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EVALUATION OF THE ACOUSTIC IMPEDANCE OF A SCREEN

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Abstract

A direct numerical computation is provided for the impedance of a screen consisting of a regular array of slits in a plane wall. The problem is solved within the framework of oscillatory Stokes flow, and results presented as a function of porosity, frequency and viscosity.

1. Introduction

The acoustic impedance of a thin permeable barrier is (Morse and Ingard [5], p. 259) a measure of the relationship between the pressure difference across the barrier and the air velocity through the barrier. Specifically, if in a time-sinusoidal flow of radian frequency σ, we have

$$\Delta p = \Gamma . V, \tag{1.1}$$

where the pressure jump is $\mathscr{R}(\Delta p\, e^{i\sigma t})$ and the normal velocity is $\mathscr{R}(Ve^{i\sigma t})$, we take Γ as the acoustic impedance. The quantities Δp, V and Γ are all complex-valued, and in general we must expect Γ to depend on the frequency σ. The real part of Γ represents dissipative or resistive effects, and the imaginary part inertial or reactive effects.

In acoustic applications, one would normally expect to have to determine Γ experimentally for any given permeable barrier. The actual geometrical fine detail of the pores through the barrier is crucial to the value of Γ, whose resistive part depends on the viscous dissipation through the pores, as in the Poiseuille or Darcy laws, and whose reactive part depends on the inertia of the oscillating columns of air in the pores. However, for sufficiently idealized pore geometry, it is possible to compute Γ theoretically, and the hope is that such computations as are presented here may have value in clarifying the dependence of the impedance on quantities such as porosity, frequency and viscosity.

It is important to note that the problem of computation of Γ is *not* a problem in acoustics, but rather one in *oscillatory low-Reynolds' number* flow of a *viscous incompressible* fluid. To neglect compressibility of the air in the neighbourhood of the pores, we only need to assume that the pore dimensions are very much smaller than the acoustic wavelength, an assumption comfortably met in most practical applications. Thus although the suggested application is to acoustics, no further attention will be paid to air compressibility, and we take the air density ρ as a constant.

Conversely, we assume that the viscosity of the air is important, if at all, *only* in the neighbourhood of the pores. We assume that the velocity amplitude of the acoustic fluctuations in the pores is small enough for linearization of the Navier–Stokes equations; that is, that the Reynolds number Va/ν is small, where a is a typical pore dimension and ν the kinematic viscosity. Under this approximation, the velocity scale V is itself not dynamically significant, all flow quantities being simply proportional to V in a linear system. The only non-dimensional parameter left in the local flow is therefore the non-dimensional frequency

$$\beta = \sigma a^2/\nu, \tag{1.2}$$

and our aim is to compute Γ as a function of β.

It is of course possible that viscosity is unimportant even in the neighbourhood of the pores. This corresponds to the inviscid or high-frequency limit $\beta \to \infty$. It is clear physically that in that limit all dissipative effects disappear, and the impedance is solely of an inertial nature. Hence Γ is a pure imaginary number, expressible in the form

$$\Gamma = -2i\sigma\rho a C, \tag{1.3}$$

where C is a non-dimensional blockage coefficient, a real number independent of pore size and frequency, and dependent only on the relative geometry of the pore arrangement. For example, if the "pores" consist of a regular array of slits of width $2a$ with centre separation $2b$, we have (for example, Tuck, [7], p. 116)

$$C = -(2/\pi\gamma)\log\sin(\tfrac{1}{2}\pi\gamma) = C(\infty, \gamma) \tag{1.4}$$

where

$$\gamma = a/b. \tag{1.5}$$

As γ decreases from unity (that is, no wall) to zero (that is, no pores), the blockage coefficient C increases monotonically from zero to plus infinity.

If $\beta \neq \infty$, we may continue to use (1.3) to define a non-dimensional impedance or blockage coefficient C, but now C is no longer a real number, and depends on frequency and viscosity via the parameter β, as well as implicitly on relative geometric parameters such as γ. Our aim in the present article is to compute $C(\beta, \gamma)$ for an array of slits, generalizing (1.4) to finite β.

If we introduce suitable planes of symmetry, we can confine attention to a "flow cell" as indicated in Fig. 1. The bottom boundary $y = 0$, $0 < x < b$ of the cell is divided into a portion $0 < x < a$ representing half of a slit, the remainder $a < x < b$ being half of a solid barrier. The side boundary $x = 0$ is a plane of symmetry through the centre of a slit and that at $x = b$ is through the centre of a barrier. By superimposing image replicates of this cell we obtain the infinite array of slits, as above. The wall thickness is taken as infinitesimal, and the slits as infinite in length, so that the flow is two-dimensional.

Although the screen geometry described above is highly idealized, it may provide a model for real permeable walls, even where the pores have a random, rather than regular character. For example, the quantity γ in our model measures the porosity, that is, the ratio between pore area and total wall area. Our results are presented as a function of this ratio, and may have relevance to randomly porous screens, whose *mean* porosity is taken as the value of γ.

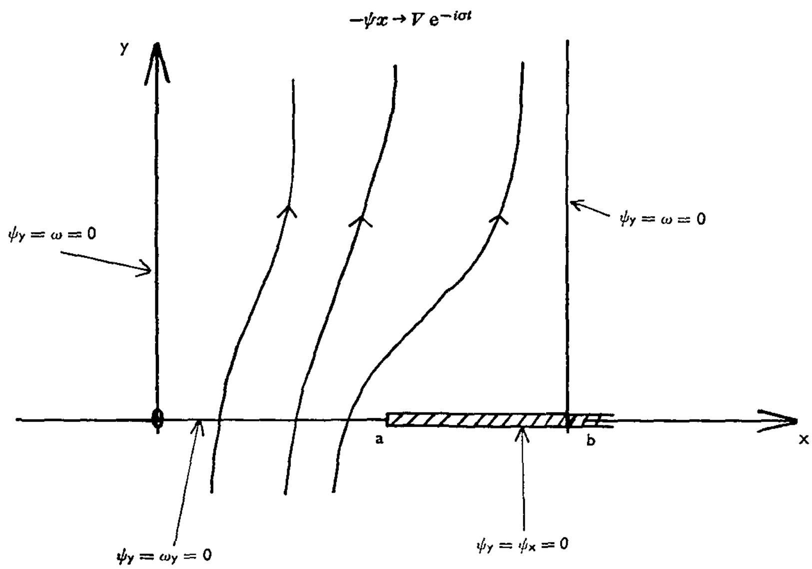

Fig. 1. Schematic drawing of an elementary flow cell.

The computations show that the inviscid approximation (1.4) is accurate for β greater than about 100. However, the magnitude of the blockage coefficient C only increases significantly above the inviscid value for β values less than about 10, as the development of a thick boundary layer tends to close off the slits. Good agreement is also demonstrated with a small-slit ($\gamma \to 0$) result of Macaskill [3]. This approximate theory is based on the assumption that each slit is alone on an otherwise impermeable wall. Nevertheless, there is good agreement with the present computations even up to $\gamma = 0.4$.

The model can readily be generalized and extended. Wall thickness can be included with only computational difficulty, and corresponding three-dimensional problems can be attacked by similar methods. A simple generalization is to asymmetric regular arrays, in which either the slits or the solid-wall portions of the

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screen are not all of the same width, but alternate between two different values. This may, in the suggested application to random porosity, enable some account to be taken of the variance as well as the mean of the pore distribution.

2. Mathematical formulation, far-field boundary condition

The velocity vector $\mathbf{u}(x,y)\,e^{i\sigma t}$ is assumed to satisfy the linearized Navier–Stokes equation

$$i\sigma\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}, \tag{2.1}$$

where $p(x,y)\,e^{i\sigma t}$ is the pressure. Since we are also assuming a (locally) incompressible fluid, we can use a stream function $\psi(x,y)\,e^{i\sigma t}$ such that $\mathbf{u} = (\psi_y, -\psi_x)$, and (2.1) then implies that

$$\nabla^4\psi = \alpha^2\nabla^2\psi, \tag{2.2}$$

where $\alpha^2 = i\sigma/\nu$.

Incompressibility, together with (2.1), also implies that the pressure amplitude $p(x,y)$ is harmonic, i.e. $\nabla^2 p = 0$. The harmonic conjugate $q(x,y)$ to $p(x,y)$, that is, the solution of the Cauchy–Riemann equations

$$p_x = q_y, \quad p_y = -q_x, \tag{2.3}$$

is related to the vorticity

$$\omega = -\nabla^2\psi; \tag{2.4}$$

specifically

$$q = -\mu(\omega + \alpha^2\psi), \tag{2.5}$$

where $\mu = \rho\nu$ is the dynamic viscosity.

Vorticity ω is generated at the wall $y = 0$ due to viscosity and decays (exponentially) as $|y| \to \infty$. Hence, at a sufficiently great distance from the wall, the flow becomes irrotational with $\omega = 0$, $\nabla^2\mathbf{u} = 0$, and is described by a velocity potential proportional to $p(x,y)$, i.e.

$$\mathbf{u} = \nabla\left(-\frac{1}{i\sigma\rho}p(x,y)\right), \quad |y| \to \infty. \tag{2.6}$$

We suppose that this velocity potential corresponds to a uniform oscillatory flow $\mathbf{u} \to V\mathbf{j}$, setting

$$-\frac{1}{i\sigma\rho}p(x,y) \to Vy + \text{constant}, \quad |y| \to \infty. \tag{2.7}$$

This irrotational flow at a great distance (relative to the pore size) from the wall is the outer approximation to the inner flow through the pores, and must match with the inner approximation to the outer acoustic flow field. Thus V has the significance of a velocity amplitude normal to the wall, and the "constant" in (2.7) can be determined from the pressure jump Δp across the wall. In fact, (2.7) must take the form

$$p(x,y) \to -i\sigma\rho Vy \pm \tfrac{1}{2}\Delta p, \quad y \to \pm\infty, \tag{2.8}$$

or, using (1.1) and (1.3),

$$-\frac{1}{i\sigma\rho}p(x,y)\to V(y\pm aC), \quad y\to\pm\infty. \tag{2.9}$$

Equation (2.9) plays the role of a boundary condition at infinity, but the quantity C is to be determined from our solution.

The remaining boundary conditions are the no-slip condition $\mathbf{u}=0$ on the "solid" portion of the boundary $y=0$, that is,

$$\psi_x(x,0)=\psi_y(x,0)=0, \quad a<x<b, \tag{2.10}$$

and symmetry conditions on the other boundaries. The side boundaries $x=0,b$ are stream surfaces on which

$$\psi_y(0,y)=\psi_y(b,y)=0, \quad y>0, \tag{2.11}$$

about which the flow and hence the vorticity is antisymmetric, so that

$$\omega(0,y)=\omega(b,y)=0, \quad y>0. \tag{2.12}$$

The "gap" portion of the boundary $y=0$ is a plane of symmetry on which

$$\psi_y(x,0)=0, \quad 0<x<a \tag{2.13}$$

(that is, vanishing x-wise velocity component) and

$$\omega_y(x,0)=0, \quad 0<x<a, \tag{2.14}$$

which (by (2.3), (2.5)) is equivalent to continuity of pressure across the gap.

3. Derivation of integral equation

We make use of a Green's function $G(x,y,\xi)$ which satisfies (2.2) everywhere except at $(x,y)=(\xi,0)$, and write

$$\psi(x,y)=\int_0^a m(\xi)\,G(x,y,\xi)\,d\xi, \tag{3.1}$$

where $m(\xi)$ is a source strength to be determined and is proportional to the velocity through the gap. One possible choice for G is the isolated source function $G(x,y,\xi)=G_0(x-\xi,y)$, where

$$G_0(x,y)=-\frac{2}{\pi\alpha^2}\int_0^\infty [\lambda+\sqrt{(\lambda^2+\alpha^2)}]\,(\lambda e^{-y\sqrt{(\lambda^2+\alpha^2)}}-\sqrt{(\lambda^2+\alpha^2)}\,e^{-\lambda y})\sin\lambda x\frac{d\lambda}{\lambda} \tag{3.2}$$

(Tuck [6]), which also satisfies the zero-velocity conditions (2.10) everywhere on $y=0$, except at the source point $x=0$. In order to satisfy in addition the side boundary conditions (2.11), (2.12), we need to use an infinite array of such isolated sources G_0, writing

$$G(x,y,\xi)=\sum_{k=-\infty}^{\infty}[G_0(x-\xi+2bk,y)+G_0(x+\xi+2bk,y)]. \tag{3.3}$$

Convergence of this series is proved in the Appendix.

The pressure field corresponding to $\psi = G_0$ is $p = P_0$, where

$$P_0(x,y) = \frac{2}{\pi}\mu\left[\alpha^2\log\sqrt{(x^2+y^2)}+\frac{x^2-y^2}{(x^2+y^2)^2}-\int_0^\infty\sqrt{(\lambda^2+\alpha^2)}\,e^{-\lambda y}\cos\lambda x\,d\lambda\right]. \tag{3.4}$$

Corresponding to G, we have

$$P(x,y,\xi) = \sum_{k=-\infty}^{\infty}[P_0(x-\xi+2bk,y)+P_0(x+\xi+2bk,y)+\bar{P}_k], \tag{3.5}$$

where $\bar{P}_k$ is a constant included to cancel the diverging leading term of (3.4) as $k\to\pm\infty$, namely

$$\bar{P}_k = -\frac{4}{\pi}\mu\alpha^2\log 2b|k|, \quad k\neq 0. \tag{3.6}$$

The remaining constant $\bar{P}_0$ can be chosen arbitrarily. We choose

$$\bar{P}_0 = \frac{4}{\pi}\mu\alpha^2\log\pi/b+2\mu\alpha/b \tag{3.7}$$

in which case (see Appendix) we have

$$P(x,y,\xi)\to\frac{2\mu\alpha^2}{b}y+O(e^{-\pi y/b}), \quad y\to\infty. \tag{3.8}$$

The integral representation (3.1) for ψ implies a corresponding representation for the pressure

$$p(x,y) = \int_0^a m(\xi)P(x,y,\xi)\,d\xi+\tfrac{1}{2}\Delta p, \tag{3.9}$$

where the arbitrary constant

$$\tfrac{1}{2}\Delta p = -i\sigma\rho aCV = -\mu\alpha^2 aCV \tag{3.10}$$

is appropriate for satisfaction of (2.8), in view of the limit (3.8). In fact, (3.8) implies as $y\to+\infty$,

$$p(x,y)\to\frac{2\mu\alpha^2}{b}\int_0^a m(\xi)\,d\xi\,.\,y+\tfrac{1}{2}\Delta p, \tag{3.11}$$

and hence from (2.8),

$$-2\int_0^a m(\xi)\,d\xi = bV. \tag{3.12}$$

This is a normalization of the source strength m, which expresses continuity, since the quantity on the left of (3.12) is the net entering flux across the gap, and that on the right the net departing flux at $y=+\infty$.

The pressure-field representation (3.9) is valid only for $y>0$, but may if we wish be extended to $y<0$ as an *odd* function of y. But in general this would require a pressure jump across the gap $y=0$, $0<x<a$, violating the boundary condition (2.14). Hence we obtain our required integral equation to determine the unknown

$m(\xi)$ by setting $p(x,0) = 0$ over the gap, that is,

$$\int_0^a m(\xi)P(x,0,\xi)\,d\xi = \mu\alpha^2 aCV, \quad 0 < x < a. \tag{3.13}$$

The problem has thus reduced to solving equations (3.12), (3.13) simultaneously for the unknown quantities $m(x)$ and C.

The problem can be further simplified if we adopt a non-dimensionalization as follows. We set

$$\xi = a\xi^*, \quad x = ax^*, \tag{3.14}$$

$$m(\xi) = CVm^*(\xi^*) \tag{3.15}$$

and

$$P(x,0,\xi) = \mu\alpha^2 P^*(x^*,\xi^*). \tag{3.16}$$

For brevity, we omit the stars henceforth. Thus the integral equation (3.13) becomes

$$\int_0^1 m(\xi)P(x,\xi)\,d\xi = 1, \tag{3.17}$$

and the normalization integral (3.12) gives

$$-2CVa\int_0^1 m(\xi)\,d\xi = bV$$

or

$$C = -\left[2\gamma\int_0^1 m(\xi)\,d\xi\right]^{-1}. \tag{3.18}$$

Once we have solved (3.17) for $m(\xi)$, (3.18) provides the required output for the impedance parameter C.

The normalized kernel $P(x,\xi)$ can be written

$$P(x,\xi) = -Q'(x-\xi) - Q'(x+\xi), \tag{3.19}$$

where (see Appendix)

$$Q'(x) = \left(1 - \frac{1}{i\beta}\frac{d^2}{dx^2}\right)\left[-\frac{\gamma}{\sqrt{(i\beta)}} + \frac{2}{\pi}\log 2\left|\sin\frac{\pi}{2}\gamma x\right| - \frac{2}{\pi}\sum_{k=-\infty}^{\infty} K_0\left(\sqrt{(i\beta)}\left|x + \frac{2k}{\gamma}\right|\right)\right] \tag{3.20}$$

is an even function of its explicit argument x, depending implicitly on the two non-dimensional parameters β, γ, defined by (1.2), (1.5). For later use we require the (odd) function obtained by integrating (3.20) namely

$$Q(x) = -\gamma x(i\beta)^{-\frac{1}{2}} - (2/\pi^2\gamma)f(\pi\gamma x) - (\gamma/i\beta)\cot(\pi\gamma x/2)$$
$$+ (2/\pi)(i\beta)^{-\frac{1}{2}}\sum_{k=-\infty}^{\infty} g(\beta^{\frac{1}{2}}(x + 2k/\gamma)), \tag{3.21}$$

where f, g are odd functions defined for $\theta > 0$ by

$$f(\theta) = -\int_0^\theta \log\sin t/2\,dt \tag{3.22}$$

and

$$g(\theta) = K_0'(i^{\frac{1}{2}}\theta) - i^{\frac{1}{2}}\int_0^\theta K_0(i^{\frac{1}{2}}t)\,dt. \tag{3.23}$$

The function $f(\theta)$ is Clausen's integral (Abramowitz and Stegun [1], p. 1005), and $g(\theta)$ can be written in terms of Kelvin functions (Abramowitz and Stegun [1], p. 379) and their integrals.

The integral equation (3.17) is very singular; indeed the kernel $P(x, \xi)$ has a *double* pole at $\xi = x$, of the form (see Appendix)

$$P(x, \xi) \to (4/\pi i\beta)(x-\xi)^{-2}, \quad \xi \to x \tag{3.24}$$

and the integral (3.17) is formally divergent. In the Appendix we show that the integral can nevertheless be given a finite interpretation, and that direct numerical methods, as outlined in the following section, can be used. The situation is quite comparable to that encountered in lifting-surface aerodynamics (Ashley and Landahl [2], p. 148), and the interpretation used is that of the Hadamard finite part (Mangler [4]).

4. Numerical procedure

We assume that the unknown source strength $m(\xi)$ in (3.1), which is proportional to the normal velocity across the gap, is a slowly varying function of ξ, except at the edge $\xi = 1$ between gap and barrier. We therefore divide the gap $0 < \xi < 1$ into a set of N segments $\xi_i < \xi < \xi_{j+1}$, on each of which we approximate $m(\xi) = \text{constant} = m_j$. Thus (3.1) becomes

$$\sum_{j=1}^{N} m_j \int_{\xi_j}^{\xi_{j+1}} P(x, \xi)\,d\xi = 1, \quad 0 < x < 1. \tag{4.1}$$

We choose to satisfy (4.1) at a set of N points $x = x_i$, so that

$$\sum_{j=1}^{N} A_{ij}\, m_j = 1, \quad i = 1, 2, \ldots, N, \tag{4.2}$$

where

$$A_{ij} = \int_{\xi_j}^{\xi_{j+1}} P(x_i, \xi)\,d\xi. \tag{4.3}$$

$$= [Q(x_i - \xi) - Q(x_i + \xi)]_{\xi_j}^{\xi_{j+1}}.$$

The problem is therefore solved, upon solution by standard methods of the matrix-vector equation

$$A\mathbf{m} = \mathbf{1}, \tag{4.5}$$

where

$$A = [A_{ij}] \tag{4.6}$$

$$\mathbf{m} = [m_j], \tag{4.7}$$

and all elements of the vector $\mathbf{1}$ are of unit value.

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In principle the above procedure is valid for any dissection $\{\xi_j\}$ of the interval (0, 1) and any ordered set of collocation points $\{x_i\}$, and should converge to the solution of (3.17) as $N \to \infty$, providing at the same time all interval lengths $\xi_{j+1} - \xi_j \to 0$. In fact we choose

$$\xi_j = \sin \frac{\pi}{2}\left(\frac{j-1}{N}\right), \quad j = 1, 2, \ldots, N+1 \tag{4.8}$$

and

$$x_i = \sin \frac{\pi}{2}\left(\frac{i-\frac{1}{2}}{N}\right), \quad i = 1, 2, \ldots, N. \tag{4.9}$$

This non-uniform spacing accommodates the expected square-root singularity in the normal velocity as $\xi \to 1$, since the interval size tends to zero correspondingly.

The formula (4.4) enables direct computation of the matrix elements A, using the formula (3.21) for the odd function $Q(x)$. This requires sub-routines for the special functions $f(\theta)$, $g(\theta)$ defined by (3.22), (3.23), neither of which causes any numerical difficulty. For Clausen's integral $f(\theta)$ we use direct trapezoidal integration, whereas for $g(\theta)$ we use series representations for the Kelvin function and for the integral of the modified Bessel function K_0.

5. Computed results

The preceding numerical procedure was coded in Fortran on the Adelaide University CDC 6400 computer. The program was run with various different numbers of mesh points and it was found that a 40-point mesh gave satisfactory accuracy.

The blockage coefficient C was obtained for various values of the parameters β, γ. In Fig. 2, the magnitude of C, suitably scaled, is plotted against $\sqrt{\beta}$, for values of γ ranging from 0.1 to 0.8. As might be expected, for fixed β, the blockage coefficient increases as γ decreases. That is, there is greater impedance when γ is small, since the flow is restricted as γ decreases. Indeed in the limit $\gamma \to 0$ we would expect no flow at all, and thus the blockage coefficient would be infinite. The results are consistent with this. At the other end of the scale as $\gamma \to 1$, the blockage coefficient should approach zero. This corresponds to a completely unrestricted flow. Again, the computed results tend to confirm this.

For fixed γ, we should expect the blockage coefficient to decrease as the non-dimensional Reynolds number β increases, due to the fact that increase in β corresponds to a decrease in viscosity. Lower viscosity implies less impedance to flow through the slits and in Fig. 2 we can see the obstruction to the flow decreasing as β increases.

For both large and small β, the present results can be compared with work done elsewhere. Very large Reynolds' number β corresponds to inviscid flow. In Fig. 3

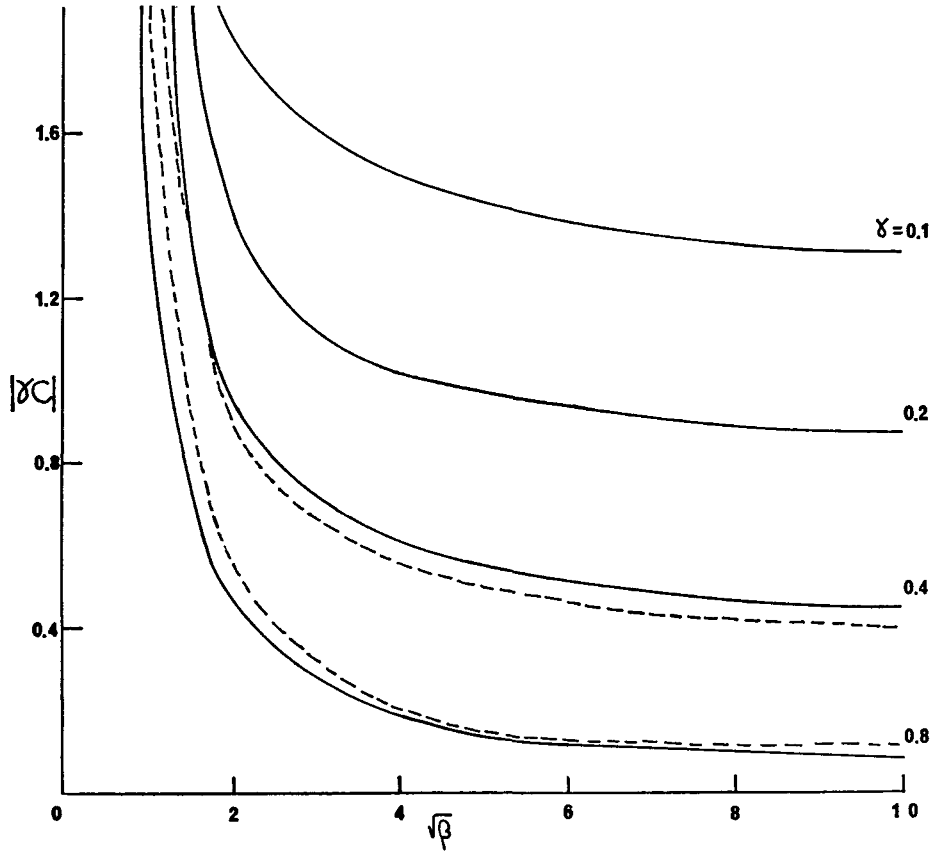

Fig. 2. The variation of the magnitude of the scaled blockage coefficient $|\gamma C|$ with the square root of the Reynolds' number, $\sqrt{\beta}$, for various values of γ. The dashed lines are the approximations for small γ, using the single-split result; for $\gamma = 0.1$ and 0.2 the two results are indistinguishable.

we compare the result (1.4) for inviscid flow in a channel, with the present work. The magnitude of the blockage coefficient is plotted against γ for various values of β. It can be seen that for values of β greater than about one hundred, there is excellent agreement between the viscous and inviscid results over the whole range of γ. This means that inviscid theory may be safely applied to flow situations where the Reynolds is of the order of one hundred or more. For very small β, which corresponds to very viscous Stokes flow, there are no general results for arrays of slits.

For small γ, the flow situation can be compared with the problem for a single slit, and it can be shown that, as $\gamma \to 0$ at fixed β,

$$\gamma C \to -(2/\pi)\log(\pi\gamma s/a) - \gamma(i\beta)^{-\frac{1}{2}}, \tag{5.1}$$

where the quantity $s = s(\beta)$ is the "effective size" of a single slit in a wall. Rough numerical values of $s(\beta)$ were given by Tuck ([7], p. 155), and improved computations, using a numerical method similar to that of the present paper, have recently been made by Macaskill [3]. In Fig. 2, the magnitude of the right-hand side of (5.1) is compared with $|\gamma C|$, using Macaskill's [3] computations for $s(\beta)$. These results

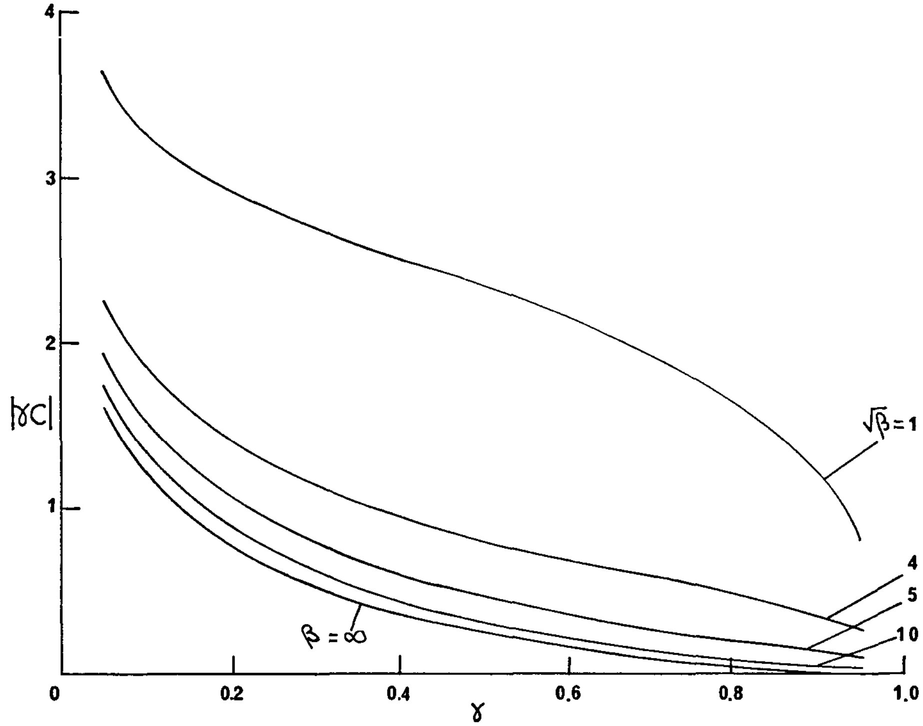

Fig. 3. The variation of the magnitude of the scaled blockage coefficient $|\gamma C|$ with γ, for various Reynolds' numbers β.

are shown dotted. For $\gamma = 0.1$ and $\gamma = 0.2$ the small-γ approximation is indistinguishable from the exact computations, and the accuracy remains within about 10% up to $\gamma \sim 0.4$, which is somewhat surprising, as this represents quite large holes. Even at $\gamma = 0.8$ the agreement appears good for the magnitude except for small β, but there is a much larger error in the phase.

Appendix

Properties of kernel functions

From the result of Tuck [6], namely as $r \to \infty$,

$$G_0(r\cos\theta, r\sin\theta) \to 1 - \frac{2}{\pi}\theta + \frac{2}{\pi\alpha}\frac{\cos\theta}{r} + \frac{2}{\pi\alpha^2}\frac{\sin 2\theta}{r^2} + O(r^{-3}), \tag{A1}$$

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we have as $|x|\to\infty$ for fixed y,

$$G_0(x,y)\to \operatorname{sgn} x+\frac{2}{\pi\alpha}(1-\alpha y)\,x^{-1}+O(x^{-3}). \tag{A2}$$

Similarly, as $r\to\infty$,

$$P_0(r\cos\theta, r\sin\theta)\to\frac{2\mu\alpha^2}{\pi}\log r-\frac{2\mu\alpha}{\pi}\frac{\sin\theta}{r}+\frac{2\mu}{\pi}\frac{\cos 2\theta}{r^2}+O(r^{-3}) \tag{A3}$$

and hence as $|x|\to\infty$ for fixed y,

$$P_0(x,y)\to\frac{2\mu\alpha^2}{\pi}\log|x|+O(x^{-2}). \tag{A4}$$

The x-argument of the terms in the series (3.3), (3.5) is

$$x\pm\xi+2bk\to 2bk\left(1+\frac{x\pm\xi}{2b}k^{-1}\right) \tag{A5}$$

and hence these terms are asymptotically given by

$$G_0\to\operatorname{sgn} k+\frac{1}{\pi\alpha b}(1-\alpha y)\,k^{-1}+O(k^{-2}), \tag{A6}$$

$$P_0\to\frac{2\mu\alpha^2}{\pi}\log 2b|k|+\frac{\mu\alpha^2}{b}(x\pm\xi)\,k^{-1}+O(k^{-2}). \tag{A7}$$

Because the $O(1)$ and $O(k^{-1})$ terms of (A6) are odd in k, they cancel as $k\to\pm\infty$ and the series (3.3) then converges, since the net summand is $O(k^{-2})$. Similarly the k^{-1} term of (A7) cancels, so that if we add the constant $\bar{P}_k$ given by (3.6), the residual summand is $O(k^{-2})$ and (3.5) also converges.

In order to derive the limit (3.8) of P as $y\to\infty$ we consider (A3), which (unlike A4) is uniformly valid as $y\to\infty$, and observe that the error term $O(r^{-3})$ is actually (see Tuck [6]) of order $\partial^2/\partial y^2(\sin\theta/r)$. Thus as $y\to\infty$, we have

$$P(x,y,\xi)\to P_\infty(x,y,\xi)-\frac{1}{\alpha}\frac{\partial}{\partial y}P_\infty(x,y,\xi)+O\left(\frac{\partial^2 P_\infty}{\partial y^2}\right), \tag{A8}$$

where

$$P_\infty(x,y,\xi)=\sum_{k=-\infty}^{\infty}\left[\frac{2\mu\alpha^2}{\pi}\log\sqrt{[(x-\xi+2bk)^2+y^2]}\right.$$
$$\left.+\frac{2\mu\alpha^2}{\pi}\log\sqrt{[(x+\xi+2bk)^2+y^2]}+\bar{P}_k\right]. \tag{A9}$$

The series in (A9) can be summed explicitly. In fact P_∞ is just the velocity potential for an irrotational source between two walls, and we have (see Tuck [7], p. 119)

$$P_\infty(x,y,\xi)=\bar{P}_0+\frac{4\mu\alpha^2}{\pi}\log\frac{2b}{\pi}$$
$$+\frac{2\mu\alpha^2}{\pi}\mathscr{R}\left[4\sin\frac{\pi}{2b}(x-\xi+iy)\sin\frac{\pi}{2b}(x+\xi+iy)\right] \tag{A10}$$

Thus as $y \to +\infty$, we have

$$P_\infty \to \bar{P}_0 + \frac{4\mu\alpha^2}{\pi}\log\frac{b}{\pi} + \frac{2\mu\alpha^2}{b}y + O(e^{-\pi y/b}), \tag{A11}$$

and hence

$$P(x,y,\xi) = \bar{P}_0 + \frac{4\mu\alpha^2}{\pi}\log\frac{b}{\pi} + \frac{2\mu\alpha^2}{b}y - \frac{1}{\alpha}\left(\frac{2\mu\alpha^2}{b}\right) + O(e^{-\pi y/b}) \tag{A12}$$

$$= \frac{2\mu\alpha^2}{b}y + O(e^{-\pi y/b}), \tag{A13}$$

providing

$$\bar{P}_0 = -\frac{4\mu\alpha^2}{\pi}\log\frac{b}{\pi} + \frac{2\mu\alpha}{\pi}, \tag{A14}$$

which is equation (3.7) of the text.

An alternative form of the representation (3.4) for the kernel P_0 is

$$\frac{\pi}{2\mu}P_0(x,y) = \mathscr{R}_\alpha\left[\alpha^2\log z + z^{-2} - \left(\alpha^2 - \frac{d^2}{dz^2}\right)F(\alpha z)\right], \tag{A15}$$

where $z = x + iy$ and

$$F(z) = K_0(z) + \frac{2i}{\pi}(I_0(z) - \mathbf{L}_0(z)), \tag{A16}$$

K_0 and I_0 being modified Bessel functions and $\mathbf{L}_0$ a modified Struve function (Abramowitz and Stegun [1], p. 495). The symbol "$\mathscr{R}_\alpha$" implies that in taking the real part, α is assumed (temporarily) real. Since $I_0, \mathbf{L}_0$ are analytic near $z = 0$, as $z \to 0$,

$$F(z) \to K_0(z) + O(1)$$

$$\to -\log z + O(1)$$

and hence

$$\frac{\pi}{2\mu}P_0(x,y) \to \mathscr{R}\left[z^{-2} - \frac{d^2}{dz^2}(\log z) + O(\log z)\right]$$

that is,

$$P_0(x,y) \to \frac{4\mu}{\pi}\frac{x^2+y^2}{(x^2+y^2)^2} + O[\log\sqrt{(x^2+y^2)}]$$

$$= \frac{4\mu}{\pi x^2} \quad \text{on } y = 0. \tag{A17}$$

For use in the numerical computation we need only values on $y = 0$, namely

$$\frac{\pi}{2\mu}P_0(x,0) = \alpha^2\log x + x^{-2} - \left(\alpha^2 - \frac{d^2}{dx^2}\right)K_0(\alpha x),$$

which implies that

$$P(x,0,\xi) = \frac{2\mu}{\pi}\left(\alpha^2 - \frac{\partial^2}{\partial\xi^2}\right)\left[\frac{\pi}{b\alpha} + \log 4\left|\sin\frac{\pi}{2b}(x-\xi)\sin\frac{\pi}{2b}(x+\xi)\right| - \sum_{k=-\infty}^{\infty} K_0(\alpha|x-\xi+2bk|) + K_0(\alpha|x+\xi+2bk|)\right].$$

The corresponding non-dimensional form of the kernel, subject to the scaling given by (3.14), (3.16), is simply obtained by dividing by $\mu\alpha^2$ and then setting $\alpha = \sqrt{(i\beta)}$, $b = 1/\gamma$, in the above formula, leading to the expression quoted in equations (3.19), (3.20) of the text.

However, care must be exercised when the limit $y \to 0$ is taken in the above determination of $P(x, 0, \xi)$. If we simply set $y = 0$ and then subsequently let $\xi \to x$, the dominant term in the complete kernel $P(x, 0, \xi)$ is obtained from the $k = 0$ term of the sum (3.5), and hence, from (A17), as $\xi \to x$,

$$P(x,0,\xi) \to (4\mu/\pi)(x-\xi)^{-2}, \tag{A20}$$

as in equation (3.24) of the text. Since the integral in (3.13) is then divergent in the ordinary sense, the process of setting $y = 0$ directly cannot be legitimate.

An appropriate interpretation is obtained by integrating by parts before allowing y to tend to zero. We first define a new function $\bar{Q}(x, y, \xi)$ by the relation $P = \partial\bar{Q}/\partial\xi$. Then we have

$$\int_0^a m(\xi)P(x,y,\xi)\,d\xi = [m(\xi)\,\bar{Q}(x,y,\xi)]_0^a - \int_0^a m'(\xi)\,\bar{Q}(x,y,\xi)\,d\xi. \tag{A21}$$

The integrated part of (A21) vanishes at $\xi = 0$ by symmetry, and at $\xi = a$ if $m(a) = 0$, as is required physically. In the remaining integral, we *can* let $y \to 0$ legitimately, providing we use a Cauchy principal-value interpretation, since $\bar{Q}$ possesses only a single-pole singularity as $\xi \to x$, that is,

$$\bar{Q}(x,0,\xi) \to (4\mu/\pi)(x-\xi)^{-1}. \tag{A22}$$

Thus we have

$$\lim_{y\to 0}\int_0^a m(\xi)P(x,y,\xi)\,d\xi = \lim_{\epsilon\to 0}\left(\int_0^{x-\epsilon} + \int_{x+\epsilon}^a\right) m'(\xi)\,\bar{Q}(x,0,\xi)\,d\xi. \tag{A23}$$

Although (A23), a standard Cauchy principal-value integral, can be treated numerically as it stands, it is more convenient for our numerical purposes to use the original unknown $m(\xi)$ rather than its derivative $m'(\xi)$, and the original kernel P rather than $\bar{Q}$. We can now return to the original variables by a reverse integration by parts, once y has been set to zero. Thus

$$\lim_{y\to 0}\int_0^a m(\xi)P(x,y,\xi)\,d\xi = \lim_{\epsilon\to 0}\left\{-[m(\xi)\,\bar{Q}(x,0,\xi)]_0^{x-\epsilon}-[m(\xi)\,\bar{Q}(x,0,\xi)]_{x+\epsilon}^{a}\right.$$
$$\left.+\left(\int_0^{x-\epsilon}+\int_{x+\epsilon}^{a}\right)m(\xi)P(x,0,\xi)\,d\xi\right\} \tag{A24}$$
$$=\lim_{\epsilon\to 0}\left\{(-8\mu/\pi\varepsilon)\,m(x)+\left(\int_0^{x-\varepsilon}+\int_{x+\varepsilon}^{a}\right)\right.$$
$$\left.\times\, m(\xi)P(x,0,\xi)\,d\xi\right\}, \tag{A25}$$

using $\bar{Q}(x,0,0)=m(a)=0$, and using (A22) to evaluate (in the limit as $\varepsilon\to 0$) the contributions from the integrated parts of (A24) at $\xi = x\pm\varepsilon$. Equation (A25) is precisely the Hadamard interpretation [4] of the double-pole divergent integral in (3.13).

The fact that the "naïve" numerical method of Section 4, which apparently ignores the divergent character of the integral, nevertheless gives the right result according to the interpretation (A25), can be explained as follows. The only potential difficulty is with the diagonal elements of the matrix A defined by (4.4), that is, with contributions from an interval (ξ_i, ξ_{i+1}) containing the singular point $\xi = x$ (and the interval $|\xi - x|<\varepsilon$), on which $m(\xi)$ takes the constant value $m_i = m(x)$. Now according to (A25)

$$\lim_{y\to 0}\int_{\xi_i}^{\xi_{i+1}} m(\xi)P(x,y,\xi)\,d\xi = m_i\lim_{\epsilon\to 0}\left\{(-8\mu/\pi\varepsilon)+(4\mu/\pi)\,[(x-\xi)^{-1}]_{\xi_i}^{x-\epsilon}\right.$$
$$+(4\mu/\pi)\,[(x-\xi)^{-1}]_{x+\epsilon}^{\xi_{i+1}}+\int_{\xi_i}^{\xi_{i+1}}[P(x,0,\xi)$$
$$\left.-(4\mu/\pi)(x-\xi)^{-2}]\,d\xi\right\}, \tag{A26}$$

where we have subtracted off the most-singular portion (A20) of the kernel, and integrated it explicitly before allowing ε to vanish. Now on performing the limit $\varepsilon\to 0$, we have

$$\lim_{y\to 0}\int_{\xi_i}^{\xi_{i+1}} m(\xi)P(x,y,\xi)\,d\xi = m_i\left\{(4\mu/\pi)\,[(x-\xi_i)^{-1}-(x-\xi_{i+1})^{-1}]\right.$$
$$\left.+\int_{\xi_i}^{\xi_{i+1}}[P(x,0,\xi)-(4\mu/\pi)(x-\xi)^{-2}]\,d\xi\right\}. \tag{A27}$$

The integral in (A27) is convergent, and can be expressed in terms of the anti-derivative $\bar{Q}(x,0,\xi)-(4\mu/\pi)(x-\xi)^{-1}$, the last term of which exactly cancels the integrated part of (A27). Thus, finally,

$$\lim_{y\to 0}\int_{\xi_i}^{\xi_{i+1}} m(\xi)P(x,y,\xi)\,d\xi = m_i[\bar{Q}(x,0,\xi)]_{\xi_i}^{\xi_{i+1}}. \tag{A28}$$

The coefficient of m_i in (A28) is a dimensional form of the diagonal element A_{ii} of (4.4), the dimensionless function Q used there being defined by

$$\bar{Q}(x,0,\xi) = \mu\alpha^2 a[Q(x^*-\xi^*) - Q(x^*+\xi^*)]. \tag{A29}$$

Thus the naïve discretization of the divergent integral (for example, in (4.1) when $j = i$) leads to the same result as is obtained by careful interpretation and removal of the most singular contributions.

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