41

Lattice gauge theory

41.1 Introduction

In this chapter, we shall discuss very briefly the main idea behind the lattice approach in QCD. More detailed discussions and some introductions can be found in different textbooks on lattice gauge theories [489] and some non-specialized reviews. (see e.g., Yndurain's book [46] or Dosch's review [51]). More recent reviews on the lattice results can be found in different contributions at the annual Lattice conferences (*Nucl. Phys. B* (Proc, Suppl.)). The starting point is the *Euclidian* generating functional:

$$Z = \int \mathcal{D}\psi(x)\mathcal{D}\bar{\psi}(x)\exp\left\{-\mathcal{S} \equiv \int d^4x\, \mathcal{L}_{\rm QCD}\right\}, \tag{41.1}$$

where the QCD action $\mathcal{S}$ is positive, thus providing the convergence factor. It is convenient to write the Lagrangian in a matrix notation:

$$G_{\mu\nu} \equiv \sum_a \frac{\lambda_a}{2} G^a_{\mu\nu}\ , \tag{41.2}$$

where λ_a are the generators of the $SU(3)_c$ gauge transformation group:

$$U(x) = \exp\left\{ig\frac{\lambda_a}{2}A^a(x)\right\}\ . \tag{41.3}$$

Therefore, it reads:

$$\mathcal{L}_{\rm QCD}(x) = \frac{1}{2}\sum_{\mu\nu} G^2_{\mu\nu}(x) + \bar{\psi}(x)(\partial^\mu\gamma_\mu + m)\psi(x) - ig\bar{\psi}\gamma_\mu A^\mu(x)\psi(x)\ , \tag{41.4}$$

where, in this notation, the gauge transformations become:

$$\begin{aligned} A_\mu(x) &\to U^{-1}(x)A_\mu(x)U(x) + \frac{i}{g}U^{-1}(x)\partial_\mu(x)U(x) \\ G_{\mu\nu}(x) &\to U^{-1}(x)G_{\mu\nu}(x)U(x)\ , \\ \psi(x) &\to U^{-1}(x)\psi(x)\ . \end{aligned} \tag{41.5}$$

Next we introduce the essential ingredients for the *lattice formulation* of QCD. Here, one expects that all expressions introduced below are well-defined, and, *in principle*, can be evaluated numerically. This feature has made lattice gauge theory one of the most important non-perturbative methods for QCD. The functional integral introduced before has to be understood as the limiting value of a high-dimensional volume integral where the fields at the lattice points $i, j, \ldots$ are the integration variables. For definiteness, we shall consider a finite hypercube lattice, with lattice spacing a and volume $V = (Na)^4$ with periodic boundary conditions. The physical (continuum) limit is reached for $V \to \infty$ first and after $a \to 0$. The lattice provides a *regularization* as a is finite, such that UV divergences do not occur. As long as N is bounded from above, IR divergences are prevented. The UV divergences will reappear as $1/a$ or/and $\log a$, when one goes to the continuum limit, where $a \to 0$.

- A point on the lattice is denoted by its coordinates in units of a, i.e. by the integers: $(n) \equiv (n_1, n_2, n_3, n_4)$, representing the point with coordinates $X = (an_1, an_2, an_3, an_4)$.
- The neighbour of the point (n) in the μ-direction is denoted by $(n + \mu)$.
- The link from point n to its neighbour in the μ-direction, $n + \mu$ is denoted by $(n, n + \mu)$. Its plays an essential rôle in the lattice.

41.2 Gluons on the lattice: the Wegner–Wilson action

- An element of the gauge group is attached to each link, while its inverse is attached to the link in the opposite direction [490,491]:

$$(n, n+\mu) \to U(n, n+\mu)\,, \qquad (n+\mu, n) \to U^{-1}(n, n+\mu)\,, \tag{41.6}$$

where the group elements $U(n, n+\mu)$ can be expressed by the generators $\lambda_a/2$ of the group as:

$$U(n, n+\mu) = \exp\left\{ig\frac{a\lambda_a}{2}A^a_\mu a(n)\right\}\,. \tag{41.7}$$

- In a local gauge theory, an element of the gauge group is attached to each point on the lattice:

$$U(n) = \exp\left\{ig\frac{\lambda_a}{2}\Lambda^a(n)\right\}\,. \tag{41.8}$$

The gauge transformation for the group element $U(n, n+\mu)$ is defined as:

$$U(n, n+\mu) \to U(n)U(n, n+\mu)U^{-1}(n+\mu)\,, \tag{41.9}$$

where one may notice that there is no inhomogeneous term on the lattice version of gauge transformation.
- The continuum limit is achieved by connecting quantities attached to neighbouring lattice points through the Taylor expansion and retaining the lowest-order contribution in the lattice spacing a:

$$U(n+\mu) = U(x) + a\partial_\mu U(x) + \mathcal{O}(a^2)\,. \tag{41.10}$$

Using the expansion:

$$U(n, n+\mu) = 1 + iag\frac{\lambda_a}{2}A^a(x) + \mathcal{O}(a^2)\,, \tag{41.11}$$

the gauge tranformation in Eq. (41.9), becomes the one of the continuum limit in Eq. (41.5).

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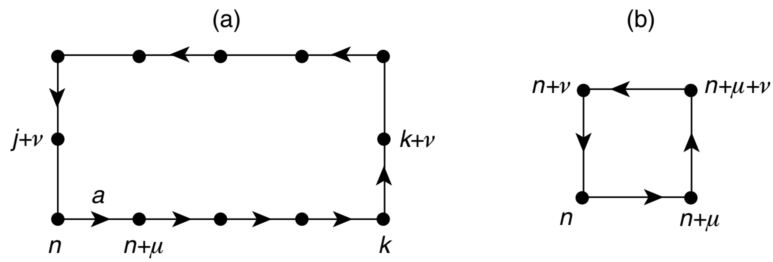

Fig. 41.1. Plaquette: a group element $U(n, n+\mu)$ is attached to each link.

- The Wegner–Wilson loop [490,491] corresponds to the product of group elements $U(n, n+\mu)$ along a closed contour L. It is defined as:

$$W[L] = U(n, n+\mu)U(n+\mu, n+\mu+\lambda)\cdots U(n+\nu, n)\,. \tag{41.12}$$

Using the property $UU^{-1} = 1$ and the fact that the trace is cyclic: $\mathrm{TrABC} = \mathrm{TrBCA} = \ldots$, it is easy to show that, under the gauge transformations in Eq. (41.9):

$$TrW[L] \qquad \text{is gauge invariant}\,. \tag{41.13}$$

- A *Plaquette* is the simplest non-trivial Wegner–Wilson loop, which is the product of four group elements attached to a square with a sidelength a and lattice points as corners (see Fig. 41.1).

$$\begin{aligned}\mathcal{P}(n,\mu,\nu) &= U(n+\mu, n+\mu+\nu)U(n+\mu+\nu, j+\nu)U(j+\nu, j)U(j, j+\mu)\\ &= U(n+\mu, n+\mu+\nu)U^{-1}(n+\nu, j+\mu+\nu)U^{-1}(j, j+\nu)U(j, j+\mu)\\ &= e^{[igaA_\nu(n+\mu)]}e^{[-igaA_\mu(n+\nu)]}e^{[-igaA_\nu(n)]}e^{[-igaA_\mu(n)]}\end{aligned} \tag{41.14}$$

Using the Campbell–Hausdorff formula:

$$e^{ax}e^{ay} \simeq e^{ax+ay+a^2[x,y]+\mathcal{O}(a^3)}\,, \tag{41.15}$$

with each pair of the previous exponentials, one obtains:

$$\begin{aligned}\mathcal{P}(n,\mu,\nu) = e^{\{-iag(A_\nu(n+\mu)-A_\mu(n+\nu))+a^2g^2[A_\nu(n+\mu),A_\mu(n+\nu)]/2+i\mathcal{O}(a^3)\}}\\ \times e^{\{-iag(A_\nu(n)-A_\mu(n))+a^2g^2[A_\nu(n),A_\mu(n)]/2+i\mathcal{O}(a^3)\}}\,,\end{aligned} \tag{41.16}$$

Using the Taylor expansion:

$$A_\mu(j+\nu) = A_\mu(n) + a\partial_\nu A_\mu(n) + \mathcal{O}(a^2)\,, \tag{41.17}$$

and applying again Eq. (41.15), one can deduce the form of the plaquette in the continuum limit:

$$\mathcal{P}(n,\mu,\nu) = e^{\{ia^2g^2[G_{\mu\nu}(n)+\mathcal{O}(a)]\}}\,, \tag{41.18}$$

with the usual definition of the field tensor:

$$G_{\mu\nu}(x) \equiv G^a_{\mu\nu}\frac{\lambda_a}{2} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig[A_\nu, A_\mu]\,. \tag{41.19}$$

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In terms of the plaquette, one can now define a positive real and gauge-invariant action on the lattice:

$$\mathcal{S}_g = -\frac{1}{g^2}\sum_n\sum_{\mu<\nu}\mathbf{Tr}\{\mathcal{P}(n,\mu,\nu)+\mathcal{P}^\dagger(n,\mu,\nu)\}\,. \tag{41.20}$$

It is customary to express the action in terms of the variable:

$$\beta \equiv \frac{2N_c}{g^2}\,. \tag{41.21}$$

In the continuum limit, one can write:

$$\begin{aligned}\mathcal{S}_g &= -\frac{1}{g^2}\sum_n\sum_{\mu<\nu}2\mathrm{Re}\,\mathbf{Tr}\exp\{ia^2gG_{\mu\nu}(n)+\mathcal{O}(a)\}\\ &= -\frac{1}{g^2}\sum_n\sum_{\mu,\nu}\mathrm{Re}\,\mathbf{Tr}\left\{1+ia^2gG_{\mu\nu}(n)-\frac{1}{2}a^4g^2G_{\mu\nu}(n)G^{\mu\nu}(n)\right\}\,,\end{aligned} \tag{41.22}$$

where the sum over μ,ν gets a factor $1/2$ because $\mu\nu$ and $\nu\mu$ define the same plaquette. Using the fact that $\mathrm{Tr}\lambda_a = 0$, one recovers the usual continuum action given in Eq. (41.4):

$$\mathcal{S}_g = \frac{1}{2}\sum_n\sum_{\mu,\nu}a^4G^{\mu\nu}(n)G_{\mu\nu}(n)+\mathcal{O}(a^6)+\text{constant}\,. \tag{41.23}$$

The vacuum expectation value of a function of the fields $F[U(n,n+\mu)]$ is:

$$\langle F[U(n,n+\mu)]\rangle = \frac{1}{\int \mathcal{D}U\, e^{\mathcal{S}_g}}\int \mathcal{D}U\, e^{\mathcal{S}_g}F[U(n,n+\mu)]\,, \tag{41.24}$$

where the invariant measure on the group attached to the link is:

$$\mathcal{D}U \equiv \prod_{n,\nu}dU(n,n+\nu)\,. \tag{41.25}$$

For an Abelian group, the measure is:

$$dU(n,n+\nu) = d(aA_\nu(n)) \qquad \text{with} : -\pi/a \le A_\nu(n) \le \pi/a\,. \tag{41.26}$$

For a non-Abelian $SU(N)_c$ group, one has:

$$dU(n,n+\nu) = \sqrt{\det\left[\frac{1-\cos aA_\nu(n)}{(aA_\nu(n))^2}\right]}\prod_{b=1}^{N_c^2-1}d\big(aA_\nu^b(n)\big)\,. \tag{41.27}$$

41.3 Quarks on the lattice

In this section, we turn to the less understood subject of the formulation of quarks (fermions) on the lattice, where the complications are already present at the free-field level. Since fermions obey Pauli exclusion principle, they are described at the classical level by anti-commuting variables forming the so-called *Grassmann algebra*, which anticommute themselves but commute with complex numbers. To each lattice points, with coordinates (n),

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are attached $N_c \times 8$ anticommuting quantities:

$$\psi_\alpha^c(n)\,, \qquad \bar\psi_\alpha^c(n)\,, \tag{41.28}$$

where the spinor α runs from 1 to 4, the colour index c from 1 to N_c. The field transform as:

$$\psi_\alpha^c(n) \to U_{cc'}\psi_\alpha^{c'}(n)\,, \qquad \bar\psi_\alpha^c(n) \to U_{cc'}^{-1}\psi_\alpha^{c'}(n)\,, \tag{41.29}$$

Therefore terms like:

$$\bar\psi_\alpha^c(n)\psi_\alpha^c(n)\,, \qquad \bar\psi_\alpha^c(n+\mu)U_{cc'}(n+\mu,n)\psi_\alpha^{c'}(n)\,, \tag{41.30}$$

are gauge invariant. It is usual to start from the free continuum Lagrangian in Eq. (41.4):

$$\mathcal{L}_{\text{free}} = \bar\psi(x)(\partial\gamma_\mu + m)\psi(x)\,, \tag{41.31}$$

which possesses a $SU(n_f)_L \times SU(n_f)_R$ global symmetry in the massless limit $m = 0$. As in previous section, one introduces a four-dimensional hypercubic lattice of N^4 sites. To each site n, one associates an independent four-component spinor variable:

$$\psi_n \equiv \psi(an) \to \psi(x) \tag{41.32}$$

characterizing the quark fields. For simplifying the lattice action, one defines the derivative symetrically:

$$\partial_\mu\psi \to \frac{1}{2a}(\psi_{n+\mu} - \psi_{n-\mu})\,. \tag{41.33}$$

Therefore, the lattice action reads:

$$\mathcal{S}_{\text{free}} = \sum_{n,k}\bar\psi_n M_{nk}\psi_k \tag{41.34}$$

with:

$$M_{nk} = \frac{1}{2}a^3\sum_\mu \gamma_\mu(\delta_{k,n+\mu} - \delta_{k,n-\mu}) + a^4 m\delta_{nk}\,. \tag{41.35}$$

Now, one can put this action into a path integral:

$$Z_{\text{free}} = \int \mathcal{D}\psi\mathcal{D}\bar\psi\; e^{-\mathcal{S}}\,, \tag{41.36}$$

where:

$$\mathcal{D}\psi \equiv \prod_k d\psi_k\,, \tag{41.37}$$

after a relatively long, though straightforward manipulation, one finds:

$$Z_{\text{free}} = \frac{(-1)^{2N+1}}{(2N+1)!}\,\det M\,. \tag{41.38}$$

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The quark propagator can be obtained by inverting M, which one can do with the help of a (finite) Fourier transform:

$$(M^{-1})_{nk} = a^{-4}(2N+1)^{-4} \sum_j \tilde{M}_j^{-1} \exp\left\{ \frac{2i\pi}{2N+1} \sum_\mu j_\mu (n-k)_\mu \right\} , \quad (41.39)$$

and by using the relation:

$$\sum_{j_\mu=-N}^{N} \exp \frac{2i\pi}{2N+1} j_\mu (n-k)_\mu = (2N+1)\delta_{n_\mu k_\mu} . \quad (41.40)$$

Therefore, one finds:

$$S(j) = \tilde{M}_j^{-1} = \left(m + \frac{i}{a} \sum_\mu \gamma_\mu \sin \frac{2\pi j_\mu}{2N+1} \right)^{-1} . \quad (41.41)$$

In the case of a large lattice, one has:

$$\frac{2\pi j_\mu}{2N+1} \equiv a p_\mu , \quad (41.42)$$

which leads to the p-space propagator:

$$S(p) = \left(m + \frac{i}{a} \sum_\mu \gamma_\mu \sin a p_\mu \right)^{-1} . \quad (41.43)$$

Replacing the sum over j by integrals:

$$\frac{1}{2N+1} \sum_{j_\mu=-N}^{N} \to a \int_{-\pi/a}^{+\pi/a} \frac{dp_\mu}{2\pi} , \quad (41.44)$$

Equation (41.39) becomes:

$$(M^{-1})_{nk} = \int_{-\pi/a}^{+\pi/a} \frac{d^4 p}{(2\pi)^4} \frac{e^{i \sum_\mu p^\mu (an-ak)_\mu}}{m + (i/a)\sum_\mu \gamma_\mu \sin a p_\mu} . \quad (41.45)$$

In the continuum limit ($a \to 0,\ an \to x,\ ak \to y$), this previous equation becomes:

$$(M^{-1})_{nk} \to S(x-y) = \int_{\infty}^{+\infty} \frac{d^4 p}{(2\pi)^4} \frac{e^{i \sum_\mu p^\mu (x-y)_\mu}}{m + i \sum_\mu \gamma_\mu p_\mu} , \quad (41.46)$$

which is the Euclidian propagator. However, by analysing Eq. (41.43), for example in the case $m = 0$, one can see that, for finite a, it has too many poles as the denominator vanishes for $p_\mu = 0$ and $p_\mu = \pi/a$. On the hypercube lattice, one has $2^4 = 16$ poles instead of one! This *fermion doubling* is catastrophic as one loses asymptotic freedom, the existence of the $U(1)$ anomaly (the 16 fermions contribute with alternate signs to the anomaly triangle), ... Several solutions to this fermion doubling problem have been proposed in the literature [489]. One of the most popular is the one proposed by Wilson [492]. It consists

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of adding to the Lagrangian a quadratic term:

$$\mathcal{L}_q^W = m\bar{\psi}_n\psi_n + \frac{4r}{a}\bar{\psi}_n\psi_n + \frac{1}{2a}\sum_\mu\{(r+\gamma_\mu)\psi_{n+\mu} + (r-\gamma_\mu)\psi_{n-\mu}\}\,, \quad (41.47)$$

where r is arbitrary. In the large lattice limit, the corresponding p-space propagator is:

$$S^W(p) = \left(m + \frac{1}{a}\sum_\mu\left[i\gamma_\mu \sin ap_\mu + \frac{r}{a}(1-\cos ap_\mu)\right]\right)^{-1}\,. \quad (41.48)$$

One can notice that for small momentum, the new term is of the order of a and thus drops out. When a component p is near π/a, the addition increases the mass of the unwanted state by $2r/a$:

$$m + \frac{r}{a}\sum_\nu(1-\cos ap_\nu) = m + \frac{2rn_\pi}{a}\,, \quad (41.49)$$

where the sum ν runs over $ap_\nu = \pi$, and n_π is the number of extra particles. Therefore, in the continuum limit, all extra states have infinite mass and then decouple. Only one species of *physical particle* mass m survives for $ap_\mu = 0$. However, it was shown [493] that the propagator in Eq. (41.48) breaks chiral invariance. One hopes that, working with Wilson fermions, one can recover chiral symmetry in the continuum limit.

41.4 Quark and gluon interactions

Now, one can formulate the quark and gluon interactions on the lattice. In the case of Abelian theory:

$$\begin{aligned}\mathcal{L}_{\rm free} + \mathcal{L}_{A\psi} &= m\bar{\psi}_n\psi_n \\ &\quad + \frac{1}{2a}\bar{\psi}_n\sum_\mu\gamma_\mu[U(n,n+\mu)\psi_{n+\mu} - U(n-\mu,n)\psi_{n-\mu}]\,, \end{aligned} \quad (41.50)$$

which is invariant under the gauge transformation in Eq. (41.9) of the link matrices ($g\lambda_a/2 \equiv e$ electric charge). Using the expansions:

$$\lim_{a\to 0} U(n,n+\mu) = 1 - iagA_\mu + \mathcal{O}(a^2)\,, \quad (41.51)$$

and:

$$\lim_{a\to 0}\psi(n+\mu) = \psi(n) + a\partial_\mu\psi(n) + \mathcal{O}(a^2)\,, \quad (41.52)$$

it is easy to show that the previous Lagrangian gives the correct continuum limit:

$$\lim_{a\to 0}\{\mathcal{L}_{\rm free} + \mathcal{L}_{A\psi}\} = m\bar{\psi}(x)\psi(x) + \bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - ie\bar{\psi}(x)\gamma^\mu A_\mu\psi(x)\,. \quad (41.53)$$

In QCD, the interaction between quarks and gluons can be introduced as in the Abelian case. For Wilson fermions, the action reads:

$$\mathcal{S}_{gq} = a^4 \sum_n \bar{\psi}_n \left(m + \frac{4r}{a}\right) \psi_n + \frac{1}{2a} \sum_{n,\mu} \bar{\psi}_n [(r+\gamma_\mu) U(n, n+\mu) \psi_{n+\mu} + (r-\gamma_\mu) U^{-1}(n-\mu, n) \psi_{n-\mu}] \,. \tag{41.54}$$

The continuum limit of the action can also be obtained:

$$\lim_{a\to 0} \mathcal{S}_{gq} = a^4 \left\{ m \sum_n \bar{\psi}_n \psi_n + \frac{1}{2} \sum_{n,\nu} [\bar{\psi}_n \gamma_\mu \partial^m u \psi_n - \partial^\mu \bar{\psi}_n \gamma_\mu \psi_n - ig\bar{\psi}_n \gamma_\mu \psi_n A^\mu] + \frac{r}{2a} \partial_\mu [\bar{\psi}_n \gamma^\mu \psi_n] \right\} . \tag{41.55}$$

The last term vanishes after summation over n (integration over x), such that the continuum limit reproduces the usual QCD action in Eq. (41.4). Therefore, the corresponding full generating functional for Wilson fermions is:

$$Z = \int \mathcal{D}U \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, e^{-(\mathcal{S}_g + \mathcal{S}_{gq})} \,, \tag{41.56}$$

where the measures have been defined in Eqs. (41.25) and (41.37). One should notice that unlike the continuum case, the gauge-fixing term is not necessary to obtain some vacuum expectation values (except the gluon propagator or some gauge-dependent quantities), as Eq. (41.56) averages over all gauges. In order to define the Green's functions, one has to define the integration over the Grassmann variables. which obey the following general properties:

$$\int d\eta_1 d\eta_2 \cdots d\eta_n \; (\eta_1 \eta_2 \cdots \eta_n) = 1, \text{ all other integrals are zero.} \tag{41.57}$$

For instance, one has:

$$\int d\eta_1 d\eta_2 \eta_1 \eta_2 = 1 = -\int d\eta_1 d\eta_2 \eta_2 \eta_1 \,,$$
$$\int d\eta_1 d\eta_2 \eta_2 = 0 = \int d\eta_1 \eta_2 \,. \tag{41.58}$$

With the previous properties, any analytic function of the Grassmann variables can be integrated. This can be done by Taylor-expanding it and then by applying Eq. (41.57). For instance, the integral over the Grassmann algebra with four generators $\bar{\eta}_j, \eta_j, \; j = 1, 2$ reads:

$$\int \prod_{j=1}^{2} d\eta_j d\bar{\eta}_j \exp\left[\sum_{i,j=1}^{2} \bar{\eta}_i A_{ij} \eta_j\right] = \int \prod_{j=1}^{2} d\eta_j d\bar{\eta}_j [\bar{\eta}_1 A_{11} \eta_1 \bar{\eta}_2 A_{22} \eta_2 + \bar{\eta}_1 A_{12} \eta_2 \bar{\eta}_2 A_{21} \eta_1] = A_{11} A_{22} - A_{12} A_{21} = \det A \,. \tag{41.59}$$

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Collecting the different results in the previous section, the vacuum expectation value of a function of the gauge and fermion fields is defined as:

$$\langle F[A_m u^a(p), \bar{\psi}(n)\psi(k)]\rangle = \frac{1}{\mathcal{N}}\int \mathcal{D}U\mathcal{D}\bar{\psi}^c_\alpha(n)\mathcal{D}\psi^c_\alpha(n)F[A_m u^a(p), \bar{\psi}(n)\psi(k)]e^{-\mathcal{S}_{\rm latt}} \tag{41.60}$$

with:

$$\mathcal{N} = \int \mathcal{D}U\mathcal{D}\bar{\psi}^c_\alpha(n)\mathcal{D}\psi^c_\alpha(n)e^{-\mathcal{S}_{\rm latt}} \ : \qquad \mathcal{S}_{\rm latt} = \mathcal{S}_g + \mathcal{S}_{gq}\ . \tag{41.61}$$

From the lattice action, one can, for example, derive different Feynman rules on the lattice. For example, the propagators can be obtained from the quadratic terms of the fields entering into the action. The quark propagator has been already given in the previous section (see e.g. Eq. (41.48) for the Wilson fermion). In the Feynman gauge, the gluon propagator is:

$$D^{cb}_{\mu\nu}(p) = \delta^{cb}\delta_{\mu\nu}\frac{1}{2a^{-2}\sum_\rho(1-\cos ap_\rho)}\ . \tag{41.62}$$

Feynman rules for the vertices are more involved as the interactions are non-polynomial functions of the fields, and there are infinite numbers of vertices associated with higher powers of the lattice spacing a. More discussions can be found in [494].

41.5 Some applications of the lattice

A large spectrum of the lattice applications can be found in the different references given in the introduction of this chapter. Here, we shall limit with very few examples as an illustration of the method.

41.5.1 The QCD coupling and the weak coupling regime

We have noticed that for finite a, QCD on the lattice is UV finite, such that we do not worry to distinguish between bare and renormalized quantities. Hower, for $a \to 0$, loop diagrams become divergent in the *weak coupling limit*, and the lattice can be considered as a regularization procedure with the cut-off $1/a \to \infty$. To leading order of pQCD, the QCD coupling reads:

$$g^2(a) = \frac{4\pi^2}{\beta_1 \log \Lambda_{\rm latt} a}\ . \tag{41.63}$$

The scale $\Lambda_{\rm latt}$ can be related to the one of the $\overline{MS}$ scheme by simply evaluating one-loop renormalization for α_s, including constant terms using the two different schemes and by equating. The lattice calculation has been done in [495] but is quite cumbersome due to the peculiarity of the lattice regularization (Lorentz invariance, ...). For $n_f = 0$ fermions, one

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obtains to one loop:

$$\Lambda_{\rm latt} \simeq \frac{\Lambda^0_{\rm mom}}{83.5} \simeq \frac{\Lambda^0_{\overline{MS}}}{39} . \tag{41.64}$$

The present values are (see previous chapters and [16]):

$$\Lambda^0_{\overline{MS}} \approx 400 \text{ MeV} \Longrightarrow \Lambda_{\rm latt} \approx 10 \text{ MeV} , \tag{41.65}$$

showing that $\Lambda_{\rm latt}$ has a very small value. From Eq. (41.63), one can also derive the leading-order relation between the lattice spacing a and $\Lambda_{\rm latt}$:

$$a = \Lambda^{-1}_{\rm latt} e^{\frac{4\pi^2}{\beta_1 g^2(a)}} \tag{41.66}$$

valid for small a and for weak coupling:

$$a\Lambda_{\rm latt},\ g^2(a) \ll 1 . \tag{41.67}$$

In order to check if one has reached the continuum limit from the numerical analysis, one should see if the lattice results behave as predicted by the renormalization group equation.

41.5.2 Wilson loop, confinement and the strong coupling regime

Here one considers the Green's function of a pair of a static infinitely heavy ($m \to \infty$) quark and anti-quark at lattice points j and $j + n\mu$. A gauge-invariant function of such a state is given by:

$$J(k) = \bar\psi_k U(k, k+\nu) \cdots U(k + (n-1)\nu, k + n\nu)\psi(k + n\nu) . \tag{41.68}$$

Its propagation in the Euclidian space–time is described by the Green's function:

$$G(k,l) = \langle J(l)^\dagger J(k)\rangle , \tag{41.69}$$

where the lattice point l is displaced with respect to k by r units in four-direction. Since, in the action, the fermionic variables $\bar\psi\psi$ occur quadratically, hence the integration is Gaussian, such that the integration over the fermion fields will not pose (in principle) any problem. It is possible to show that for $m \to \infty$, the Green's function behaves as:

$$G(k,l) \sim \left(\frac{p}{m}\right)^{2n} \langle Tr W[L]\rangle_U \ : \qquad p \equiv a^3/2 \tag{41.70}$$

where $W(L)$ is the rectangular Wilson loop with corners $k, k+\nu, l, l+\nu$, and $\langle\ldots\rangle_U$ corresponds to the vacuum expectation value in Eq. (41.24) over the gauge field U. One can sketch the derivation of this result by considering the integration over the fermion fields at the point k. In the integrand one has from Eq. (41.68) the term $\bar\psi(k)$. The integral will not vanish if one has an additional factor $\psi(k)$, which one can obtain by expanding the action $e^{-\mathcal{S}_{\rm latt}}$. This expansion leads, among others, to the term: $p\bar\psi(k+\mu)(r+\gamma_\mu)U^{-1}(k, k+\mu)\psi(k)$. After fermion integration at the point k, the fermion field $\bar\psi(k)$ is no longer present, but now we have a fermion field at the position $k + \mu$ and the previous factor $p \ldots$. We thus hopped with the fermion field from k to $k+\mu$, such that we may hop from k to j, and from $j+\nu$ to $k+\nu$. For other points, we need to expand the mass term in $e^{-\mathcal{S}_{\rm latt}}$, which yields a factor m for each points. The final factor $(p/m)^{2n}$ and the group elements U attached to the links of the loop are obtained after dividing by the normalization factor $\mathcal{N}$.

On the other hand, one knows that the Wilson loop measures the response of the gauge fields to an external quark-like source passing around its perimeter. For a timelike loop, this represents the production of a quark pair at the earliest time, moving them along the world lines dictated by the sides of the loop, and then annihilating at the latest time. If the loop is a rectangle of dimensions T and R, a transfer matrix argument suggests that for large T:

$$\lim_{T\to\infty}\langle W[L]\rangle_U = -\exp[-E(R)T] \tag{41.71}$$

where $E(R)$ is the static quark–anti-quark energy separated by a distance R. In the *strong coupling regime* $1/g^2 \to 0$, one obtains to leading order:

$$\langle W[L]\rangle_U \sim \left(\frac{1}{g}\right)^{RT/a^2}, \tag{41.72}$$

showing that in that approximation the static energy of the quarks increases linearly with the spatial distance R:

$$\lim_{R\to\infty} E(R) = \sigma R\ , \tag{41.73}$$

where σ is called *the string tension* and characterizes long-distance physics effects. Therefore a separation of the two quarks would need infinite energy. Unfortunately, this result is also obtained for Abelian theory. Since we do not observe confinement in QED, we have to assume that there is a phase transition between the confining phase in the strong-coupling regime and the deconfined phase in the weak-coupling regime. There is no formal proof that such a transition does not exist in non-Abelian QCD. A numerical evaluation of the expectation value $\langle W[L]\rangle_U$ indicates that the area law in Eq. (41.72) is also verified for weak coupling, strongly indicating that *confinement is a consequence of the QCD-Lagrangian*. Phenomenologically, the string tension can be related to the slope of the Regge trajectory if one uses a string model for describing the hadrons [496]:

$$\alpha' = (2\pi\sigma)^{-1}\ , \tag{41.74}$$

where using the phenomenological value $\alpha' \simeq 1\,\text{GeV}^{-2}$, one finds:

$$\sigma \simeq (400\ \text{MeV})^2\ . \tag{41.75}$$

Using the previous equations, one can notice that this quantity is proportional to the QCD coupling g^2, i.e. $\Lambda_{\rm lattice}$. This is a remarkable feature as one is able to relate a long-distance (σ) to a short-distance ($\Lambda_{\rm latt}$) quantities.

41.5.3 Some other applications and limitations of the lattice

Some observables like hadron masses, . . . can also be obtained by calculating numerically Green's functions of interpolating fields on the lattice. In so doing, let us consider the vector current:

$$J_\mu(x) = \bar{\psi}(k)\gamma_u\psi(k) , \tag{41.76}$$

which has the quantum number of the ρ-meson. After a rotation in the Euclidian space–time, the two-point correlator reads (we omit indices for simplicity):

$$\Pi(T) \equiv \langle J(T)J(0)\rangle = \langle J(0)e^{-HT}J(0)\rangle \tag{41.77}$$

Inserting a complete set of energy eigenstates and taking the large T limit, one may select the lowest ground state ρ-meson contribution:

$$\Pi(T) = \sum_n |\langle J(0)|n\rangle|^2 e^{-E_n T} \to |\langle J(0)|0\rangle|^2 e^{-E_0 T} , \tag{41.78}$$

where E_0 is equal to the ρ-meson mass M_ρ. In this way, one can recover the whole hadron spectrum, . . . However, in practice, there are many difficulties and questions which the *lattice experimentalists* should clearly answer. Besides the usual statistical and finite size (about 1% if the lattice size $L \geq 3$ fermi, and $m_\pi L \geq 6$) errors inherent to the numerical lattice calculations, which can be minimized using modern technology, there are still large uncertainties related to the uses of field theory on the lattice.

- When one approximates the functional integral by a product of Riemann integrals, when do we reach the continuum limit ? The renormalization group analysis shows that one should expect an exponential dependence of the lattice spacing on the coupling constant. This can be reached if the lattice spacing a is relatively small like the coupling g.
- However, if the lattice spacing a is small say a fraction of a fermi, the lattice should be large enough in order to accommodate a hadron of a typical size of one fermi. Therefore, the lattice should at least have 4×10^4 lattice points. Since for $SU(3)$, we have, for each lattice point, eight groups of integrations and 24 fermionic integrations, it is clear that one needs very sophisticated integraltion methods. However, even with these sophisticated integration methods, one has to do some approximations, as an exact evaluation of the fermionic integrals are not possible with most of the present computers.
- In the case of (*quenched approximation*), one ignores quark loops, thus simplifying the evaluation of the integral, but with a brutal non-inclusion of the fermion determinant into the action. This implies a modification of chiral symmetry (χS) for $m_q = 0$ as well as the disappearance of the QCD anomaly: $M_{\eta'} \approx m_\pi$. At present, some progress towards including active quark flavours has been achieved by some groups.
- Another obstacle is the small values of the light quark masses. Generally, one evaluates the Green's functions at large mass and then extrapolates the results to zero quark mass values with the help of the mass dependence expected from chiral perturbation theory (ChPT) (see next section). For a typical value of the lattice spacing $1/a \simeq 2$ GeV, and keeping the condition $m_\pi L \geq 6$, one requires $L/a \geq 90$ in order to avoid finite volume effects. At present the lattice size L/a is about 32 (quenched) and about 24 (unquenched) which is far below this limit.

- There are also discretization errors specific to each lattice actions, which are $\mathcal{O}(a)$ for the Wilson (explicit breaking of χS) and domain walls (extra fifth dimension for preserving χS) actions. The errors are $\mathcal{O}(a^2)$ for the staggered (reduction of quark couplings with high-momenta gluons) and $\mathcal{O}(a\alpha_s)$ for the Clover (inclusion of the mixed quark-gluon operator) actions.
- There are also errors due to the mixing of different operators at finite a.
- How good is the separation of the ground state from the rest of the spectra in the large Euclidian time limit if the mass splitting between the ground state and the first radial excitation is accidentally small?

The list of difficulties which we have given is not exhaustive but lattice experts know all of them completely. These difficulties will have to be resolved before reliable lattice results on the hadron and QCD parameters, will be available. We hope that such difficulties can be solved gradually in the future. However, it is unfortunate that most non-lattice experts and especially experimentalists *blindly* use the present lattice results without asking about their reliability, although this is, however, difficult to quantify by non-experts in the field. Some lattice results will be presented in subsequent chapters as a comparison with the QCD spectral-sum rules results.