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CONTINUOUS CHAOTIC FUNCTIONS OF AN INTERVAL HAVE GENERICALLY SMALL SCRAMBLED SETS

IVAN MIZERA

It is shown that continuous self-mappings of a compact interval, chaotic in the sense of Li and Yorke, have generically, in the uniform topology, only scrambled sets which are nowhere dense and of zero Lebesgue measure.

Let f be a continuous function from a compact interval I into itself. A set S is called scrambled, if for every $x, y \in S$, $x \neq y$, and for every periodic point p of f

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0, \tag{1}$$

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0, \tag{2}$$

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| > 0, \tag{3}$$

where f^n denotes the n-th iterate of f (see [**9**]). A function f is called chaotic in the sense of Li and Yorke when it has an uncountable scrambled set. This property is equivalent to the existence of a non-empty perfect "uniformly" scrambled set ([**6**], [**14**]). The property "being chaotic" is generic ([**1**], [**2**], [**8**], [**11**]). There are known examples of functions with scrambled sets of the first and of the second Baire category ([**5**]). If a scrambled set has the Baire property, it must be first category - hence it cannot be residual ([**3**], [**4**]). A scrambled set can have a positive Lebesgue measure arbitrarily close to the measure of the whole interval ([**7**], [**12**], [**13**]). It can even have a full measure ([**10**]).

The size of the scrambled set - in the sense of the category or of the measure - reflects somehow the degree of "chaos". However, in this paper we show that large scrambled sets are not typical for chaotic functions. We consider the space $C^0(I, I)$ of continuous functions of an interval into itself, endowed with the usual topology of uniform convergence. We prove the following

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THEOREM. *There is a first Baire category set* $M \subseteq C^0(I,I)$ *such that any* $f \in C^0(I,I) \setminus M$ *has only scrambled sets (if any) which are nowhere dense in* I *and of zero Lebesgue measure. In other words, mappings from* $C^0(I,I)$ *have generically only nowhere dense zero measure scrambled sets.*

Combining the above mentioned results with the Theorem, we obtain

COROLLARY. *Continuous self-mappings of an interval are generically chaotic but only with nowhere dense zero measure scrambled sets.*

We prove the Theorem by several Lemmas. We may assume that $I = [0,1]$. Let λ denote the Lebesgue measure on I. For $n = 1,2,3,\ldots,$ let $0 < \zeta_n < 4^{-n}$. For $i = 1,2,3,\ldots,2^n$ let $a(n,i) = (i-1)2^{-n}$, $a(n,2^n+1) = 1$, $b(n,i) = i2^{-n} - \zeta_n$. Let $I(n,i) = [a(n,i),\ b(n,i)]$, $I_n = \bigcup_i I(n,i)$. Let $A_n = \{f \in C^0(I,I) :\ f(I_n) \subseteq \operatorname{int} I_n\}$. Immediately we have

LEMMA 1. *For every* n, A_n *is open in* $C^0(I,I)$.

Let $B_n = \bigcup_{k \geqslant n} A_k$.

LEMMA 2. *For every* n , B_n *is open and dense in* $C^0(I,I)$.

PROOF: By Lemma 1, B_n is open. Let $U_\varepsilon(f)$ be an ε -neighbourhood of f in $C^0(I,I)$. By the uniform continuity of f there exists an integer $k \geqslant n$ such that $2^{-k} < \varepsilon/4$ and

$$|f(x) - f(y)| < \varepsilon/4 \quad \text{whenever} \quad |x-y| \leqslant 2^{-k}. \tag{4}$$

Now we construct a function $g \in A_k \cap U_\varepsilon(f)$ in two steps. ∎

First define an auxiliary function $h\colon I_k \to I_k$ such that for every i , h is constant on $I(k,i)$, $h(I_k) \subseteq \operatorname{int} I_k$ and

$$|h(a(k,i)) - f(a(k,i))| < \varepsilon/8. \tag{5}$$

We put $f(a(k,i))$ for the image of $I(k,i)$ under h , provided $f(a(k,i)) \in \operatorname{int} I_k$. Otherwise, we put for the image of $I(k,i)$ a slightly perturbed value: $f(a(k,i))+2^{-(k+1)}$ or $f(a(k,i)) - 2^{-(k+1)}$. Note that the gaps between the adjacent intervals of I_k have each length $\zeta_k < 4^{-k} \leqslant 2^{-(k+1)}$, while the length of each of these intervals is $2^{-k} - \zeta_k > 2^{-(k+1)}$. Hence, if $f(a(k,i)) \notin \operatorname{int} I_k$, then at least one of the perturbed values belongs to $\operatorname{int} I_k$.

In the second step, let g be a continuous extension of h such that g is linear on every interval contiguous to I_k and $g(1) = f(1)$. Clearly $g \in A_k$, so it remains

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only to show that $|f(x) - g(x)| < \varepsilon$ for all $x \in I$. This is true for $x = 1$; let $x \in [a(k,i), a(k,i+1))$ for some i. We have

$$|f(x) - g(x)| \leqslant |f(x) - f(a(k,i))| + |f(a(k,i)) - g(a(k,i))| + |g(a(k,i)) - g(x)|. \tag{6}$$

By (5), the middle term on the right of (6) is less than $\varepsilon/8$. By the linearity of g, (4) and (5) we get

$$\begin{aligned}|g(a(k,i)) - g(x)| &\leqslant |g(a(k,i)) - g(a(k,i+1))| \\ &\leqslant \varepsilon/8 + |f(a(k,i)) - f(a(k,i+1))| + \varepsilon/8 < \varepsilon/2.\end{aligned}$$

Now, summarizing all this, we have $|f(x) - g(x)| < \varepsilon$, and this completes the proof.

LEMMA 3. *Let $f \in A_n$, let S be a scrambled set for f. If $S \cap I_n \neq \emptyset$, then*

$$\limsup_{k\to\infty} |f^k(u) - f^k(v)| < 2^{-n} \quad \text{for all } u, v \in S. \tag{7}$$

PROOF: Let $w \in S \cap I_n$. Then $f^k(w) \in \operatorname{int} I_n$ for every $k \geqslant 1$ since $f(I_n) \subseteq \operatorname{int} I_n$. Since actually $f(I_n) \subset \operatorname{int} I_n$, $\operatorname{dist}\{f(I_n), I \setminus I_n\} > 0$, hence (2) for $x = u$, $y = w$ implies that for any sufficiently large k both $f^k(u)$ and $f^k(w)$ lie in the same component of I_n (dependent on k). By the same argument applied to v and w we obtain that for sufficiently large k both $f^k(u)$ and $f^k(v)$ lie in the same component of I_n, dependent on k. This implies (7).

Now we can complete the proof of the Theorem. Put $M = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. By Lemma 2 M is residual in $C^0(I,I)$. Let $f \in M$, let S be a scrambled set for f. There exists an infinite, increasing sequence $\{n_i\}_{i=1}^{\infty}$ of integers such that $f \in A_{n_i}$ for every i. By Lemma 3, there exists an integer s such that $S \cap I_{n_i} = \emptyset$ for all $n_i > s$ – since otherwise (1) would be violated. Let $J = \bigcup_{n_i > s} \operatorname{int} I_{n_i}$. For every n, $\lambda(I \setminus \operatorname{int} I_n) < 2^{-n}$, hence J has full measure. This implies J is dense; clearly it is open. Since $S \subseteq I \setminus J$, we have that $\lambda(S) = 0$ and S is nowhere dense in I. ∎

REMARK 1: We want to call attention to the following fact, implied by Lemmas 2 and 3. Let M_ε be the set of all $f \in C^0(I,I)$ such that either all scrambled sets for f have Lebesgue measure less than ε, or for every scrambled set S (for f) is $\limsup_{n\to\infty} |f^n(x) - f^n(y)| < \varepsilon$ for all $x, y \in S$. Then M_ε contains an open dense subset - its complement in $C^0(I,I)$ is nowhere dense. In other words, functions from M_ε are generically stable.

REMARK 2: The results admit a straight forward generalisation to the case of continuous self-mappings of the circle. They can be also extended to cover the n-dimensional case - for continuous functions of I^n into I^n.

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Department of Mathematics
Komenský University
842 15 Bratislava
Czechoslovakia

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