Proceedings of the Edinburgh Mathematical Society (1980), **23**, 249–260 ©

THE NATURAL PARTIAL ORDER ON A REGULAR SEMIGROUP

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(Received 12th July 1978)

It is well-known that on an inverse semigroup S the relation $\leqq$ defined by $a \leqq b$ if and only if $aa^{-1} = ab^{-1}$ is a partial order (called the natural partial order) on S and that this relation is closely related to the global structure of S (cf. (**1**, §7.1), (**10**)). Our purpose here is to study a partial order on regular semigroups that coincides with the relation defined above on inverse semigroups. It is found that this relation has properties very similar to the properties of the natural partial order on inverse semigroups. However, this relation is not, in general, compatible with the multiplication in the semigroup. We show that this is true if and only if the semigroup is pseudo-inverse (cf. (**8**)). We also show how this relation may be used to obtain a simple description of the finest primitive congruence and the finest completely simple congruence on a regular semigroup.

1. The natural partial order

In this paper we use, whenever possible, the notations of (**2**). For the definitions of bi-ordered sets and related concepts we refer the reader to (**7**). We shall also use notations established in (**7**) with the following exceptions. We shall write $\mathcal{R} = \omega^r \cap (\omega^r)^{-1}$ and $\mathcal{L} = \omega^l \cap (\omega^l)^{-1}$ instead of $\stackrel{r}{\approx}$ and $\stackrel{l}{\approx}$ respectively. Further, as an economy measure, we shall use "basic products" (Clifford (**2**, Equation (1.3))) instead of τ-mappings $\tau^r(e)$, $\tau^l(e)$ etc.

Recall that for any semigroup S, the relations

$$L_x \leqq L_y \text{ if and only if } S^1x \subseteq S^1y,$$
$$R_x \leqq R_y \text{ if and only if } xS^1 \subseteq yS^1$$

are partial orders on $S/\mathcal{L}$ and $S/\mathcal{R}$ respectively and that

$$H_x \leqq H_y \text{ if and only if } L_x \leqq L_y, R_x \leqq R_y$$

is a partial order on $S/\mathcal{H}$. When S is regular, the last relation may also be defined as

$$H_x \leqq H_y \text{ if and only if } x \in ySy. \tag{1.1}$$

Proposition 1.1. *Let S be a regular semigroup. For $x, y \in S$, define*

$$x \leqq y \text{ if and only if } R_x \leqq R_y \text{ and } x = fy \text{ for some } f \in E(R_x) \tag{1.2}$$

where for any $X \subseteq S$, $E(X)$ denotes the set of idempotents in X. Then the relation $\leqq$ is a partial order on S whose restriction to $E(S)$ is the relation ω.

Proof. It is evident that the relation defined above is reflexive. Suppose that $x \leqq y \leqq z$. Then $R_x \leqq R_z$ and $x = fy$, $y = gz$ for some $f \in E(R_x)$, $g \in E(R_y)$. Since $f \in E(gS)$ we have $gf = f$, $fg \in E(R_x)$ and $x = fy = (fg)z$. Therefore $x \leqq z$. If $x \leqq y \leqq x$, then $R_x = R_y$ and so the equality $x = fy$ for some $f \in E(R_x)$ implies $x = y$. This proves that $\leqq$ is a partial order on S. The last statement is obvious from (1.2) and the definition of the relation ω on $E(S)$ (cf. (**7**)).

The relation defined by (1.2) on a regular semigroup S is called the *natural partial order* on S. Notice that when S is an inverse semigroup this relation coincides with the usual partial order on S (cf. (**1**, §7.1)). Though our definition is typically one-sided, the relation $\leqq$ is in fact self-dual as is clear from the following proposition.

Proposition 1.2. *Let x and y belong to a regular semigroup S. Then the following statements are equivalent.*

(a) $x \leqq y$.
(b) *For every $f \in E(R_y)$ there exists $e \in E(R_x)$ such that $e\omega f$ and $x = ey$.*
(c) *For every $f' \in E(L_y)$ there exists $e' \in E(L_x)$ such that $e'\omega f'$ and $x = ye'$.*
(d) *$H_x \leqq H_y$ and for some [for all] $y' \in i(y)$, $xy'x = x$.*

Proof. (a) $\Rightarrow$ (b). Let $e \in E(R_x)$ be the idempotent such that $x = ey$ and choose $f \in E(R_y)$. Since $R_e = R_x \leqq R_y = R_f$ we have $e\mathcal{R}e_1 = ef\omega f$ and $e_1 y = efy = ey = x$.

(b) $\Rightarrow$ (c). Choose $f' \in E(L_y)$ and $y' \in i(y) \cap R_{f'}$. By (b) there exists $e \in E(R_x)$ such that $e\omega yy'$ and $x = ey$. If $e' = y'ey$ then $e' \in (L_x)$, $e'\omega y'y = f'$ and $ye' = (yy')ey = ey = x$.

(c) $\Rightarrow$ (d). Let $y' \in i(y)$. By (c) there exists $e' \in E(L_x)$ such that $e'\omega y'y$ and $x = ye'$. Then $x = ye' = ye'y'y \in ySy$ and so $H_x \leqq H_y$. Also $xy'x = xy'ye' = xe' = x$.

(d) $\Rightarrow$ (a). From $H_x \leqq H_y$ it follows that $R_x \leqq R_y$. Assume that $xy'x = x$ for some $y' \in i(y)$. Then $e = xy' \in E(R_x)$ and $x \in ySy$. Since $y'y$ is a right identity of every element in ySy it follows that $ey = xy'y = x$; that is $x \leqq y$.

Corollary 1.3. *Let S be a regular semigroup. Then we have the following.*

(a) *$e \in E(S)$, $x \in S$ and $x \leqq e$ implies $x \in E(S)$.*
(b) *$x, y \in S$, $x\mathcal{R}y$ and $x \leqq y$ implies $x = y$.*
(c) *$x_i \leqq y$, $i = 1, 2$ and $H_{x_1} \leqq H_{x_2}$ implies $x_1 \leqq x_2$.*
(d) *Let $e\omega f$. Then for each $(y, y') \in R_f \times L_f$ with $y' \in i(y)$, there exists a unique pair $(x, x') \in R_e \times L_e$ with $x' \in i(x)$ such that $x'x = y'ey$, $x \leqq y$ and $x' \leqq y'$.*

Proof. Statements (a) and (b) are obvious from the definition. (c) If $y' \in i(y)$. by Proposition 1.2 (b), there exist $e_i \in E(R_{x_i})$ such that $e_i\omega yy'$ and $x_i = e_i y$, $i = 1, 2$. Then $x_2' = y'e_2 \in i(x_2)$ and since $H_{x_1} \leqq H_{x_2}$ we have $e_1\omega^r e_2$. Therefore $x_1 x_2' x_1 = e_1 yy'e_2 e_1 y = e_1 y = x_1$ and so, by Proposition 1.2(d), $x_1 \leqq x_2$.

(d) Let $e\omega f$ and $(y, y') \in R_f \times L_f$ with $y' \in i(y)$. Then the pair $(ey, y'e) \in R_e \times L_e$ satisfies the required conditions. To prove the uniqueness assume that $(x, x') \in R_e \times L_e$ also satisfies these conditions. Then $xx' = e = eyy'e$, $x'x = y'ey$ and hence $x\mathcal{H}ey$ and $x'\mathcal{H}y'e$. By (c) we conclude that $x = ey$ and $x' = y'e$.

A non-zero element of a regular semigroup S is *primitive* if it is minimal among the non-zero elements of S. If S does not have zero then a primitive element of S is a minimal element of S. The fact that the restriction of the natural order of S to $E(S)$ is the relation ω implies that, for idempotents, the foregoing definition is equivalent to the usual definition (cf. (**1**)). Moreover, if x is a primitive element of S, then it follows from Proposition 1.2 that every element of D_x is also primitive. Recall that a primitive semigroup is a regular semigroup in which every non-zero idempotent is primitive and that a completely simple semigroup is a primitive semigroup without zero. Thus we have the following:

Theorem 1.4. *A regular semigroup S is primitive if and only if every non-zero element of S is primitive. Therefore, a regular semigroup S without zero is completely simple if and only if the natural partial order on S is the identity relation.*

Theorem 1.5. *Let T be a regular subsemigroup of a regular semigroup S. Then the natural partial order on T is the restriction of the natural partial order on S to T.*

Proof. Let $x, y \in T$. If $x \leqq y$ in T then it is clear that $x \leqq y$ in S also. Conversely assume that $x \leqq y$ in S. Since T is regular, y has an inverse y' in T. Let $f = yy'$ and $e = xy'$. Then $f, e \in E(T)$, $e\omega f$ and $x = ey$. Now $e\mathcal{R}x$ in S and since Green's relation $\mathcal{R}$ in T is the restriction of the corresponding relation in S to T, it follows that $e\mathcal{R}x$ in T. Therefore by Proposition 1.2(b), $x \leqq y$ in T.

By the trace of a regular semigroup S we mean the partial algebra $S(*)$ defined as follows:

$$x^*y = \begin{cases} xy & \text{if} \quad xy \in R_x \cap L_y; \\ \text{undefined otherwise.} \end{cases} \tag{1.3}$$

The following result shows the relation between the natural partial order on S and the global structure of S. The reader may compare this with Lemma 1 of (**4**).

Theorem 1.6. *Let $x_1, x_2, \ldots, x_n$ be elements of a regular semigroup S and $x = x_1x_2 \ldots x_n$. Then there exist $y_1, y_2, \ldots, y_n \in S$ such that*

(1) $y_i \leqq x_i$, $i = 1, 2, \ldots, n$;
(2) $x = y_1{}^*y_2{}^*\ldots{}^*y_n$.

If the product $x_1{}^*x_2{}^*\ldots{}^*x_n$ exists in $S(*)$ and if $y_1, y_2, \ldots, y_n$ are elements in S satisfying (1) *and* (2), *then $x_i = y_i$, $i = 1, 2, \ldots, n$.*

Proof. The proof is by induction on the integer n. If $n = 2$, then for $e \in E(L_{x_1})$, $f \in E(R_{x_2})$ and $h \in \mathcal{S}(e, f)$, by Proposition 1.1 and Theorem 2 of (**7**), we have $x_1h \leqq x_1$, $hx_2 \leqq x_2$ and $x_1x_2 = (x_1h)^*(hx_2)$. Further, if $x_1{}^*x_2$ exists in $S(*)$ and if y_1 and y_2 satisfy conditions (1) and (2), then by Corollary 1.3(b) and its dual $x_1 = y_1$ and $x_2 = y_2$.

Now suppose that the result holds for all integers less than or equal to n and consider $x = x_0x_1 \ldots x_n$ where the product $x_1{}^*x_2{}^*\ldots{}^*x_n$ exists in $S(*)$. Then there exist $e_0, e_1, \ldots, e_n \in E(S)$ such that $x_i \in R_{e_{i-1}} \cap L_{e_i}$, $i = 1, 2, \ldots, n$. Choose $e \in E(L_{x_0})$, $h \in \mathcal{S}(e, e_0)$ and define $h_0 = he_0$ and $h_i = x_i'h_{i-1}x_i$, $i = 1, 2, \ldots, n$ where x_i' is the inverse of x_i in $L_{e_{i-1}} \cap R_{e_i}$. Then $h_i\omega e_i$ for all i and $y_i = h_{i-1}x_i \leqq x_i$, $h_i \in L_{y_i} \cap R_{y_{i+1}}$ for $i = 1, \ldots, n-1$. From this it follows that, if $1 \leqq i \leqq j$, then

$$y_i{}^* \ldots {}^* y_j = h_{i-1}x_i \ldots x_j = x_i \ldots x_jh_j \leqq x_i \ldots x_j.$$

Now if $y_0 = x_0 h$ then $y_0 \leqq x_0$ and by Theorem 2 of (**7**) we have

$$\begin{aligned} x_0 x_1 \dots x_n &= y_0{}^* h x_1 \dots x_n \\ &= y_0{}^* h_0 x_1 \dots x_n \\ &= y_0{}^* y_1{}^* \dots {}^* y_n. \end{aligned}$$

If $x_0{}^* \dots {}^* x_n$ exists in $S(*)$ and if $y_0, \dots, y_n$ satisfy (1) and (2) then

$$x_0{}^* \dots {}^* x_n = x_0{}^* z = y_0{}^* u$$

where $z = x_1{}^* \dots {}^* x_n$ and $u = y_1{}^* \dots {}^* y_n$. But $y_0 \leqq x_0$ and $u \leqq z$ and so we have $y_0 = x_0$, $u = z$. By induction it now follows that $y_i = x_i$ for $i = 1, 2, \dots, n$.

Remark. By arguments similar to the foregoing we can prove the following statement.

Let S be a regular semigroup and for $i = 1, 2, \dots, n$, let $x_i \in S$ and $x'_i \in i(x_i)$. Then there exist $y_i \in S$ and $y'_i \in i(y_i)$ such that

(1)″ $y_i \leqq x_i$ *and* $y'_i \leqq x'_i$ *for* $i = 1, 2, \dots, n$; *and*
(2)″ $x_1 x_2 \dots x_n = y_1{}^* \dots {}^* y_n$, $y'_n{}^* \dots {}^* y'_1 \in i(x_1 x_2 \dots x_n)$.

We shall say that a mapping ϕ of a quasi-ordered set X into a quasi-ordered set Y *reflects quasi-orders* if for all $y, y' \in X\phi$ with $y' \leqq y$ and $x \in X$ with $x\phi = y$ there exists x' such that $x' \leqq x$ and $x'\phi = y'$. An important property of homomorphisms of regular semigroups is that they preserve and reflect natural partial orders. To prove this we need the following lemma which is an immediate consequence of Theorem 1 of (**7**).

Lemma 1.7. *Let S be a regular semigroup and $e, e', f, f' \in E(S)$. Then we have the following.*

(i) *If $e\omega^r f$ then*

$$\mathcal{S}(e, f) = \omega^r(f) \cap E(L_e), \quad \mathcal{S}(f, e) = \omega(f) \cap E(R_e).$$

Dually if *$e\omega^l f$ then*

$$\mathcal{S}(e, f) = \omega(f) \cap E(L_e), \quad \mathcal{S}(f, e) = \omega^l(f) \cap E(R_e).$$

(ii) *If $e \mathcal{L} e'$ and $f \mathcal{R} f'$ then $\mathcal{S}(e, f) = \mathcal{S}(e', f')$.*

Theorem 1.8. *Let $\phi: S \to S'$ be a homomorphism of regular semigroups. Then ϕ preserves and reflects natural partial orders of S and S′.*

Proof. Since ϕ preserves the relation ω and products, it follows that it preserves the natural partial order.

Now consider $e', f' \in E(S\phi)$ with $e'\omega f'$. Then there exist $e, f \in E(S)$ such that $e\phi = e'$ and $f\phi = f'$. Choose $h \in \mathcal{S}(e, f)$ and $g \in \mathcal{S}(f, e)$. By Theorem 5 of (**7**), $h\phi \in \mathcal{S}(e', f')$ and $g\phi \in \mathcal{S}(f', e')$. Since $e'\omega f'$, by Lemma 1.7 (i), $e' \mathcal{L} h\phi f'$. Also $(hf)\phi = h\phi f' = h\phi$. Dually, $e' \mathcal{R} g\phi \omega f'$ and $(fg)\phi = g\phi$. Hence by Lemma 1.7 (ii), $\mathcal{S}((hf)\phi, (fg)\phi) = \mathcal{S}(e', e') = \{e'\}$. Hence if $k \in \mathcal{S}(hf, fg)$ then $k\omega f$ and by Theorem 5 of (**7**), $k\phi = e'$. Now let $u, v \in S\phi$ with $u \leqq v$ and let $y \in S$ with $y\phi = v$. If $f \in E(R_y)$ then $f\phi = f' \in E(R_v)$ and so by Proposition

1.2(b), there exists $e' \in E(R_u)$ such that $e'\omega f'$ and $u = e'v$. By the result proved above, there exists $e \in \omega(f)$ such that $e\phi = e'$. Then $x = ey \leqq y$ and $x\phi \leqq e'v = u$.

From Theorems 1.5 and 1.8 we have the following.

Corollary 1.9. *Let $\phi: S \to S'$ be a homomorphism of regular semigroups and $u_1, u_2, \ldots, u_n$ be elements in $S\phi$ such that $u_1 * u_2 \ldots * u_n$ exists in $S'(*)$. Then for every choice of elements $x_i \in S$ such that $x_i\phi = u_i$, $i = 1, \ldots, n$ we can find $y_i \in S$ such that $y_i \leqq x_i$, $y_i\phi = u_i$ and $y_1 * y_2 * \ldots * y_n$ exists in $S(*)$.*

2. Green's relations and the natural partial order

Let X be a partially ordered set and ρ be an equivalence relation on X. We say that ρ is *reflecting* if for all $x, y, z \in X$ with $x \leqq y\rho z$ there exists $y' \in X$ such that $x\rho y' \leqq z$. A reflecting equivalence relation is *compatible* with the partial order on X if, for all $x, y \in X$ with $x \leqq y$, $x \leqq u \leqq y$ implies $u\rho x$. In particular, if no two distinct ρ-related elements are comparable with respect to $\leqq$ then ρ is clearly compatible with $\leqq$; in this case we say that ρ is *strictly compatible* with $\leqq$.

Suppose that ρ is a reflecting equivalence relation on X. Then the relation

$$\begin{gathered}\rho(x) \leqq \rho(y) \quad \text{if and only if, for every } y' \in \rho(y) \\ \text{there exists } x' \in \rho(x) \text{ such that } x' \leqq y',\end{gathered} \tag{2.1}$$

is a quasi-order on X/ρ and the canonical map $\rho^*: X \to X/\rho$ is order preserving and reflecting. On the other hand, if $f: X \to Y$ is an order preserving and reflecting map of a partially ordered set X to a quasi-ordered set Y, then one easily checks that

$$\ker f = \{(x, y) : xf = yf\}$$

is a reflecting equivalence relation on X. Further, ρ is compatible with $\leqq$ if and only if the relation defined by (2.1) is a partial order.

Lemma 2.1. *Let X be a partially ordered set and ρ be a reflecting equivalence relation on X. Then the relation*

$$\bar{\rho} = \{(x, y) : \text{ there exist } x', y' \text{ in } X \text{ with } x\rho x' \leqq y,\ y\rho y' \leqq x\}$$

is the finest compatible equivalence relation on X containing ρ.

Proof. It is clear that $\bar{\rho}$ is reflexive and symmetric. To prove transitivity assume that $x\bar{\rho}y$ and $y\bar{\rho}z$. Then there exist x' and y' such that $x\rho x' \leqq y\rho y' \leqq z$. Since ρ is reflecting there exists x'' such that $x\rho x'\rho x'' \leqq y' \leqq z$. Similarly there exists z'' such that $z\rho z'' \leqq x$ and hence $x\bar{\rho}z$.

Suppose that $x \leqq y\bar{\rho}z$. Then there exists z' such that $y\rho z' \leqq z$. Since ρ is reflecting, we can find x' with $x\rho x' \leqq z' \leqq z$. Therefore $x\bar{\rho}x' \leqq z$ and so $\bar{\rho}$ is reflecting. To prove that $\bar{\rho}$ is compatible assume that $x\bar{\rho}y$ and $x \leqq u \leqq y$. Then $u \leqq y\bar{\rho}x$ and so there exists u' with $u\bar{\rho}u' \leqq x$. Since $x \leqq u$, it follows that $u\bar{\rho}x$.

Finally let σ be a compatible equivalence relation containing ρ. If $x\bar{\rho}y$ then there exists x' such that $x\rho x' \leqq y$ and so $x'\bar{\rho}y$. This implies that there exists y' with $y\rho y' \leqq x' \leqq y$. Hence

$$\sigma(y) = \sigma(y') \leqq \sigma(x') = \sigma(x) \leqq \sigma(y).$$

Therefore $x\sigma y$.

Let S be a regular semigroup. By a reflecting or compatible equivalence relation on S we mean a relation having the corresponding property with respect to the natural order on S. Theorem 1.8 implies that every congruence on S is compatible. Similarly Proposition 1.2, Corollary 1.3(b) and its dual shows that Green's relations $\mathcal{R}$ and $\mathcal{L}$ are strictly compatible. But $\mathcal{H}$ is not, in general, reflecting. We prove below that $\mathcal{D}$ is reflecting and that $\bar{\mathcal{D}} = \mathcal{J}$. First we prove the following.

Lemma 2.2. *Let x and y be elements of a regular semigroup S. Then we have the following.*

(1) *If $x \leqq y$ then for every $y' \in D_y$ there exists $x' \in D_x$ such that $x' \leqq y'$. If $x \neq y$, we may choose x' so that $x' \neq y'$. Hence a $\mathcal{D}$-class containing elements x, y such that $x \leqq y$ and $x \neq y$, does not contain minimal elements.*

(2) *$x \in SyS$ if and only if there exists y' such that $x\mathcal{D}y' \leqq y$. Hence if D_y contains minimal elements then $D_y = J_y$ and every element of J_y is minimal.*

Proof. (1) Let $f \in E(R_y)$. Then by Proposition 1.2 (b), there exists $e \in E(R_x)$ such that $e\omega f$ and $x = ey$. Let y' be any other element in D_y and $f' \in E(R_{y'})$. Choose $z \in R_f \cap L_{f'}$, and let z' be its inverse in $L_f \cap R_{f'}$. Then $e' = z'ez \leqq z'z = f'$ and $e' \in D_x$. Hence $x' = e'y' \leqq y'$ and $x' \in D_x$. If $x \neq y$ then by Corollary 1.3(b), $e \neq f$ and so $e' = z'ez \neq z'z = f'$. Consequently $x' \neq y'$. If $D_x = D_y = D$ and $x \neq y$, by the foregoing, for every $y' \in D$ there exists $x' \in D$ such that $x' \leqq y'$ and $x' \neq y'$. Therefore D cannot contain minimal elements.

(2) If $x \in SyS$ then $x = uyv$ for some $u, v \in S$ and by Theorem 1.6 we can find $u' \leqq u$, $v' \leqq v$ and $y' \leqq y$ such that $x = u' * y' * v'$. In particular, u', v', $y' \in D_x$ and so $x\mathcal{D}y' \leqq y$. Conversely if y' exists satisfying this condition, then $y' \in yS$ and so $SxS = Sy'S \subseteqq SyS$. Finally assume that D_y contains minimal elements and that $x \in J_y$. Then there exist y' and y'' such that $x\mathcal{D}y' \leqq y$ and $y\mathcal{D}y'' \leqq y'$. By Statement (1), every element of D_y is minimal and so, $y'' = y' = y$. Therefore $x \in D_y$.

Corollary 2.3. *Let D be a $\mathcal{D}$-class of a regular semigroup S. Then D contains minimal elements if and only if it does not contain a bicyclic semigroup.*

Proof. If D contains a bicyclic semigroup then it is clear that it contains distinct idempotents e and f with $e\omega f$ and so by Lemma 2.2, D does not contain minimal elements. Conversely if D does not contain minimal elements then by Lemma 2.2, D contains distinct idempotents e and f such that $e\omega f$. If $x \in R_e \cap L_f$ and if x' is the inverse in $L_e \cap R_f$ then it is routine to check that the semigroup generated by elements x and x' is a bicyclic semigroup contained in D.

Lemmas 2.1 and 2.2 yield the following.

Theorem 2.4. *Let S be a regular semigroup. Then Green's relation $\mathcal{D}$ is reflecting and $\bar{\mathcal{D}} = \mathcal{J}$. In particular, $\mathcal{D} = \mathcal{J}$ if and only if $\mathcal{D}$ is compatible.*

Corollary 2.5. *A regular semigroup S is $[0-]$ simple if and only if for all $x, y \in S$ $[x, y \in S\backslash\{0\}]$ there exists $x_1 \in S$ such that $x\mathcal{D}x_1 \leqq y$.*

The class of regular semigroups for which $\mathscr{D}$ is strictly compatible is also of interest. If S has this property then no $\mathscr{D}$-class of S contains a bicyclic semigroup by Corollary 2.3. Since any bicyclic subsemigroup of S must be contained in a $\mathscr{D}$-class of S, it follows that S does not contain bicyclic subsemigroups and so S is completely semisimple (cf. [11]). Conversely if S is completely semisimple, then it does not contain bicyclic subsemigroups and so by Corollary 2.3, every $\mathscr{D}$-class of S contains minimal elements. Therefore, $\mathscr{D}$ is strictly compatible. Thus

Theorem 2.6. *A regular semigroup S is completely semisimple if and only if Green's relation $\mathscr{D}$ on S is strictly compatible.*

Corollary 2.7. *A regular semigroup S is a union of groups if and only if S is completely semisimple and every $\mathscr{D}$-class of S is a subsemigroup of S.*

Proof. The "only if" part of the assertion is well-known. If S is completely semisimple and if $\mathscr{D}$-classes of S are subsemigroups then every $\mathscr{D}$-class of S is a bisimple regular subsemigroup of S and by Theorem 1.5, the natural partial order on these semigroups are identity relations. Therefore every $\mathscr{D}$-class of S is completely simple (by Theorem 1.4).

We have already observed that congruences on regular semigroups are compatible with the natural partial order. Those congruences that are strictly compatible may be characterised as follows.

Theorem 2.8. *Let ρ be a congruence on a regular semigroup S. Then ρ is strictly compatible if and only if $\rho(e)$ is a completely simple subsemigroup of S for all $e \in E(S)$.*

Proof. Suppose that ρ is strictly compatible and $e \in E(S)$. *If* $\rho^*: S \to S/\rho$ is the canonical homomorphism, $x \in \rho(e)$, $f \in E(R_x)$ and $g \in E(L_x)$, then $f\rho^* \mathscr{R} e\rho^* \mathscr{L} g\rho^*$ and so, by Lemma 1.5,

$$\mathscr{S}(g\rho^*, f\rho^*) = \mathscr{S}(e\rho^*, e\rho^*) = \{e\rho^*\}.$$

By Theorem 5 of (7), $\mathscr{S}(g, f)\rho^* \subseteqq \mathscr{S}(g\rho^*, f\rho^*)$ and so $\mathscr{S}(g, f) \subseteqq \rho(e)$. Therefore if $h \in \mathscr{S}(g, f)$, then xh, $hx \in \rho(e)$, $xh \leqq x$ and $hx \leqq x$. Since ρ is strictly compatible, this implies that $x = xh \mathscr{L} h \mathscr{R} hx$; that is, $x \in H_h$. In particular, $\rho(e)$ is a regular subsemigroup of S and by Theorem 1.5, the natural partial order on it is the identity relation. Therefore, by Theorem 1.4, $\rho(e)$ is completely simple.

Conversely suppose that $\rho(e)$ is a completely simple subsemigroup for all $e \in E(S)$ and that $(x, y) \in \rho$, $x \leqq y$. If $f \in E(R_y)$ then by Proposition 1.2, there exists $g \in E(R_x)$ such that $g\omega f$ and $x = gy$. Now for some inverse y' of y, $f = yy'$ and $g = gyy' = xy'$. Hence $(f, g) \in \rho$ and since $\rho(f)$ is completely simple, by Theorem 1.4, $f = g$. Therefore $x = y$.

The foregoing theorem in particular implies that if ρ is a proper congruence (that is, congruence which is different from the universal congruence) on a completely 0-simple semigroup S, then for every $e \in E(S)$, $\rho(e)$ is a completely simple subsemigroup of S.

3. Pseudo-inverse semigroups

It is well-known that the natural partial order on an inverse semigroup is compatible with the multiplication of the semigroup (cf. (2, Lemma 7.2)). This is not true, in general, for regular semigroups. This leads to the problem of determining the class of regular semigroups for which this holds. We prove below that it is precisely the class of semigroups studied in (**8**). We call them pseudo-inverse (p-inverse) semigroups.

A p-inverse semigroup S is a regular semigroup such that the bi-ordered set $E(S)$ is a pseudo-semilattice; that is $E(S)$ satisfies the condition that for all $e, f \in E(S)$ there exists $h \in E(S)$ such that

$$\omega^l(e) \cap \omega^r(f) = \omega(h) \tag{3.1}$$

(cf. (**8**, **9**)); this is equivalent to requiring that for all $e \in E(S)$, $\omega(e)$ is a semilattice (cf. (**8**)). Thus we have:

Theorem 3.1. *A regular semigroup S is pseudo-inverse if and only if for every $e \in E(S)$, eSe is an inverse semigroup.*

Since we are not concerned with pseudo-semilattices in this paper, the reader may take the foregoing result as the definition of p-inverse semigroups. Examples of p-inverse semigroups are numerous. Inverse semigroups are obviously p-inverse; completely 0-simple semigroups also belong to this class. Apart from these, several subclasses of the class of p-inverse semigroups such as generalised inverse semigroups (Yamada (**13**)), locally testable regular semigroups (Zalcstein (**14**)), etc. have been studied. For generalised inverse semigroups, Theorem 3.1 is due to Yamada (**13**, Theorem 1).

Theorem 3.2.

(a) *Every regular subsemigroup of a p-inverse semigroup is p-inverse.*
(b) *Every homomorphic image of a p-inverse semigroup is p-inverse.*
(c) *The direct product of a family of p-inverse semigroups is p-inverse.*

Proof. The statement (a) is clear.

(b) Let $\phi: S \to S'$ be a homomorphism of a p-inverse semigroup S onto a semigroup S'. Then by (**2**, Lemma 7.35) and (**7**, Theorem 5) S' is regular and $E(S)\phi = E(S\phi) = E(S')$. Let $e' \in E(S')$ and $e \in E(S)$ with $e\phi = e'$. Then it is clear that $(eSe)\phi \subseteq e'S'e'$. If $x' \in e'S'e'$ and if $x\phi = x'$ for some $x \in S$, then $(exe)\phi = e'x'e' = x'$ and so, $\epsilon Se)\phi = e'S'e'$. Since eSe is an inverse semigroup, so is $e'S'e'$ by (**1**, Theorem 7.36)). Thus S' is p-inverse.

(c) Let $S = \times S_i$ be the direct product of p-inverse semigroups S_i. If $P_i: S \to S_i$ is the i-th projection, then $e \in E(S)$ if and only if $e_i = eP_i \in E(S_i)$ for all i. Further for $e, f \in E(S)$, $e\omega f$ if and only if $e_i \omega f_i$ for all i. Therefore, $\omega(e)$ is the direct product of the semilattices $\omega(e_i)$ and so is a semilattice. Hence for all $e \in E(S)$, eSe is an inverse subsemigroup of S.

The following theorem gives several characterisations of p-inverse semigroups in terms of its natural partial order (cf. statements (b), (c) and (d)). All these statements are well-known properties of inverse semigroups (cf. (**1**, 7.1), (**10**)). These properties are used in (**8**) to obtain a structure theorem for p-inverse semigroups which is analogous to Schein's theorem for inverse semigroups (cf. (**10**)).

Theorem 3.3. *The following conditions on a regular semigroup S are equivalent.*

(a) *S is p-inverse.*

(b) *If $x \leqq y$ then for every $(y_1, y_2) \in L_y \times R_y$ there exists a unique pair $(x_1, x_2) \in L_x \times R_x$ such that $x_i \leqq y_i$, $i = 1, 2$.*

(c) *$x, y, u, v \in S$, $x \leqq u$, $y \leqq v$ implies $xy \leqq uv$.*
(d) *If $y \in S$, $y' \in i(y)$ and $x \leqq y$ then there exists a unique $x' \in i(x)$ such that $x' \leqq y'$.*

Proof.

(a) $\Rightarrow$ (b). Let $x \leqq y$. Then by Proposition 1.2 (b), there exist $f \in E(R_y)$, $e \in E(R_x)$ such that $e\omega f$ and $x = ey$. If $y_1 \in R_y$ then by the same result it follows that $ey_1 \in R_x$ and $ey_1 \leqq y_1$. Now if $x_1 \in R_x$ is another element with $x_1 \leqq y_1$, then there exists $e_1 \in E(R_x)$ with $e_1 \omega f$ and $x_1 = e_1 y_1$. Then $e, e_1 \in \omega(f)$ and $e \mathcal{R} e_1$. *Since* $\omega(f)$ is a semilattice, we have $e = e_1 e = ee_1 = e_1$ and so, $x_1 = ey_1$. Dually it can be shown that for every $y_2 \in L_y$ there exists a unique $x_2 \in L_x$ such that $x_2 \leqq y_2$.

(b) $\Rightarrow$ (c). Suppose that $e \in E(L_u)$ and $f \in E(R_v)$. Then by Proposition 1.2 (b) and (c) there exist $e' \in \omega(e) \cap E(L_x)$ and $f' \in \omega(f) \cap E(R_y)$ such that $x = ue'$ and $y = f'v$. Let $h \in \mathcal{S}(e, f)$ and $k \in \mathcal{S}(e', f')$. Then $k \in w^l(e) \cap \omega^r(f)$ and so by the definition of sandwich sets (cf. (**7**)), $ek\omega^r eh$. Hence $(ek).(eh) \mathcal{R} ek$. Since $(ek).(eh)$, $ek \in \omega(e)$, by (b), $(ek).(eh) = ek$ and so $ek\omega eh$. Also, $e'k$, $ek \in \omega(e)$ and $e'k \mathcal{L} ek$ and so, again by (b), $e'k = ek$. Therefore

$$xk = u(e'k) = u(ek) = uk = u(eh).(ek) = (uh).(ek).$$

Hence $xk \leqq uh$. *Dually*, $ky = kv \leqq hv$ and by Theorem 1 of (**7**), $xy = (xk).(ky) = (uk).(kv) = ukv$. Choose $u' \in i(u)$ and $v' \in i(v)$ with $u'u = e$ and $vv' = f$. Then it is routine to check that $uhu' \in E(R_{uv})$, $uku' \in E(R_{xy})$ and $uku' \omega uhu'$. Further, $(uku').uv = ukv = xy$. Hence $xy \leqq uv$.

(c) $\Rightarrow$ (d). Let $y' \in i(y)$ and $x \leqq y$. Then by Corollary 1.3(d) it follows that there exists $x' \in i(x)$ with $x' \leqq y'$. To prove that x' is unique, assume that $x'' \in i(x)$ and $x'' \leqq y'$. Then by (c), $e = xx'\omega yy' = f$ and $e' = xx''\omega f$. Also e, $e' \in \omega(f) \cap E(R_x)$ and so $e' = ee' \leqq ef = e$; that is, $e' = e$. Therefore $x' \mathcal{L} x''$. Dually $x' \mathcal{R} x''$ and so $x' = x''$.

(d) $\Rightarrow$ (a). Choose $e \in E(S)$ and suppose that $f, f' \in \omega(e)$ with $f \mathcal{R} f'$. Then $f, f' \in i(f)$, $e \in i(e)$, $f \leqq e$ and $f' \leqq e$. Hence by (d), $f = f'$. Thus the restriction of the relation $\mathcal{R}$ to $\omega(e)$ is the identity relation. Dually the restriction of the relation $\mathcal{L}$ to $\omega(e)$ is also the identity relation. Therefore $\omega(e)$ is a semilattice (cf. (**7**)). Since $E(eSe) = \omega(e)$, we conclude by Theorem 3.1 that S is p-inverse.

4. Primitive congruences on regular semigroups

As in (**1**), we write $S = S^0$ to mean that the semigroup S has a zero.

Let $S = S^0$ be a regular semigroup that is categorical at zero. Define the relation $\beta(S)$ on S as follows:

$$\beta(S) = \{(x, y)\colon \text{ for some } \ z \in S \backslash \{0\}, z \leqq x \ \text{ and } \ z \leqq y\} \cup \{(0, 0)\}$$

If S is an inverse semigroup, then this relation is the finest 0-restricted primitive congruence on S. We show below that, in general, this congruence is the congruence generated by the relation $\beta(S)$. For alternate forms of this congruence the reader is referred to (**3**) and (**5**).

Theorem 4.1. *Let $S = S^0$ be a regular semigroup which is categorical at zero. Then the congruence β^c generated by $\beta = \beta(S)$ is the finest* 0*-restricted primitive congruence on S.*

Proof. Let σ be a 0-restricted primitive congruence on S and $(x, y) \in \beta$. Then $x=0$ if and only if $y=0$ so that, if $x=0$, then clearly $(x, y) \in \sigma$. If $x \neq 0$, then $y \neq 0$ and there exists $z \neq 0$ such that $z \leqq x$ and $z \leqq y$. Since is 0-restricted, $z \neq 0$ implies $z\sigma^* \neq 0$ where σ^* denotes the canonical homorphism of S onto S/σ. By Theorem 1.8, $z\sigma^* \leqq x\sigma^*$ and $z\sigma^* \leqq y\sigma^*$ and by Theorem 1.4 we conclude that $x\sigma^* = z\sigma^* = y\sigma^*$; that is, $(x, y) \in \sigma$. Therefore $\beta \subseteqq \sigma$ and hence $\beta^c \subseteqq \sigma$. In particular, β^c is 0-restricted. To prove that S/β^c is primitive, consider $\bar{x}, \bar{y} \in S/\beta^c$ where $\bar{x}$ denotes the canonical image of $x \in S$ in S/β^c, such that $\bar{x} \leqq \bar{y}$ and $\bar{y} \neq 0$. By Theorem 1.8 we may assume that $x \leqq y$. If $x=0$, then clearly $\bar{x}=0$. If $x \neq 0$ then (x, y) and so $\bar{x}=\bar{y}$. Therefore S/β^c is primitive.

As we have already observed, for an inverse semigroup $S=S^0$ that is categorical at zero, we have $\beta(S)=\beta(S)^c$. We proceed to show that this holds for a wider class of regular semigroups. In what follows, by a *directed* subset if S we mean a subset X such that for all x, $y \in X$ there exists $z \in X$ such that $z \leqq y$, and $z \leqq x$.

Theorem 4.2. *For a regular semigroup $S=S^0$ that is categorical at zero, the following statements are equivalent.*

(a) *For every $e \in E(S)\backslash\{0\}$, $\omega(e)\backslash\{0\}$ is directed.*
(b) *$\beta(S)$ is an equivalence relation.*
(c) *$\beta(S)=\beta(S)^c$.*

Proof.

(a) $\Rightarrow$ (b). Assume that $(x, y), (y, z) \in \beta$. Then either $x=y=z=0$ or none of them is zero. In the former case $(x, z) \in \beta$. In the latter case there exist u_1, $u_2 S\backslash\{0\}$ such that $u_1 \leqq x$, $u_1 \leqq y$, $u_2 \leqq y$ and $u_2 \leqq z$. Choose $f \in E(R_y)$. Then there exist $e_i \in E(R_{u_i}) \cap \omega(f)$ such that $u_i = e_i y$, $i=1, 2$. Since $e_i \mathscr{R} u_i$ and $u_i \neq 0$, it follows that $e_i \neq 0$. Therefore there exists $g \in \omega(f)\backslash\{0\}$ such that $g\omega e_i$, $i=1,2$. Then $g\mathscr{R}gy$ and so, $gy \neq 0$ and $gy = ge_1 y = gu_1 \leqq u_1 \leqq x$, $gy = ge_2 y \leqq u_2 \leqq z$. This implies that $(x, z) \in \beta$ and so β is transitive. Since β is obviously reflective and symmetric, it is an equivalence relation.

(b) $\Rightarrow$ (c). It must be shown that β is compatible with multiplication. To this end, first consider $x, y, c \in S$ with $x \leqq y$ and $x \neq 0$. Choose $y' \in i(y)$ and let $f = yy'$ and $f' = y'y$. Then by Proposition 1.2(b) and (c), there exists $e \in \omega(f) \cap E(R_x)$ such that $x = ey = ye'$ where $e' = y'ey$. If $cy=0$ then $cx = cfx = (cy)y'x = 0$. Conversely if $cx=0$, then $cfx=0$. Since S is categorical at zero and since $fx = x \neq 0$, it follows that $cyy' = cf = 0$. Since $yy' = f \neq 0$, we have $cy=0$. Therefore when either cx or cy is zero, the other is zero and $(cx, cy) \in \beta$. Next suppose that $cx \neq 0 \neq cy$. Let $g \in E(L_c)$, $h \in \mathscr{S}(g, f)$ and $k \in \mathscr{S}(g, e)$. Then $h' = y'hy \in E(L_{cy}) \cap \omega(f')$, and $k' = y'ky \in E(L_{cx}) \cap \omega(f')$. Therefore e', h' and k' are non-zero idempotents in $\omega(f')$. Now every element of $\omega(f')\backslash\{0\}$ is β-related to f' and so $e'\beta h'\beta k'$ by (b). Hence it follows that $\omega(e') \cap \omega(h') \cap \omega(k')\backslash\{0\} \neq \square$. If $k \in \omega(e') \cap \omega(h') \cap \omega(k')\backslash\{0\}$ then $z = cyk = cye'k = cxk \leqq cx$ and $z \leqq cy$. Since $z\mathscr{L}k$, $z \neq 0$ and hence $(cx, cy) \in \beta$.

Now consider any $(u, v) \in \beta$ and $c \in S$. If $u=v=0$, then clearly $(cu, cv) \in \beta$. Otherwise there exists $z \neq 0$ such that $z \leqq u$ and $z \leqq v$. Then by the foregoing (cz, cu), $(cz, cv) \in \beta$. Since β is an equivalence relation, it follows that $(cu, cv) \in \beta$. In a similar way, it can be shown that $(uc, vc) \in \beta$. Therefore β is a congruence.

(2) $\Rightarrow$ (a). Assume that β is a congruence, $e \in E(S)\backslash\{0\}$ and $f, g \in \omega(e)\backslash\{0\}$. Then

https://doi.org/10.1017/S0013091500003801 Published online by Cambridge University Press

$(f, e), (e, g) \in \beta$ and so $(f, g) \in \beta$. This implies that there exists $z \in S\backslash\{0\}$ such that $z \leqq f$ and $z \leqq g$. By Corollary 1.3(a), $z \in E(S)$ and so $\omega(e)\backslash\{0\}$ is directed.

Corollary 4.3. *Let $S = S^0$ be a p-inverse semigroup that is categorical at zero. Then $\beta(S)$ is the finest 0-restricted primitive congruence on S.*

Proof. Let $e \in E(S)\backslash\{0\}$ and $f, g \in \omega(e)$. If $fg = 0$, then $feg = 0$ and since S is categorical at zero, either $f = fe = 0$ or $g = eg = 0$. Since $\omega(e)$ is a semilattice, this implies that $\omega(e)\backslash\{0\}$ is directed. Therefore the result follows from Theorem 4.2.

Let S be any regular semigroup (without zero). Then $S^0 = S \cup \{0\}$ is clearly categorical at zero and the set of non-zero elements of S^0 forms a subsemigroup. Therefore, the set of non-zero elements of $S^0/\beta(S^0)^c$ forms a completely simple subsemigroup and the restriction of the congruence $\beta(S^0)^c$ to S is the finest completely simple congruence on S. Thus as a corollary to Theorem 4.1 and 4.2 we have

Theorem 4.4. *Let S be a regular semigroup and define*

$$\rho(S) = \{(x, y)\colon \text{ For some } z \in S,\ z \leqq x,\ z \leqq y\}. \tag{4.2}$$

Then ρ^c is the finest congruence on S such that S/ρ^c is completely simple. $\rho = \rho^c$ if and only if every ω-ideal of $E(S)$ is directed. In particular, for a p-inverse semigroup S we have $\rho = \rho^c$.

It may be noted that Theorem 1 of **(6)** (see also **(12)**) is the specialisation of the foregoing theorem to inverse semigroups.

Acknowledgement. The author expresses his gratitude to Prof. B. M. Schein, whose comments and suggestions have led to a significant improvement in the presentation of this paper.

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