THE STABILITY OF PURE WEIGHTS UNDER CONDITIONING

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(Received 27 June, 1972)

1. Introduction. In [1], we showed how a collection of physical operations or experiments could be represented by a nonempty set $\mathscr{A}$ of nonempty sets satisfying certain conditions (irredundancy and coherence) and we called such sets $\mathscr{A}$ *manuals*. We also introduced " complete stochastic models " for the empirical universe of discourse represented by such a manual $\mathscr{A}$, namely, the so-called *weight functions* for $\mathscr{A}$. These weight functions form a convex set the extreme points of which are called *pure weights*. We also showed that there is a so-called *logic* $\Pi(\mathscr{A})$ affiliated with a manual $\mathscr{A}$ and that each weight function for $\mathscr{A}$ induces a *state* on this logic.

In practice, physical operations are usually synthesized from " simpler " or more " primitive " operations by iteration or compounding. In [8], we gave an indication of a mathematical construction whereby such compound operations can be given a perspicuous representation. Specifically, given a manual $\mathscr{A}$, one can construct from it a new manual $\mathscr{A}^c$ whose elements represent compound operations built up from the operations in the parent manual $\mathscr{A}$. In [9], we gave an indication of how the weight functions on the parent manual $\mathscr{A}$ induce (by means of so-called *transition functions*) weight functions on the compound manual $\mathscr{A}^c$.

We showed in [2] that the weight functions for the compound manual $\mathscr{A}^c$ can be transformed by certain natural *conditioning maps* into new weight functions for $\mathscr{A}^c$. In the present paper, we shall concern ourselves with the investigation of the stability of pure weight functions for $\mathscr{A}^c$ under these conditioning maps. It will be convenient to deal with *premanuals*, which are generalized manuals, rather than with manuals. Premanuals, which will shortly be defined, were first studied (under a different name) by Greechie and Miller in [4].

2. Premanuals and weight functions. By a *premanual* we mean a nonempty set $\mathscr{A}$ of nonempty sets. If $\mathscr{A}$ is such a premanual, the set $X = \bigcup \mathscr{A}$ is called the set of *outcomes* of $\mathscr{A}$. By a *weight function* for the premanual $\mathscr{A}$ we mean a real valued function ω defined on the outcome set $X = \bigcup \mathscr{A}$ and satisfying the following two conditions: (i) $0 \leqq \omega(x) \leqq 1$ for all $x \in X$. (ii) For each $E \in \mathscr{A}$, the unordered sum $\sum_{x \in E} \omega(x)$ converges to 1. In [3], Greechie has given examples of premanuals affiliated with finite orthomodular lattices which admit no weight functions whatsoever. We shall denote the set of all weight functions for the premanual $\mathscr{A}$ by $\Omega(\mathscr{A})$.

If $\mathscr{A}$ is a premanual and if α, $\beta \in \Omega(\mathscr{A})$, we define, for each real number t, a mapping $t\alpha + (1-t)\beta$ from $X = \bigcup \mathscr{A}$ to the real numbers by

$$(t\alpha + (1-t)\beta)(x) = t\alpha(x) + (1-t)\beta(x)$$

for all $x \in X$. Evidently, if $0 \leqq t \leqq 1$, then $t\alpha + (1-t)\beta \in \Omega(\mathscr{A})$; hence, in this case, we refer to $t\alpha + (1-t)\beta$ as a *convex combination* of the weight functions α and β. A weight function

https://doi.org/10.1017/S0017089500002020 Published online by Cambridge University Press

$\omega \in \Omega(\mathscr{A})$ is said to be *pure* if it cannot be written, nontrivially, as a convex combination of weight functions α and β. Specifically, ω is a pure weight function if and only if $\omega = t\alpha + (1-t)\beta$ with $\alpha, \beta \in \Omega(\mathscr{A})$ and $0 < t < 1$ implies that $\alpha = \beta$. We denote by $\Omega_p(\mathscr{A})$ the set of all pure weight functions for $\mathscr{A}$.

If $\mathscr{A}$ is a premanual and if $\alpha, \beta \in \Omega(\mathscr{A})$, we define a real number $r(\alpha, \beta)$ by the following:

$$r(\alpha, \beta) = \inf\left\{\frac{\alpha(x)}{\beta(x)} \middle| x \in X = \bigcup \mathscr{A} \quad \text{and} \quad \beta(x) \neq 0\right\}.$$

Evidently, $0 \leqq r(\alpha, \beta)$. If $1 \leqq r(\alpha, \beta)$, then $\beta(x) \leqq \alpha(x)$ for all $x \in X = \bigcup \mathscr{A}$, from which it easily follows that $\beta = \alpha$ and $r(\alpha, \beta) = 1$. In particular, then, if $\alpha \neq \beta$, $0 \leqq r(\alpha, \beta) < 1$. The following theorem generalizes a result of Greechie and Miller [4].

THEOREM 1. *Let $\alpha \in \Omega(\mathscr{A})$, where $\mathscr{A}$ is any premanual. Then α is pure if and only if $r(\alpha, \beta) = 0$ holds for all $\beta \in \Omega(\mathscr{A})$ with $\beta \neq \alpha$.*

Proof. Suppose first that α is pure, but that there exists $\beta \in \Omega(\mathscr{A})$ with $\beta \neq \alpha$ and $r(\alpha, \beta) > 0$. Put $t = (1 - r(\alpha, \beta))^{-1}$, noting that $1 < t$. Put $\mu = t\alpha + (1-t)\beta$. If there existed $y \in X = \bigcup \mathscr{A}$ with $\mu(y) < 0$, then we would have $0 \leqq \alpha(y) < \beta(y)$ and $\alpha(y)/\beta(y) < r(\alpha, \beta)$, a contradiction. It follows that $\mu(x) \geqq 0$ for all $x \in X$. If $E \in \mathscr{A}$, then $\sum_{x \in E} \mu(x) = 1$, from which it follows that $\mu \in \Omega(\mathscr{A})$. Put $s = t^{-1}$, so that $0 < s < 1$ and $\alpha = s\mu + (1-s)\beta$. Since α is pure, we conclude that $\mu = \beta$, and hence that $\alpha = \beta$, a contradiction.

Conversely, suppose that $r(\alpha, \beta) = 0$ for all $\beta \in \Omega(\mathscr{A})$ with $\beta \neq \alpha$, but that α is not pure. Then there exist $\mu, \beta \in \Omega(\mathscr{A})$ with $\mu \neq \beta$ and there exists a real number s with $0 < s < 1$ such that $\alpha = s\mu + (1-s)\beta$. Evidently $\alpha \neq \beta$; hence there exists $y \in X$ with $\beta(y) > \alpha(y) \geqq 0$, $\alpha(y)/\beta(y) < 1 - s$. However, this gives the immediate contradiction $s\mu(y) < 0$ and completes the proof.

3. Compound premanuals. Let $\mathscr{A}$ be a given premanual and $X = \bigcup \mathscr{A}$. Let $\Gamma = \Gamma(X)$ denote the free monoid (semigroup with unit 1) over the set X. An element of Γ (other than the unit 1) is uniquely expressible in the form $x_1 x_2 \ldots x_n$ with n a positive integer (called the *length* of the element) and $x_1, x_2, \ldots, x_n \in X$. We define the *length* of the unit 1 to be 0 and we denote the length of an element $a \in \Gamma$ by $|a|$. The elements of Γ of length one are naturally identified with the corresponding elements of X, so that $X \subseteq \Gamma$.

A subset A of Γ is said to be *bounded* if there is a non-negative integer n such that $|a| \leqq n$ for all $a \in A$. If A and B are subsets of Γ, we naturally define the product AB by $AB = \{ab \mid a \in A, b \in B\}$. If $a \in \Gamma$ and $B \subseteq \Gamma$, we define $aB = \{a\}B$ and $Ba = B\{a\}$.

If E and F are subsets of Γ and if there exists, for each $e \in E$, $G_e \in \mathscr{A} \cup \{\{1\}\}$ such that $F = \bigcup_{e \in E} eG_e$, we shall say that F is a *direct successor* of E. A set $\mathscr{H}$ of subsets of Γ will be called an *inductive class* provided that it satisfies the following two conditions:

(i) $\{1\} \in \mathscr{H}$;

(ii) if $E \in \mathscr{H}$ and if F is a direct successor of E, then $F \in \mathscr{H}$.

Notice that any $G \in \mathscr{A}$ is a direct successor of $\{1\}$; hence $\mathscr{A}$ is contained in any inductive class. The set of all nonempty subsets of Γ is an inductive class, and the intersection of any family

https://doi.org/10.1017/S0017089500002020 Published online by Cambridge University Press

of inductive classes is again an inductive class. We shall denote by $\mathscr{A}^c$ the intersection of the family of all inductive classes of subsets of Γ, so that $\mathscr{A} \subseteq \mathscr{A}^c$ and $\mathscr{A}^c$ is the smallest inductive class of subsets of Γ. Since $\emptyset \notin \mathscr{A}^c$, $\mathscr{A}^c$ is a premanual called the *compound premanual over* $\mathscr{A}$.

Evidently, the collection of all bounded subsets of Γ is an inductive class; hence every $E \in \mathscr{A}^c$ is bounded. A subset K of Γ is called an *abridged* set provided that, if $a, b \in K$ and if there exists $c \in \Gamma$ with $a = bc$, then $c = 1$ (so that $a = b$). We shall now prove that every $E \in \mathscr{A}^c$ is an abridged set.

THEOREM 2. *Let $\mathscr{A}$ be any premanual and let $E \in \mathscr{A}^c$. Then E is an abridged set.*

Proof. Let $X = \bigcup \mathscr{A}$ and let Γ be the free monoid over X. Let $\mathscr{H}$ denote the set of all abridged subsets of Γ. It will suffice to prove that $\mathscr{H}$ is an inductive class. Clearly, $\{1\} \in \mathscr{H}$. Suppose that $E \in \mathscr{H}$ and that F is a direct successor of E, but that $F \notin \mathscr{H}$. For each $e \in E$, there exists $G_e \in \mathscr{A} \cup \{\{1\}\}$ such that $F = \bigcup_{e \in E} eG_e$. Since $F \notin \mathscr{H}$, there exist $a, b \in F$ and $c \in \Gamma$, with $c \neq 1$ and $a = bc$. Since $a, b \in F$, there exist $d, e \in E$, $x \in G_d$ and $y \in G_e$ such that $a = dx$ and $b = ey$. Thus we have $dx = eyc$. Since E is abridged, $x \neq 1$, for otherwise $d = e(yc)$, so that $yc = 1$, $c = 1$, a contradiction. Since $c \neq 1$, we can write $c = hz$ for some $h \in \Gamma, z \in X$. The equation $dx = eyhz$, together with the facts that Γ is freely generated by X and that $x, z \in X$, implies that $z = x$; hence we have $d = eyh$. Again, since E is abridged, we must have $yh = 1$; hence, $d = e$, $y = 1$. Thus we have $e = ey = b$, and so $e \in F$. Also, $ex = dx = a \in F$; so $ex \in F$, with $x \in G_e$, $x \neq 1$. Since $G_e \in \mathscr{A} \cup \{\{1\}\}$ and $x \in G_e$ with $x \neq 1$, we have $1 \notin G_e$; hence $e \notin eG_e$. But, since $e \in F$, there must exist $k \in E$ with $e \in kG_k$. Hence $e = kw$ for some $w \in G_k$. Since E is abridged, $w = 1$ and $k = e$; hence $e \in eG_e$, a contradiction. The proof is complete.

COROLLARY 3. *Let $\mathscr{A}$ be a premanual and let $E \in \mathscr{A}^c$. For each $e \in E$, let $G_e \in \mathscr{A} \cup \{\{1\}\}$. Then, if $d, e \in E$ with $d \neq e$, it follows that $dG_d \cap eG_e = \emptyset$.*

THEOREM 4. *Let $\mathscr{A}$ be a premanual with $X = \bigcup \mathscr{A}$ and let Γ be the free monoid over X. Then $\bigcup \mathscr{A}^c = \Gamma$.*

Proof. It will suffice to show that each element $a \in \Gamma$ belongs to at least one set $E \in \mathscr{A}^c$. We prove this by induction on $|a|$. If $|a| = 0$, then $a = 1 \in \{1\} \in \mathscr{A}^c$. Suppose that the assertion is true for all $a \in \Gamma$ with $|a| = n$. Let $b \in \Gamma$ with $|b| = n+1$. Then we can write $b = ax$ with $|a| = n$ and $x \in X$. By hypothesis, there exists $E \in \mathscr{A}^c$ with $a \in E$. Since $x \in X$, there exists $G \in \mathscr{A}$ with $x \in G$. For each $e \in E$, define $G_e = G$, and note that $F = \bigcup_{e \in E} eG_e \in \mathscr{A}^c$, since $\mathscr{A}^c$ is an inductive class. Since $b = ax \in aG_a \subseteq F$, the proof is complete.

4. Weight functions for compound premanuals. For the remainder of this paper, we assume that $\mathscr{A}$ is a premanual with $\Omega(\mathscr{A}) \neq \emptyset$ and we put $X = \bigcup \mathscr{A}$. We also denote by Γ the free monoid over X. By a *transition function* for the premanual $\mathscr{A}^c$ we mean a function $f \colon \Gamma \times X \to \mathbb{R}$ such that, for every $e \in \Gamma$, $f(e, \cdot) \in \Omega(\mathscr{A})$. Thus, a transition function can be regarded as a family of weight functions for $\mathscr{A}$ indexed by the elements of Γ. If f is a transition function for $\mathscr{A}^c$, we define a real-valued function ω_f on Γ by recursion as follows:

(1) $\omega_f(1) = 1$;
(2) if $a \in \Gamma$ and $x \in X$, then $\omega_f(ax) = \omega_f(a) f(a, x)$.

In particular, we have $\omega_f(x) = f(1, x)$ for all $x \in X$. For $x_1, x_2, \ldots, x_n \in X$, $n \geqq 2$, we will then have

$$\omega_f(x_1 x_2 \ldots x_n) = f(1, x_1) \prod_{j=2}^{n} f(x_1 x_2 \ldots x_{j-1}, x_j).$$

THEOREM 5. *If f is any transition function for $\mathscr{A}^c$, then $\omega_f \in \Omega(\mathscr{A}^c)$.*

Proof. Evidently, $\omega_f(a) \geqq 0$ for all $a \in \Gamma$. Thus it will suffice to show that, for any $E \in \mathscr{A}^c$, $\sum_{e \in E} \omega_f(e) = 1$. Thus, let $\mathscr{H}$ denote the set of all sets $E \in \mathscr{A}^c$ such that $\sum_{e \in E} \omega_f(e) = 1$. It will be enough to show that $\mathscr{H}$ is an inductive class. Clearly, $\{1\} \in \mathscr{H}$. Thus, let $E \in \mathscr{H}$, and suppose that F is a direct successor of E. Then, for every $e \in E$, there exists $G_e \in \mathscr{A} \cup \{\{1\}\}$ such that $F = \bigcup_{e \in E} eG_e$. By Corollary 3, the latter is a disjoint union. Let us temporarily fix an $e \in E$ and put $G = G_e$. If $G = \{1\}$, then $\sum_{a \in eG} \omega_f(a) = \omega_f(e)$. On the other hand, if $G \neq \{1\}$, then $G \in \mathscr{A}$ and we have

$$\sum_{a \in eG} \omega_f(a) = \sum_{x \in G} \omega_f(ex) = \sum_{x \in G} \omega_f(e) f(e, x) = \omega_f(e) \sum_{x \in G} f(e, x) = \omega_f(e).$$

It follows that $\sum_{a \in F} \omega_f(a) = \sum_{e \in E} \omega_f(e) = 1$; hence $\mathscr{H}$ is an inductive class and the proof is complete.

LEMMA 6. *Let $\omega \in \Omega(\mathscr{A}^c)$ and let $a \in \Gamma$, $G \in \mathscr{A}$. Then $\sum_{x \in G} \omega(ax) = \omega(a)$.*

Proof. By Theorem 4, there exists $E \in \mathscr{A}^c$ with $a \in E$. For $e \in E$ with $e \neq a$, define $G_e = \{1\}$. Define $G_a = G$. Put $F = \bigcup_{e \in E} eG_e$, noting that $F \in \mathscr{A}^c$. Put $H = E \backslash a$. We now have

$$1 = \sum_{b \in F} \omega(b) = \sum_{e \in H} \omega(e) + \sum_{x \in G} \omega(ax) = 1 - \omega(a) + \sum_{x \in G} \omega(ax),$$

and the lemma is proved.

Suppose that f is a transition function for $\mathscr{A}^c$ and that d belongs to Γ. We then define a new transition function f/d, called *f conditioned* by d, by the following prescription:

$$(f/d)(a, x) = \begin{cases} f(d, x) & \text{if} \quad \omega_f(da) = 0, \\ f(da, x) & \text{if} \quad \omega_f(da) \neq 0, \end{cases}$$

for $a \in \Gamma$, $x \in X$.

THEOREM 7. *Let f be a transition function for $\mathscr{A}^c$ and let $d \in \Gamma$. Put $g = f/d$. Then, for any $a \in \Gamma$, we have $\omega_g(a)\omega_f(d) = \omega_f(da)$.*

Proof. The proof is by induction on $|a|$. If $|a| = 0$, then $a = 1$ and the result is evidently true. Suppose that the result holds for $|a| = n$ and let $b \in \Gamma$ with $|b| = n+1$. Then there exists $a \in \Gamma$ and $x \in X$, with $|a| = n$, $b = ax$. By hypothesis, $\omega_g(a)\omega_f(d) = \omega_f(da)$. Hence $\omega_g(b)\omega_f(d) = \omega_g(ax)\omega_f(d) = \omega_g(a)g(a, x)\omega_f(d) = \omega_f(da)g(a, x)$. Hence, if $\omega_f(da) \neq 0$, we have $\omega_g(b)\omega_f(d) = \omega_f(da)f(da, x) = \omega_f(dax) = \omega_f(db)$ as desired. Thus we can suppose that

https://doi.org/10.1017/S0017089500002020 Published online by Cambridge University Press

$\omega_f(da) = 0$. This gives $\omega_g(b)\omega_f(d) = 0$, and we are obliged to prove that $\omega_f(db) = 0$. Since $x \in X$, there exists $G \in \mathscr{A}$ with $x \in G$. By Lemma 6,

$$0 = \omega_f(da) = \sum_{y \in G} \omega_f(day) \geqq \omega_f(dax) = \omega_f(db) \geqq 0;$$

hence $\omega_f(db) = 0$ as desired.

A transition function f for $\mathscr{A}^c$ is said to be *normalized* if it satisfies the following condition: For all $a \in \Gamma$ and all $x \in X$, if $\omega_f(a) = 0$, then $f(a, x) = f(1, x)$. Suppose that $\alpha \in \Omega(\mathscr{A}^c)$ and define $f: \Gamma \times X \to \mathbb{R}$ as follows. For $a \in \Gamma$ and $x \in X$,

$$f(a, x) = \begin{cases} \alpha(x) & \text{if} \quad \alpha(a) = 0, \\ \dfrac{\alpha(ax)}{\alpha(a)} & \text{if} \quad \alpha(a) \neq 0. \end{cases}$$

As a consequence of Lemma 6, we see that f is a transition function for $\mathscr{A}^c$, and direct calculation reveals that $\omega_f = \alpha$, from which it easily follows that f is normalized. A final calculation shows that, if g is any normalized transition function for $\mathscr{A}^c$ such that $\omega_g = \alpha$, then $g = f$. Thus we have the following lemma.

LEMMA 8. *The mapping $f \mapsto \omega_f$ provides a one-to-one correspondence between normalized transition functions f for $\mathscr{A}^c$ and the set $\Omega(\mathscr{A}^c)$ of all weight functions ω_f for $\mathscr{A}^c$.*

Suppose that $\alpha \in \Omega(\mathscr{A}^c)$ and that $d \in \Gamma$. Let f be the unique normalized transition function for $\mathscr{A}^c$ such that $\omega_f = \alpha$. We can now form the conditioned transition function f/d and thence the weight function $\omega_{f/d}$. We call $\omega_{f/d}$ the weight function obtained by *conditioning* α *by* d and we introduce the notation α/d for $\omega_{f/d}$. According to Theorem 7, we have the identity $(\alpha/d)(a) \cdot \alpha(d) = \alpha(da)$ for all $a \in \Gamma$. In particular, if $\alpha(d) \neq 0$, we have

$$(\alpha/d)(a) = \frac{\alpha(da)}{\alpha(d)},$$

a formula which is analogous to the classical definition of conditional probability. An easy calculation shows that the transition function f/d is normalized, so that f/d is the unique normalized transition function corresponding to α/d according to Lemma 8.

Continuing with the above notation, we notice that from the equation $\alpha(da) = (\alpha/d)(a) \cdot \alpha(d)$ we can deduce that, if $\alpha(d) = 0$, then $\alpha(da) = 0$ holds for all $a \in \Gamma$. From this we see that, if $\alpha(d) = 0$, then we have

$$(\alpha/d)(x_1 x_2 \ldots x_n) = \alpha(x_1)\alpha(x_2) \ldots \alpha(x_n),$$

for $x_1, x_2, \ldots, x_n \in X$, $n \geqq 1$. This suggests a slight extension of the above notation. Given any normalized transition function f for $\mathscr{A}^c$, we define a transition function $f/*$ for $\mathscr{A}^c$ by $(f/*)(a, x) = f(1, x)$ for all $a \in \Gamma$ and all $x \in X$. Evidently, $f/*$ is normalized. Given any weight function $\alpha \in \Omega(\mathscr{A}^c)$, we now define $\alpha/*$ as follows. Let f be the unique normalized transition function for which $\alpha = \omega_f$, and define $\alpha/* = \omega_{f/*}$. Evidently,

$$(\alpha/*)(x_1 x_2 \ldots x_n) = \alpha(x_1)\alpha(x_2) \ldots \alpha(x_n)$$

https://doi.org/10.1017/S0017089500002020 Published online by Cambridge University Press

holds for $x_1, x_2, \ldots, x_n \in X$, $n \geqq 1$. In particular, $(\alpha/*)(ab) = (\alpha/*)(a)\cdot(\alpha/*)(b)$ holds for all $a, b \in \Gamma$ and we have the result that, if $d \in \Gamma$ with $\alpha(d) = 0$, then $\alpha/d = \alpha/*$.

5. The stability of pure weights under conditioning. In the present section, we shall prove the main theorem of this paper, namely that, if α belongs to $\Omega_p(\mathscr{A}^c)$ and if d is any element of Γ, then the conditioned weight function α/d also belongs to $\Omega_p(\mathscr{A}^c)$.

LEMMA 9. *Let $\alpha \in \Omega_p(\mathscr{A}^c)$ and let $d \in \Gamma$ with $\alpha(d) \neq 0$. Define a real-valued function β on $X = \bigcup \mathscr{A}$ by $\beta(x) = \alpha(dx)/\alpha(d)$ for all $x \in X$. Then $\beta \in \Omega_p(\mathscr{A})$.*

Proof. Suppose that $\beta \notin \Omega_p(\mathscr{A})$. By Theorem 1, there exists $\mu \in \Omega(\mathscr{A})$, with $\mu \neq \beta$ and $r(\beta, \mu) > 0$. Let f be the unique normalized transition function for $\mathscr{A}^c$ such that $\alpha = \omega_f$. Define a transition function g for $\mathscr{A}^c$ as follows.

$$g(a, x) = \begin{cases} f(a, x) & \text{if} \quad a \neq d, \\ \mu(x) & \text{if} \quad a = d, \end{cases}$$

for $a \in \Gamma$, $x \in X$.

Suppose that $\omega_g = \alpha$. Then, for any $x \in X$, $\alpha(d)f(d, x) = \omega_f(d)f(d, x) = \omega_f(dx) = \alpha(dx) = \omega_g(dx) = \omega_g(d)g(d, x) = \alpha(d)\mu(x)$; hence, since $\alpha(d) \neq 0$, $f(d, x) = \mu(x)$ holds for all $x \in X$. However, since $\alpha(d) \neq 0$, $f(d, x) = \beta(x)$ holds for all $x \in X$, and we obtain the contradiction $\mu = \beta$. Thus $\omega_g \neq \alpha$.

Since $\alpha \in \Omega_p(\mathscr{A}^c)$ and $\omega_g \in \Omega(\mathscr{A}^c)$ with $\alpha \neq \omega_g$, Theorem 1 gives $r(\alpha, \omega_g) = 0$. It follows that there exists an element $c \in \Gamma$ with $0 \leqq \alpha(c) < \omega_g(c)$, $\alpha(c)/\omega_g(c) < r(\beta, \mu)$. Evidently, $c \neq 1$; hence we can write $c = x_1 x_2 \ldots x_n$ with $n \geqq 1$ and $x_1, x_2, \ldots, x_n \in X$. For $1 \leqq j \leqq n$, define $c_j = 1$ if $j = 1$ and $c_j = x_1 x_2 \ldots x_{j-1}$ if $2 \leqq j \leqq n$. We have

$$\alpha(c) = f(c_1, x_1)f(c_2, x_2)\ldots f(c_n, x_n)$$

and

$$\omega_g(c) = g(c_1, x_1)g(c_2, x_2)\ldots g(c_n, x_n).$$

Since $\alpha(c) \neq \omega_g(c)$, there must exist a positive integer i with $1 \leqq i \leqq n$ and $f(c_i, x_i) \neq g(c_i, x_i)$. From the definition of $g(c_i, x_i)$ it follows that $c_i = d$ and $g(c_i, x_i) = \mu(x_i)$. We also have $f(c_i, x_i) = f(d, x_i) = \beta(x_i)$, since $\alpha(d) \neq 0$. For $1 \leqq j \leqq n$ with $j \neq i$, we have $c_j \neq c_i = d$; hence $g(c_j, x_j) = f(c_j, x_j)$. From the condition $0 \leqq \alpha(c) < \omega_g(c)$ we deduce that $g(c_j, x_j) > 0$ for $1 \leqq j \leqq n$ and that

$$\frac{\beta(x_i)}{\mu(x_i)} = \frac{\alpha(c)}{\omega_g(c)} < r(\beta, \mu),$$

an immediate contradiction.

COROLLARY 10. *Let $\alpha \in \Omega_p(\mathscr{A}^c)$ and let f be the unique normalized transition function for $\mathscr{A}^c$ for which $\omega_f = \alpha$. Then, for every element $a \in \Gamma$, $f(a, \cdot) \in \Omega_p(\mathscr{A})$.*

Proof. If $\alpha(a) \neq 0$, then $f(a, x) = \alpha(ax)/\alpha(a)$ for all $x \in X$; so $f(a, \cdot) \in \Omega_p(\mathscr{A})$, by Lemma 9. On the other hand, if $\alpha(a) = 0$, then $f(a, x) = f(1, x) = \alpha(x) = \alpha(1x)/\alpha(1)$ for all $x \in X$, so that, again by Lemma 9, $f(a, \cdot) \in \Omega_p(\mathscr{A})$, and the corollary is proved.

A normalized transition function f for $\mathscr{A}^c$ will be called *pure* if $f(a, \cdot) \in \Omega_p(\mathscr{A})$ holds for every $a \in \Gamma$. Corollary 10 says that, if ω_f is a pure weight function for $\mathscr{A}^c$, then f is pure. In the following theorem we shall establish the converse.

THEOREM 11. *Let f be a normalized transition function for $\mathscr{A}^c$. Then f is pure if and only if $\omega_f \in \Omega_p(\mathscr{A}^c)$.*

Proof. We know already that, if ω_f is pure, then so is f. Suppose, then, that f is pure, but that ω_f is not pure. Then, by Theorem 1, there exists a normalized transition function $g \neq f$ such that $0 < r(\omega_f, \omega_g)$. Suppose that $f(1, \cdot) \neq g(1, \cdot)$. Since $f(1, \cdot) \in \Omega_p(\mathscr{A})$, Theorem 1 gives $r(f(1, \cdot), g(1, \cdot)) = 0$; hence there exists $x \in X$ such that $0 \leqq f(1, x) < g(1, x)$ and $f(1, x)/g(1, x) < r(\omega_f, \omega_g)$. Since $\omega_f(x) = f(1, x)$ and $\omega_g(x) = g(1, x)$, the latter inequality cannot be true; hence we conclude that $f(1, \cdot) = g(1, \cdot)$.

Choose $b \in \Gamma$ with $|b|$ minimal such that $f(b, \cdot) \neq g(b, \cdot)$. Since $b \neq 1$, we can write $b = x_1 x_2 \ldots x_n$ with $n \geqq 1$ and $x_1, x_2, \ldots, x_n \in X$. Put $c_1 = 1$ and $c_j = x_1 x_2 \ldots x_{j-1}$ for $2 \leqq j \leqq n$. We have

$$\omega_f(b) = \prod_{j=1}^{n} f(c_j, x_j) = \prod_{j=1}^{n} g(c_j, x_j) = \omega_g(b),$$

since $|c_j| < n = |b|$ for all $j = 1, 2, \ldots, n$. Suppose that $\omega_f(b) = \omega_g(b) = 0$. Then, since f and g are normalized,

$$f(b, \cdot) = f(1, \cdot) = g(1, \cdot) = g(b, \cdot),$$

a contradiction. We conclude that $\omega_f(b) = \omega_g(b) \neq 0$.

Since $f(b, \cdot) \neq g(b, \cdot)$ and $f(b, \cdot) \in \Omega_p(\mathscr{A})$, then, by Theorem 1, $r(f(b, \cdot), g(b, \cdot)) = 0$. Hence there exists $x \in X$ such that

$$\frac{f(b,x)}{g(b,x)} < r(\omega_f, \omega_g) < \frac{\omega_f(bx)}{\omega_g(bx)} = \frac{\omega_f(b) f(b,x)}{\omega_g(b) g(b,x)},$$

yielding the contradiction that $f(b, x)/g(b, x)$ is less than itself. This contradiction proves the theorem.

THEOREM 12. *Let $\alpha \in \Omega_p(\mathscr{A}^c)$ and let $d \in \Gamma$. Then the conditioned weight function α/d, as well as $\alpha/*$, belong to $\Omega_p(\mathscr{A}^c)$.*

Proof. Let f be the unique normalized transition function for $\mathscr{A}^c$ such that $\omega_f = \alpha$. By Theorem 11, f is pure. From the definitions of f/d and $f/*$, we see immediately that f/d and $f/*$ are pure; hence, by Theorem 11, $\alpha/d = \omega_{f/d} \in \Omega_p(\mathscr{A}^c)$ and $\alpha/* = \omega_{f/*} \in \Omega_p(\mathscr{A}^c)$ as desired.

6. Concluding remarks. There are many known examples of " conditioning " processes which preserve the " purity " of " stochastic models ". Indeed, the classical example is obtained as follows: Let $\mathscr{B}$ denote any Boolean algebra. From $\mathscr{B}$, we construct a premanual $\mathscr{A} = \{E \subseteq \mathscr{B} \mid E$ is a finite set of pairwise disjoint nonzero elements of $\mathscr{B}$ and $\sum_{e \in E} e = 1\}$. Evidently, the weight functions in $\Omega(\mathscr{A})$ are in a natural one-to-one correspondence with the finitely additive probability measures on $\mathscr{B}$. We shall identify a weight function $\omega \in \Omega(\mathscr{A})$ with the corresponding probability measure. The pure weights now correspond to the points in the Stone space affiliated with $\mathscr{B}$. Suppose that $\omega \in \Omega(\mathscr{A})$ and that $a \in \mathscr{B}$ with $\omega(a) \neq 0$. By

" conditioning " ω by a, we can define a new weight function $\omega_a \in \Omega(\mathscr{A})$ by $\omega_a(b) = \omega(ab)/\omega(a)$ for all $b \in \mathscr{B}$. It is easy to see that, if ω is pure, so is ω_a.

A second example arises in conventional non-relativistic quantum mechanics. To construct this example, let $\mathscr{H}$ be a complex, separable, infinite-dimensional Hilbert space and let $\mathscr{D}$ be the set of all von Neumann density operators on $\mathscr{H}$. Let the premanual $\mathscr{A}$ consist of all countable sets $\{P_1, P_2, \ldots\}$ of pairwise orthogonal nonzero projection operators on $\mathscr{H}$ such that $\sum_i P_i = 1$. For each $D \in \mathscr{D}$, define the weight function ω_D by $\omega_D(P) = Tr(DP)$ for all nonzero projection operators P on $\mathscr{H}$. These weight functions are now in a natural one-to-one correspondence with the quantum mechanical states. Furthermore, the weight function ω_D corresponds to a pure state if and only if $D = D^2$. Evidently, ω_D is a pure weight if and only if it corresponds to a pure quantum mechanical state. Suppose that $D \in \mathscr{D}$ and that P is a projection operator on $\mathscr{H}$ such that $Tr(DP) \neq 0$. The usual quantum mechanical " conditioning by P " [6, p. 333; 5] converts D into $D_P = (Tr(DP))^{-1}PDP$. It is easy to check that this conditioning preserves pure weights.

In Pool's work on the logic of quantum mechanics, it is shown that (under suitable hypotheses on the event-state-operation structures under consideration) pure states are stable under conditioning by operations precisely when the quantum logic is semimodular [7].

Our Theorem 12 provides still another example of the " stability of purity under conditioning ". However, it can be shown that, if the weight functions on a compound premanual are conditioned not by outcomes, but by so-called " events ", the purity of the weight functions is not generally preserved.

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