epsilon,\eta}$ are $\{\phi_\epsilon, \theta_{0,\mu}\} = \{\phi_\epsilon, \psi_\mu\}$ for $\mu \supseteq \eta$. Now

$$\begin{aligned} g\alpha(\theta_{\epsilon,\eta}) &= g\alpha_0(\theta_{\epsilon,\eta}) \vee \textstyle\bigvee_{\mu \supseteq \eta} (g\alpha(\phi_\epsilon) \wedge g\alpha(\psi_\mu)) \\ &= g(z_{\epsilon,\eta}) \vee \textstyle\bigvee_{\mu \supseteq \eta} [g(y_\epsilon \vee \bigvee_{\omega:\sigma(\epsilon) \to \{\pm 1\}} z_{\epsilon,\omega}) \wedge g(x_\mu)] \\ &= \bar{\theta}_{\epsilon,\eta} \vee \textstyle\bigvee_{\mu \supseteq \eta} [(\bar{\phi}_\epsilon \vee \bigvee_\omega \bar{\theta}_{\epsilon,\omega}) \wedge \bar{\psi}_\mu] \\ &= \bar{\theta}_{\epsilon,\eta} \vee \textstyle\bigvee_{\mu \supseteq \eta} (\bar{\phi}_\epsilon \wedge \bar{\psi}_\mu) = \bar{\theta}_{\epsilon,\eta}. \end{aligned}$$

In the last step we used the fact that if $\mu \supseteq \eta$ then

$$\phi_\epsilon \wedge \bar{\psi}_\mu \leqq \bar{\theta}_{\epsilon,\eta}.$$

Thus we obtain $(\mathbf{a}, \mathbf{b}) \in \bar{\theta}_{\epsilon,\eta}$ from which it follows that $(a_{\epsilon,\chi}, b_{\epsilon,\chi}) \in \theta$ for all $\chi \supseteq \eta$, by the definition of $\bar{\theta}_{\epsilon,\eta}$.

Now let $T \subseteq U$, $|T| \geqq 2$. Then

$$\begin{aligned} g\alpha(\theta_{T,\eta}) &= g\alpha_0(\theta_{T,\eta}) \vee \textstyle\bigvee_{S \subset T} \bigvee_{\omega \supset \eta} g(\alpha(\phi_{T-S}) \wedge \alpha(\theta_{S,\omega})) \\ (*) \qquad &= [\textstyle\bigvee_{\epsilon \in T} \bigvee_{\omega \supseteq \eta} (\bar{\phi}_T \wedge \bar{\theta}_{\epsilon,\omega})] \vee \bigvee_{S \subset T} \bigvee_{\omega \supset \eta} g\alpha(\phi_{T-S}) \\ &\qquad \wedge g\alpha(\theta_{S,\omega}) \\ &= [\textstyle\bigvee_{\epsilon \in T} \bigvee_{\omega \supseteq \eta} (\bar{\phi}_T \wedge \bar{\theta}_{\epsilon,\omega})] \\ &\qquad \vee \textstyle\bigvee_{S \subset T} \bigvee_{\omega \supset \eta} (\bar{\phi}_{T-S} \wedge g\alpha(\theta_{S,\omega})) \end{aligned}$$

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where $\bar{\phi}_T = \bigwedge_{\epsilon \in T} \bar{\phi}_\epsilon$. Since $(\mathbf{a}, \mathbf{b}) \in g\alpha(\theta_{T,\eta})$ there is a finite sequence $\mathbf{a}, \mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{b}$ in B with each $(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)})$ in one of the summands of (*). Since the map $go\alpha$ is order preserving and $g\alpha(\phi_S) = \bar{\phi}_S$, we have

$$g\alpha(\theta_{S,\omega}) \leqq \bar{\phi}_S.$$

From this it follows that each of the summands of (*) is less than or equal to $\bar{\phi}_T$.

Let δ and ϵ be in T. By hypothesis $a_{\epsilon,\chi} = a_{\delta,\chi}$. Since $(\mathbf{a}, \mathbf{a}^{(1)}) \in \bar{\phi}_T$,

$$a_{\epsilon,\chi}\phi a_{\epsilon,x}{}^{(1)} \text{ and } a_{\delta,\chi}\phi a_{\delta,\chi}{}^{(1)}.$$

Since $\mathbf{a}^{(1)} \in B$,

$$a_{\epsilon,\chi}{}^{(1)}\psi a_{\delta,\chi}{}^{(1)}.$$

Hence

$$a_{\epsilon,\chi}{}^{(1)}\psi \wedge \phi a_{\delta,\chi}{}^{(1)}.$$

Thus $a_{\epsilon,\chi}{}^{(1)} = a_{\delta,\chi}{}^{(1)}$. Hence each $\mathbf{a}^{(i)}$ also satisfies the hypothesis of the lemma.

Suppose $(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)}) \in \bar{\phi}_T \wedge g\alpha(\theta_{S,\omega})$ for $\emptyset \neq S \subset T$. Since $S \subset T$ $\mathbf{a}^{(i)}$ and $\mathbf{a}^{(i+1)}$ satisfy the hypothesis of the lemma with S in place of T. Hence by induction

$$(a_{\epsilon,\chi}{}^{(i)}, a_{\epsilon,\chi}{}^{(i+1)}) \in \theta$$

for all $\epsilon \in S$ and all $\chi \supseteq \omega$. If $\epsilon \in T$ choose $\delta \in S$, then

$$a_{\epsilon,\chi}{}^{(i)} = a_{\delta,\chi}{}^{(i)} \text{ and } a_{\epsilon,\chi}{}^{(i+1)} = a_{\delta,\chi}{}^{(i+1)}.$$

Thus $(a_{\epsilon,\chi}{}^{(i)}, a_{\epsilon,\chi}{}^{(i+1)}) \in \theta$ for $\epsilon \in T$ and $\chi \supseteq \omega$. Suppose μ is such that $\mu \supseteq \eta$. Choose any $\chi \supseteq \omega$. Then $\chi \supseteq \omega \supseteq \eta$, and by Lemma 5.2

$$a_{\epsilon,\mu}{}^{(i)}\theta a_{\epsilon,\chi}{}^{(i)}\theta a_{\epsilon,\chi}{}^{(i+1)}\theta a_{\epsilon,\mu}{}^{(i+1)}.$$

Thus $(a_{\epsilon,\mu}{}^{(i)}, a_{\epsilon,\mu}{}^{(i+1)}) \in \theta$ for all $\epsilon \in T$ and $\mu \supseteq \eta$.

If $S = \emptyset$ then $(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)}) \in g\alpha(\phi_T \wedge \psi_\omega) = \bar{\phi}_T \wedge \bar{\psi}_\omega$. Thus

$$a_{\epsilon,\omega}{}^{(i)} = a_{\epsilon,\omega}{}^{(i+1)} \text{ for } \epsilon \in T.$$

Again by Lemma 5.2

$$(a_{\epsilon,\mu}{}^{(i)}, a_{\epsilon,\mu}{}^{(i+1)}) \in \theta \text{ for all } \epsilon \in T \text{ and all } \mu \supseteq \eta.$$

The final case is $(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)}) \in \bar{\phi}_T \wedge \bar{\theta}_{\epsilon,\omega}$ for some $\epsilon \in T$, $\omega \supseteq \eta$. This is handled in a same manner. This completes the proof.

(5.4) THEOREM. *Let c be the element of B each of whose coordinates is c, and let* $\mathbf{d}$ *be the element each of whose coordinates is d. Then* $(\mathbf{c}, \mathbf{d}) \notin g\alpha(\theta_{U,\emptyset})$.

Proof. If $(\mathbf{c}, \mathbf{d}) \in g\alpha(\theta_{U,\emptyset})$ then by Lemma 5.3 we would have $(c, d) \in \theta$, a contradiction.

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(5.5) LEMMA. *Fix* $T \subseteq U$ *and* $\eta\colon \sigma(T) \rightarrow \{\pm 1\}$. *Let* $\mathbf{a}, \mathbf{b} \in B$ *be such that for each* $\omega\colon \sigma(T) \rightarrow \{\pm 1\}$, $\omega \neq \eta$ *either*

$$(1) \quad a_{\epsilon,\chi} = b_{\epsilon,\chi} = c \quad \forall \epsilon \in T \quad \forall \chi \supseteq \omega \quad \textit{and}$$
$$\alpha_{\epsilon,\chi} = b_{\epsilon,\chi} = a \quad \forall \epsilon \notin T \quad \forall \chi \supseteq \omega$$

or

$$(2) \quad a_{\epsilon,\chi} = b_{\epsilon,\chi} = d \quad \forall \epsilon \in T \quad \forall \chi \supseteq \omega \quad \textit{and}$$
$$a_{\epsilon,\chi} = b_{\epsilon,\chi} = b \quad \forall \epsilon \notin T \quad \forall \chi \supseteq \omega.$$

Moreover assume

$$(3) \quad a_{\epsilon,\chi} = c,\ b_{\epsilon,\chi} = d \quad \forall \epsilon \in T \quad \forall \chi \supseteq \eta \quad \textit{and}$$
$$a_{\epsilon,\chi} = a,\ b_{\epsilon,\chi} = b \quad \forall \epsilon \notin T \quad \forall \chi \supseteq \eta.$$

Then $(\mathbf{a}, \mathbf{b}) \in g\beta(\rho_{T,\eta})$.

Proof. The hypothesis of the lemma partitions the rows of the "matrices" $\mathbf{a}$ and $\mathbf{b}$ into two blocks, T and $U - T$. It partitions the columns of $\mathbf{a}$ and $\mathbf{b}$ according to their restriction to $\sigma(T)$; i.e., $\chi \sim \mu$ if $\chi|_{\sigma(T)} = \mu|_{\sigma(T)}$. This situation is represented in the following figure.

$$\begin{array}{cc} & \begin{array}{cccc} \chi \supseteq \omega & \chi \supseteq \omega' & \cdots & \chi \supseteq \eta \end{array} \\ \mathbf{a} = \begin{array}{c} T \\ U - T \end{array} & \left[\begin{array}{c|c|c|c} c & d & & c \\ \hline a & b & & a \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} \chi \supseteq \omega & \chi \supseteq \omega' & \cdots & \chi \supseteq \eta \end{array} \\ \mathbf{b} = \begin{array}{c} T \\ U - T \end{array} & \left[\begin{array}{c|c|c|c} c & d & & d \\ \hline a & b & & b \end{array}\right] \end{array}$$

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To prove the lemma, induct on T. Suppose $T = \emptyset$. Then

$$g\beta(\rho_{\hat{\emptyset},\eta}) = g\beta_0(\rho_{0,\eta}) = \bigwedge_{\epsilon\in U} \bigwedge_{\substack{\mu\in U\\ \mu\not\supseteq\eta}} \bar{\pi}_{\epsilon,\mu} \wedge \bigwedge_{\epsilon\notin T} \bigwedge_{\omega:\sigma(\epsilon)\to\{\pm1\}} \bar{\theta}_{\epsilon,\omega}.$$

Recall that $(\mathbf{a}, \mathbf{b}) \in \bar{\pi}_{\epsilon,\mu}$ if and only if $a_{\epsilon,\mu} = b_{\epsilon,\mu}$ and that $(\mathbf{a}, \mathbf{b}) \in \bar{\theta}_{\epsilon,\omega}$ if and only if $(a_{\epsilon,\mu}, b_{\epsilon,\mu}) \in \theta$ for all $\mu \supseteq \omega$. From this it is easy to check that for $\mathbf{a}$, $\mathbf{b}$ satisfying 1), 2), and 3), $(\mathbf{a}, \mathbf{b}) \in g\beta(\rho_{0,\eta})$.

Let $T \subseteq U$ be arbitrary. Since $\beta(\tau_\epsilon) = \beta_0(\tau_\epsilon)$ we have

$$g\beta(\rho_{T,\eta}) = g\beta_0(\rho_{T,\eta}) \wedge \bigwedge_{S\subset T} [\bigvee_{\epsilon\in T-S} (\bigwedge_{\delta\neq\epsilon} \bigwedge_{\epsilon\in U} \bar{\pi}_{\delta,\mu}) \vee \bigvee_{\omega\supseteq\eta} g\beta(\rho_{S,\omega})].$$

Calculations similar to those above show that $(\mathbf{a}, \mathbf{b}) \in g\beta_0(\rho_{T,\eta})$.

Let $\mathbf{a}^{(1)} \in B$ be defined by $a_{\epsilon,\chi}{}^{(1)} = a$ for all $\epsilon \in T - S$ and $\chi \supseteq \eta$; for each ω: $\sigma(T) \to \{\pm1\}$, $\omega \neq \eta$ and each $\epsilon \in T - S$ and each $\chi \supseteq \omega$, $a_{\epsilon,\chi}{}^{(1)} = a$ if $a_{\epsilon,\chi} = c$ and $a_{\epsilon,\chi}{}^{(1)} = b$ if $a_{\epsilon,\chi} = d$, and $a_{\epsilon,\chi}{}^{(1)} = a_{\epsilon,\chi}$ for all other ϵ and χ (see Figure 5.4). Similarly $\mathbf{b}^{(1)}$ is defined by $b_{\epsilon,\chi}{}^{(1)} = b$ if $\epsilon \in T - S$ and $\chi \supseteq \eta$, and for $\omega \neq \eta$ and each $\epsilon \in T - S$ and each $\chi \supseteq \omega$, $b_{\epsilon,\chi}{}^{(1)} = a$ if $b_{\epsilon,\chi} = c$ and $b_{\epsilon,\chi}{}^{(1)} = b$ if $b_{\epsilon,\chi} = d$; $b_{\epsilon,\chi}{}^{(1)} = b_{\epsilon,\chi}$ otherwise. The reader can verify that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}^{(1)}$, $\mathbf{b}^{(1)}$ are actually in B. Moreover it is straightforward to check that $(\mathbf{a}, \mathbf{a}^1)$ and $(b, b^{(1)})$ are in $\bigvee_{\epsilon\in T-S} (\bigwedge_{\delta\neq\epsilon} \bigwedge_{\mu\in U} \bar{\pi}_{\delta,\mu})$.

$$\begin{array}{cc}
 & \begin{array}{cccc} \chi\supseteq\omega & \chi\supseteq\omega' & \cdots & \chi\supseteq\eta \end{array} \\
\begin{array}{r} S \\ T-S \\ \mathbf{a}^{(1)} = \\ \\ \end{array} &
\left[\begin{array}{c|c|c|c}
c & d & & c \\ \hline
a & b & & a \\ \hline
a & b & & a
\end{array}\right]
\end{array}$$

FIGURE 5.4

Since $S \subset T$, $\sigma(S) \supseteq \sigma(T)$. Thus if we partition the columns χ according to their restriction to $\sigma(S)$ we get a finer partition than when we partition them according to their restriction to $\sigma(T)$.

Let $\nu_1, \ldots, \nu_m$ be all the maps $\sigma(S) \to \{\pm1\}$ such that $\nu_i \supseteq \eta$. Let $\mathbf{a}^{(2)}$ be defined by $a_{\epsilon,\chi}{}^{(2)} = b$ if $\epsilon \in S$ and $\chi \supseteq \nu_1$; and $a_{\epsilon,\chi}{}^{(2)} = b$ if $\epsilon \notin S$ and $\chi \supseteq \nu_1$, and $a_{\epsilon,\chi}{}^{(2)} = a_{\epsilon,\chi}{}^{(1)}$ otherwise (see Figure 5.5). By the inductive hypothesis

$$(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}) \in g\beta(\rho_{S,\nu_1}).$$

Continuing in this way we see that

$$(a^{(1)}, b^{(1)}) \in \bigvee_{\nu\supseteq\eta} \beta(\rho_{S,\nu}).$$

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Thus $(\mathbf{a}, \mathbf{b}) \in g\beta(\rho_{T,\eta})$, proving the lemma.

$$\mathbf{a}^{(2)} = \begin{array}{c} \begin{array}{ccccc} \chi \supseteq \omega & \chi \supseteq \omega' & \cdots & \chi \supseteq \nu_1 \supseteq \eta & \end{array} \\ \left[\begin{array}{c|c|c|c:c} c & d & & d & c \\ \hline a & b & & b & a \\ \hline a & b & & b & a \end{array}\right] \end{array}$$

FIGURE 5.5

(5.6) THEOREM. *Let* $\mathbf{c}$ *and* $\mathbf{d}$ *be as in Theorem* 5.4. *Then* $(\mathbf{c}, \mathbf{d}) \in g\beta(\rho_{U,0})$. *Thus* ζ_n *fails in* $\Theta(B)$.

Proof. This theorem follows immediately from Lemma 5.5 and Theorem 5.4.

6. The main results. Now that all calculations have been completed we can state our main results. The first theorem and its corollaries follow directly.

(6.1) THEOREM. *Let* $\mathscr{K}$ *be an arbitrary variety of algebras; then* $\mathscr{K}$ *is not congruence modular if and only if* $\mathbf{Con}(\mathscr{P}) \subseteq \mathbf{Con}(\mathscr{K})$.

(6.2) COROLLARY. $\mathbf{Con}(\mathscr{P})$ *is the least non-modular congruence variety.*

(6.3) COROLLARY. *For a variety of algebras,* $\mathscr{K}$, *the following are equivalent:*

(1) $\mathscr{K}$ *is congruence modular*

(2) $\mathbf{Con}(\mathscr{K}) \vDash \zeta_n$ *for some* $n < \omega$

(3) $\mathbf{Con}(\mathscr{P}) \nsubseteq \mathbf{Con}(\mathscr{K})$

(4) $L_n \notin \mathbf{Con}(\mathscr{K})$ *for some* $n < \omega$.

The second main result shows us that at least "$\vDash_c$ mod" is (very strongly) compact.

(6.4) THEOREM. *Let* Σ *be a set of lattice identities; then the following are equivalent:*

(1) $\sum \vDash_c \text{mod}$

(2) $\exists \delta \in \Sigma \delta \vDash_c \text{mod}$

(3) $\exists \delta \in \Sigma \; \exists n < \omega \;\; \delta \vdash \zeta_n$

(4) $\exists \delta \in \Sigma \; \exists n < \omega \;\; L_n \nvDash \delta$

(5) $\mathbf{Con}(\mathscr{P}) \nvDash \Sigma$

Proof. By our previous results we have (5) $\Leftrightarrow$ (4) $\Leftrightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Now if $\mathbf{Con}(\mathscr{P}) \vDash \Sigma$ then P is a variety of algebras whose congruence lattices satisfy Σ but not mod. Therefore $\Sigma \nvDash_c$ mod.

Note that in (2), (3) and (4) the δ (and n) remain the same so that we can also state the result for $\Sigma = \{\delta\}$.

(6.5) COROLLARY. *For a lattice identity δ, the following are equivalent:*
(1) $\delta \vDash_c$ mod
(2) $\delta \vDash \zeta_n$ *for some* $n < \omega$
(3) $L_n \nvDash \delta$ *for some* $n < \omega$.

7. Equations satisfied by $\Theta(\mathscr{P})$**.** In view of the theorems of § 6 a better understanding of the lattices $\Theta(\mathscr{P})$ is important. In this section we investigate lattice identities holding in $\Theta(\mathscr{P})$ and give some applications. Equation (7.1.1) below is from [**20**]. Equations (7.1.2)-(7.1.5) are McKenzie's splitting equations for N_6, $Q_0{}^d$, Q_1, Q_4 [**22**].

(7.1) THEOREM. *The following hold in* $\Theta(\mathscr{P})$ *and thus in* $\mathbf{Con}(\mathscr{P}) = \mathbf{HSP}\Theta(\mathscr{P})$.

(1) $x \wedge (y \vee z) \leqq y \vee (x \wedge (z \vee (x \wedge y)))$.

(2) $$y \wedge [(x \wedge (w \vee (x \wedge z))) \vee (z \wedge (w \vee (x \wedge z)))] \leqq x \vee \{[x \vee y \vee (w \wedge (x \vee z))] \wedge [z \vee (w \wedge (x \vee y))]\}.$$

(3) $$y \wedge ([x \wedge (z \vee (x \wedge y))] \vee (y \wedge z)) \leqq [x \wedge (y \vee (z \wedge (x \vee y)))] \vee (z \wedge (x \vee y)).$$

(4) $$x \wedge [(x \wedge y) \vee (z \wedge (w \vee (x \wedge y \wedge z)))] \leqq (x \wedge y) \vee [(z \vee w) \wedge (x \vee (w \wedge (x \vee y)))].$$

(5) $$y \wedge [z \vee (y \wedge (x \vee (y \wedge z)))] \leqq x \vee [(x \vee y) \wedge (z \vee (x \wedge (y \vee z)))].$$

The dual of (7.1.3) *fails in* $\Theta(\mathscr{P})$. *However, the duals of all of the other equations hold in* $\Theta(\mathscr{P})$.

Proof. As we saw in Section 1 it suffices to show these identities hold in $\Theta(P(\mathbf{S}, A))$ when A is finite. Hence let x, y, z be congruences of $P(\mathbf{S}, A)$, A finite. Let $x = (x_*; x_a, a \in A)$, $y = (y_*; y_a, a \in A)$ and $z = (z_*; z_a, a \in A)$ be the congruence representations of x, y, and z.

(7.2) LEMMA. *Let f_x: $A \to A$ by letting $f_x(a)$ be the least element of A congruent to a modulo x_*. Then*

(1) $f_{x\vee y}(a) = f_x(a) \wedge f_y(a)$

(2) $f_{x\wedge y}(a) = f_x(a) \vee f_y(a)$

(3) $f_x(f_y(a)) = f_y(f_x(a)) = f_{x\vee y}(a) = f_x(a) \wedge f_y(a)$

(4) $f_{x\wedge(y\vee z)} = f_{(x\wedge y)\vee(x\wedge z)}$

To see this notice that since A is a finite Boolean algebra, there is a unique $a_x \in A$ such that $x_* = \theta(a_x, 1)$. Since

$$(x \vee y)_* = x_* \vee y_* = \theta(a_x \wedge a_y, 1),$$

by elementary properties of Boolean algebras, we have

$$a_{x\vee y} = a_x \wedge a_y.$$

Similarly

$$a_{x \wedge y} = a_x \vee a_y.$$

Now (1), (2), (3) follow easily from the fact that $f_x(a) = a_x \wedge a$. To see (4) notice that f_x only depends on x_*. Now using the distributivity of $\Theta(A)$ and (2.7)

$$\begin{aligned}[x \wedge (y \vee z)]_* &= x_* \wedge (y_* \vee z_*) \\ &= (x_* \wedge y_*) \vee (x_* \wedge z_*) = [(x \wedge y) \vee (x \wedge z)]_*.\end{aligned}$$

Hence (4) holds, i.e., we can use the distributive law on the subscript of f.

In order to prove (7.1.1)–(7.1.5) we must show that the a-component of the left-hand side is less than or equal to the a-component of the right-hand side for $a \in A$ and that the same holds for the $*$-component. The latter is easy: $x \rightarrow x_*$ is a lattice homomorphism from $\Theta(P(\mathbf{S}, A))$ to $\Theta(A)$ and $\Theta(A)$ is distributive.

In what follows we use plus and juxtaposition or dot in place of join and meet in order to simplify the notation. Let $a \in A$. Then

$$\begin{aligned}(x(y + z))_a &= x_a(y + z)_a \\ &= x_a\kappa_b{}^a(y_b + z_b) \\ &= x_a \cdot \kappa_b{}^a(x_b) \cdot \kappa_b{}^a(y_b + z_b) \\ &= x_a \cdot \kappa_b{}^a(x_b(y_b + z_b)) \\ &= x_a \cdot \kappa_b{}^a(x_by_b + x_bz_b)\end{aligned}$$

where

$$b = f_{y+z}(a) = f_y(a) \cdot f_z(a),$$

and we have used $x_a \leqq \kappa_b{}^a(x_b)$ by $(R1)$, and the distributivity of $\Theta(\mathbf{S}(b))$. Now

$$\begin{aligned}[y + x(z + xy)]_a &= \kappa_c{}^a(y_c + (x(z + xy))_c) \\ &= \kappa_c{}^a(y_c + x_c(z + xy)_c) \\ &= \kappa_c{}^a(y_c + x_c \cdot \kappa_d{}^c(z_d + x_dy_d))\end{aligned}$$

where

$$\begin{aligned}c &= f_{y+x(z+xy)}(a) = f_{y+xz}(a) \text{ and} \\ d &= f_{z+xy}(c) = f_{z+xy+y+xy}(a) = f_{z+y}(a) = b,\end{aligned}$$

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using this and $(R1)$ we have

$$\begin{aligned}
[y + x(z + xy)]_a &= \kappa_c{}^a(y_c + x_c \cdot \kappa_b{}^c(z_b + x_b y_b)) \\
&= \kappa_c{}^a(y_c + x_c \cdot \kappa_b{}^c(x_b) \cdot \kappa_b{}^c(z_b + x_b y_b)) \\
&= \kappa_c{}^a(y_c + x_c \cdot \kappa_b{}^c(x_b z_b + x_b y_b)) \\
&\geqq \kappa_c{}^a(y_c) + \kappa_c{}^a(x_c) \cdot \kappa_c{}^a(\kappa_b{}^c(x_b z_b + x_b y_b)) \\
&\geqq x_a \cdot \kappa_b{}^a(x_b z_b + x_b y_b)
\end{aligned}$$

proving (7.1.1).

To see (7.1.3) note

$$\begin{aligned}
&[y(x(z + xy) + yz)]_a \\
&\quad = y_a \cdot \kappa_b{}^a(x_b \cdot \kappa_c{}^b(z_c + x_c y_c) + y_b z_b) \\
&\quad \leqq \kappa_b{}^a(y_b) \cdot \kappa_b{}^a(x_b \cdot \kappa_c{}^b(z_c + x_c y_c) + y_b z_b) \\
&\quad = \kappa_b{}^a(y_b x_b \cdot \kappa_b{}^a(z_c + x_c y_c) + y_b z_b) \\
&\quad \leqq \kappa_b{}^a(y_b x_b + y_b z_b)
\end{aligned}$$

where $b = f_{xz+xy+yz}(a)$ and $c = f_{z+xy}(b) = f_{z+xy}(a)$. Now letting $d = f_{y+xz}(b) = f_{y+xy}(a)$ and $e = f_{x+y}(a) = f_{x+y}(d) = f_{x+y}(b)$ we have

$$\begin{aligned}
&[x(y + z(x + y) + z(x + y))]_a \\
&\quad = \kappa_b{}^a(x_b \cdot \kappa_d{}^b(y_d + z_d \cdot \kappa_e{}^d(x_e + y_e)) + z_b \cdot \kappa_e{}^b(x_e + y_e)) \\
&\quad \geqq \kappa_b{}^a(x_b \cdot \kappa_d{}^b(y_d) + z_b \kappa_e{}^b(y_e)) \\
&\quad \geqq \kappa_b{}^a(x_b y_b + z_b y_b)
\end{aligned}$$

proving (7.1.3).

The proofs of (7.1.2), (7.1.4), (7.1.5), their duals and the dual of (7.1.1) are similar. The dual of (7.1.3) is the splitting equation for Q_0 [**22**] pictured in Figure 7.1.

Q_0

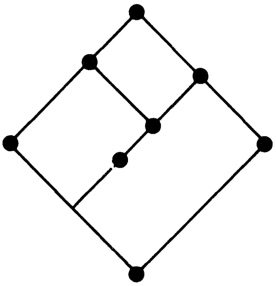

FIGURE 7.1

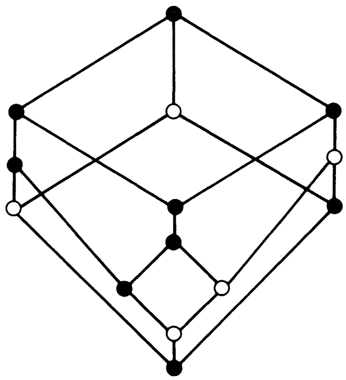

FIGURE 7.2

Q_0 is a sublattice of the lattice diagrammed in Figure 7.2. This lattice is the congruence lattice of $P(\mathbf{S}, A)$ where A is the four element Boolean

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algebra and $\mathbf{S}(a) = \mathbf{2}$ for each $a \in A$. Hence $Q_0 \in \mathrm{Con}(\mathscr{P})$. Thus the dual of (7.1.3) must fail in $\mathrm{Con}(\mathscr{P})$, completing the proof.

A lattice is *semidistributive* if it satisfies the following law and its dual.

$(SD_\wedge)$: $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$.

(7.3) COROLLARY. *All the lattices of* $\mathrm{Con}(\mathscr{P}) = \mathbf{HSP}\Theta(\mathscr{P})$ *are semidistributive.*

Proof. The proof follows from the fact that (7.1.1) implies $(SD_\wedge)$, which is easy to verify (cf. [**20**]).

(7.4) COROLLARY. *Those covers of the variety generated by* N_5 *which give rise to covers of* $\mathrm{Con}(\mathscr{P})$ *are precisely the varieties generated by* M_3, $L_1 - L_6$, *and* $L_8 - L_{12}$ *in the notation of* [**20**].

Proof. By Jonsson's Theorem [**16**] it suffices to show M_3, $L_1 - L_6$, $L_8 - L_{12}$ are not in $\mathrm{Con}(\mathscr{P})$ while L_7, $L_{13} - L_{15}$ are. M_3 and $L_1 - L_5$ are not semidistributive and hence are not in $\mathrm{Con}(\mathscr{P})$ by Corollary 7.3. By [**22**] the splitting equation of L_6 is (7.1.2), of L_8 is (7.1.3), of L_9 is (7.1.4), of L_{11} is (7.1.5). The splitting equations of L_{10} and L_{12} are the duals of (7.1.4) and (7.1.5) respectively. The corollary now follows from Theorem 7.1.

(7.5) COROLLARY. $\mathrm{Con}(\mathscr{P})$ *is not self-dual. Moreover its dual is not a congruence variety.*

Proof. Since $\mathbf{Con}(\mathscr{P})$ satisfies (7.1.3) but not its dual, it is not self dual. If its dual were a congruence variety, by our main result we would have

$$\mathbf{Con}(\mathscr{P}) \subseteq \mathbf{Con}(\mathscr{P})^{\mathrm{dual}}.$$

This would make $\mathbf{Con}(\mathscr{P})$ self-dual, a contradiction.

The next result shows that $\mathscr{P}$ has 4-permutable congruences (cf. [**14**]), i.e., its congruence lattices have type III joins (cf. [**1**]).

(7.6) THEOREM. $\mathscr{P}$ *has* 4-*permutable congruences.*

Proof. Let q_1, q_2, q_3 be the following polynomials in the language of $\mathscr{P}$.

$$\begin{aligned} q_1(x, y, z) &= x(yz^+)^+ \\ q_2(x, y, z) &= (xy')'(zy')'(xz)' \\ q_3(x, y, z) &= z(yx^+)^+. \end{aligned}$$

By [**14**] it suffices to show that the following identities hold in $\mathscr{P}$:

$$\begin{aligned} q_1(x, z, z) &= x \\ q_1(x, x, z) &= q_2(x, z, z) \\ q_2(x, x, z) &= q_3(x, z, z) \\ q_3(x, x, z) &= z. \end{aligned}$$

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Let $x, z \in P(\mathbf{S}, A) \in \mathscr{P}$. Let $x = (a, s)$ and $z = (b, t)$. Then

$$\begin{aligned} q_1(x, z, z) &= (a, s)((b, r)(b, t)^+)^+ \\ &= (a, s)(b, t)(b', 1))^+ \\ &= (a, s)(0, t)^+ \\ &= (a, s)(1, 1) = (a, s) = x. \end{aligned}$$

The verification of the other identities is similar.

(7.7) THEOREM. *There are $2^{\aleph_0}$ nonmodular congruence varieties.*

Proof. If $\mathscr{V}_1$ and $\mathscr{V}_2$ are varieties of algebras, possibly of different similarity types, let $\mathscr{V}_1 \otimes \mathscr{V}_2$ be their product as defined in Definition 1.7 of [**26**]. It follows easily from Corollary 1.13 of [**26**] that

$$\mathrm{Con}(\mathscr{V}_1 \otimes \mathscr{V}_2) = \mathrm{Con}(\mathscr{V}_1) \vee \mathrm{Con}(\mathscr{V}_2).$$

Thus congruence varieties are closed under finite joins (but not under intersections, see [**8**]). There are $2^{\aleph_0}$ modular congruence varieties, e.g. [**15**]. By (7.3) $\mathrm{Con}(\mathscr{P})$ contains no modular nondistributive lattices. Joining the modular congruence varieties with $\mathrm{Con}(\mathscr{P})$ gives $2^{\aleph_0}$ nonmodular congruence varieties which are all distinct by the above remarks and Jónsson's Theorem.

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