

10

Lattice gauge theory

Perturbation theory applied to QCD predicts that the normally strong interactions among quarks and gluons become weak at high temperatures and densities on account of asymptotic freedom. This leads to a state known as quark–gluon plasma. The perturbative analysis of QCD was the subject of the last two chapters. At low temperatures and densities quarks and gluons are not observed individually but only as color-neutral objects, hadrons, on account of confinement. Then hadrons are the relevant degrees of freedom, just as atoms and molecules are the relevant degrees of freedom in biological physics. Nuclear matter and hot hadronic matter are the subjects of the next two chapters. The standard computational method for studying QCD in the transitional region is lattice gauge theory.

Lattice gauge theory is a field of intellectual study in itself. It is not possible in one chapter to cover it in all detail, not the least reason being that it is numerically quite involved. We will introduce the basic theoretical ideas and the main numerical results. As the field is evolving owing to rapid increases in computational power, these results will no doubt be superseded in the near future. Nevertheless, the main conclusions should stand the test of time.

The formulation of nonabelian gauge theories on a spacetime lattice in Euclidean space was introduced by Wilson [1] with the purpose of studying quark confinement. The infinite-dimensional functional integral that defines a quantum field theory becomes a finite-dimensional integral when the lattice has a finite extent in space and time and is therefore unambiguously defined. It is natural to expect that there is a unique continuum limit when the lattice spacing a goes to zero, at least for asymptotically free theories. The argument is that the bare coupling $g(a)$ becomes small in this limit, and the long-distance properties of the theory should be insensitive to the details of the ultraviolet cutoff introduced by the lattice.

After Creutz [2] demonstrated that the functional integral of lattice gauge theory could be evaluated with the help of Monte Carlo numerical techniques, lattice gauge theory became the method of choice for the calculation of observables that are beyond the reach of perturbation theory. This includes most properties of individual hadrons as well as their interactions at low energies.

It is natural to apply the lattice technique to the study of deconfinement and chiral symmetry restoration at finite temperature. The first such studies were made by Polyakov [3] and by Susskind [4] using the Hamiltonian formulation of lattice gauge theory. They could show that deconfinement disappears at high temperature but were unable to prove that this phenomenon persists in the continuum limit, nor could they compute a critical temperature. This was first achieved in SU(2) gauge theory by means of numerical calculations by McLerran and Svetitsky [5] and by Kuti, Polónyi, and Szlachányi [6].

10.1 Abelian gauge theory

As a warm-up to full QCD let us consider how to define a Hamiltonian on a discrete spatial lattice for a pure gauge theory that has Abelian gauge theory as its continuum limit. The procedure is not unique since one can always add terms to the discretized theory which vanish in the limit that the lattice spacing goes to zero. In fact, this arbitrariness can be both a boon and a bane, as we shall see.

Consider a cubic lattice with spacing a . Label each *site* of a lattice with a vector $\mathbf{x} = (x_1, x_2, x_3)$. There are six unit lattice vectors: $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_{-1}, \mathbf{n}_{-2}, \mathbf{n}_{-3}$, where \mathbf{n}_1 points in the positive x_1 direction, \mathbf{n}_{-1} points in the negative x_1 direction, and so on. A directed *link* is defined by the pair of vectors (\mathbf{x}, \mathbf{n}) ; it starts from the site \mathbf{x} and goes to the neighboring site $\mathbf{x} + a\mathbf{n}$ (see Figure 10.1). The lattice may be finite or infinite in extent, meaning that there are either a finite or a countably infinite number of degrees of freedom. A continuum field theory, by contrast, has an uncountably infinite number of degrees of freedom no matter whether the box is finite or infinite in extent.

It is natural to associate links with dynamical degrees of freedom. Let us define this degree of freedom to be

$$U(\mathbf{x}, \mathbf{n}) = \exp[i\phi(\mathbf{x}, \mathbf{n})] \quad (10.1)$$

Each link has a mate: $(\mathbf{x}, \mathbf{n}) \Leftrightarrow (\mathbf{x} + a\mathbf{n}, -\mathbf{n})$. The link and its mate should not have independent degrees of freedom associated with them, and so it is natural to require that

$$U(\mathbf{x} + a\mathbf{n}, -\mathbf{n}) = U^\dagger(\mathbf{x}, \mathbf{n}) \quad (10.2)$$

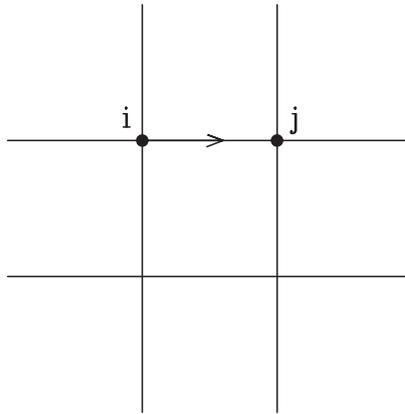


Fig. 10.1. Neighboring lattice sites i and j connected by a directed link.

and so

$$\phi(\mathbf{x} + a\mathbf{n}, -\mathbf{n}) = -\phi(\mathbf{x}, \mathbf{n}) \quad (10.3)$$

It is also natural to associate each link with an electric flux $E(\mathbf{x}, \mathbf{n})$. Clearly we should require that

$$E(\mathbf{x} + a\mathbf{n}, -\mathbf{n}) = -E(\mathbf{x}, \mathbf{n}) \quad (10.4)$$

The theory is quantized by demanding that E be the momentum canonically conjugate to the variable ϕ :

$$[\phi(\mathbf{x}, \mathbf{n}), E(\mathbf{x}, \mathbf{n})] = i \quad (10.5)$$

Since ϕ is an angle, E has integers for its spectrum of eigenvalues. We are dealing with a compact $U(1)$ gauge theory.

The electric contribution to the Hamiltonian H_{electric} must be proportional to

$$\sum_{\text{links}} \frac{E^2}{2a}$$

where the factor $1/a$ gives the term the proper dimension, and the factor one-half is inserted for the conventional reasons of normalization.

The magnetic field energy is less obvious. The independent degrees of freedom have already been defined and so it must be expressed in terms of them. The coordinate ϕ is the natural starting point. One defines a *plaquette* Γ to be a square made of four links connected head to tail (see Figure 10.2). The variable U associated with this plaquette is

$$U(\Gamma) = U(i)U(j)U(k)U(l) = \exp \{i[\phi(i) + \phi(j) + \phi(k) + \phi(l)]\} \quad (10.6)$$

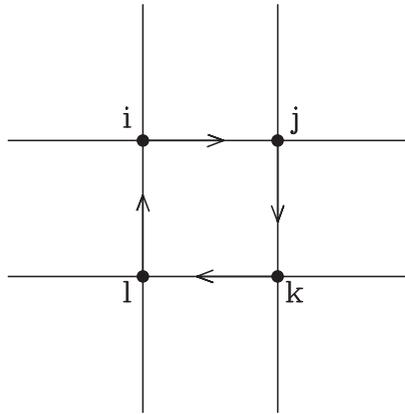


Fig. 10.2. A plaquette.

where the four sides of the plaquette are labeled $ijkl$, with the head of l connected to the tail of i . By going around in a closed loop like this, one will obtain a curl in the limit $a \rightarrow 0$.

The Hamiltonian is defined to be

$$H = \sum_{\text{links}} \frac{g^2 E^2}{2a} - \sum_{\text{plaquettes}} \frac{1}{2ag^2} [U(\Gamma) + U^\dagger(\Gamma) - 2] \quad (10.7)$$

Notice that a dimensionless coupling constant g has been used in this definition of H . The fact that the coefficient of the electric and magnetic field energies can depend on g^2 should not be surprising. It does represent the strength of interactions because the Hamiltonian is not quadratic in the independent dynamical variables ϕ but has terms to all orders in ϕ . The particular normalization is chosen, with hindsight, to reproduce the Abelian theory in the continuum limit.

To calculate the continuum limit of the magnetic energy, consider a single plaquette with corners labeled $abcd$. The quantity

$$U_{ab} = e^{i\phi_{ab}} \quad (10.8)$$

is associated with the link ab . Define a vector potential A_i via

$$\phi_{ab} = g(\mathbf{x}_b - \mathbf{x}_a)_i A_i \left(\frac{\mathbf{x}_a + \mathbf{x}_b}{2} \right) \quad (10.9)$$

We do the same for the four links comprising the plaquette, namely, ab , bc , cd , da . In the limit that the lattice spacing becomes very small we can

expand the exponentials in a power series:

$$\begin{aligned}
 H_{\text{magnetic}}(\Gamma) &= -\frac{1}{2ag^2} \left[U(\Gamma) + U^\dagger(\Gamma) - 2 \right] \\
 &= -\frac{1}{2ag^2} \left[i(\phi_{ab} + \phi_{bc} + \phi_{cd} + \phi_{da}) \right. \\
 &\quad \left. - \frac{1}{2}(\phi_{ab} + \phi_{bc} + \phi_{cd} + \phi_{da})^2 + \dots + \text{c.c.} \right] \\
 &\approx \frac{1}{2a} \left[(\mathbf{x}_b - \mathbf{x}_a)_i A_i \left(\frac{\mathbf{x}_a + \mathbf{x}_b}{2} \right) + (\mathbf{x}_c - \mathbf{x}_b)_i A_i \left(\frac{\mathbf{x}_b + \mathbf{x}_c}{2} \right) \right. \\
 &\quad \left. + (\mathbf{x}_d - \mathbf{x}_c)_i A_i \left(\frac{\mathbf{x}_c + \mathbf{x}_d}{2} \right) + (\mathbf{x}_a - \mathbf{x}_d)_i A_i \left(\frac{\mathbf{x}_a + \mathbf{x}_d}{2} \right) \right]^2
 \end{aligned}
 \tag{10.10}$$

Now Taylor-expand the vector potentials about the center of the plaquette. For example, if the link ab points in the direction \mathbf{n}_2 and the link bc points in the direction \mathbf{n}_1 then

$$A_i \left(\frac{\mathbf{x}_a + \mathbf{x}_b}{2} \right) \approx A_i \left(\frac{\mathbf{x}_a + \mathbf{x}_b + \mathbf{x}_c + \mathbf{x}_d}{4} \right) - \frac{1}{2} a \frac{\partial A_i}{\partial x_1} \left(\frac{\mathbf{x}_a + \mathbf{x}_b + \mathbf{x}_c + \mathbf{x}_d}{4} \right)
 \tag{10.11}$$

and similarly for the other terms. The result is that the magnetic Hamiltonian for this plaquette is

$$H_{\text{magnetic}}(\Gamma) = \frac{1}{2} a^3 \left(\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} \right)^2
 \tag{10.12}$$

which is proportional to the square of the third component of the curl of the vector potential, otherwise known as the magnetic field.

Summing over all plaquettes results in the continuum limit for the magnetic part of the Hamiltonian,

$$H_{\text{magnetic}} = \frac{1}{2} \int d^3x \mathbf{B}^2(\mathbf{x})
 \tag{10.13}$$

since $a^3 \sum_{\mathbf{x}} \rightarrow \int d^3x$ in the continuum limit. The physical electric field \mathbf{E} associated with the link ab is defined to be $g\mathbf{n}_2 E(\mathbf{x}_a, \mathbf{n}_2)/a^2$, which has both the correct dimensions and direction. The continuum limit of the electric part of the Hamiltonian is therefore

$$H_{\text{electric}} = \frac{1}{2} \int d^3x \mathbf{E}^2(\mathbf{x})
 \tag{10.14}$$

The resulting continuum theory is a free-field theory in the absence of electric charges. Note, however, that the original lattice theory is a fully

interacting theory with interactions to all orders in the vector potential. The range of these interactions is of the order of the lattice spacing a .

Since this is a Hamiltonian formulation, only physical states obeying Gauss's law should be included when calculating the partition function:

$$Z(\beta) = \sum_{\substack{\text{physical} \\ \text{states } \psi}} \langle \psi | e^{-\beta H} | \psi \rangle \quad (10.15)$$

These states $|\psi\rangle$ should satisfy

$$\sum_{\mathbf{n}} E(\mathbf{x}, \mathbf{n}) |\psi\rangle = 0 \quad (10.16)$$

for each site \mathbf{x} , assuming that there are no electric charges in the system. To impose Gauss's law, we insert a factor

$$\delta \left(\sum_{\mathbf{n}} E(\mathbf{x}, \mathbf{n}) \right) = \int_{-\pi}^{\pi} \frac{d\alpha(\mathbf{x})}{2\pi} \exp \left(i\alpha(\mathbf{x}) \sum_{\mathbf{n}} E(\mathbf{x}, \mathbf{n}) \right) \quad (10.17)$$

at each site. This will take care of the restriction to physical states automatically.

Let us study the theory in the strong-coupling limit, $g^2 \gg 1$. This is the extreme opposite of the weak-coupling limit, where perturbation theory can be applied. In strong coupling we can drop the magnetic energy and keep only the electric. Imposing Gauss's law by use of the Dirac delta function leads to the expression

$$Z = \prod_{\mathbf{x}} \int_{-\pi}^{\pi} \frac{d\alpha(\mathbf{x})}{2\pi} \prod_{\text{links at } \mathbf{x}} \left(\sum_E \exp \left\{ -\frac{\beta g^2}{2a} E^2(\mathbf{x}, \mathbf{n}) + i[\alpha(\mathbf{x}) - \alpha(\mathbf{x} + \mathbf{n})] E(\mathbf{x}, \mathbf{n}) \right\} \right) \quad (10.18)$$

Here and from now on E represents the eigenvalues (integers) of the operator. To understand the nature of the strong-coupling phase, insert a pair of static immobile charges, one of charge g located at $\mathbf{x} = \mathbf{0}$ and the other of charge $-g$ located at $\mathbf{x} = \mathbf{R}$. Then Gauss's law becomes

$$\begin{aligned} \sum_{\mathbf{n}} E(\mathbf{0}, \mathbf{n}) &= 1 \\ \sum_{\mathbf{n}} E(\mathbf{R}, \mathbf{n}) &= -1 \\ \sum_{\mathbf{n}} E(\mathbf{x}, \mathbf{n}) &= 0 \quad \text{for } \mathbf{x} \neq \mathbf{0}, \mathbf{R} \end{aligned} \quad (10.19)$$

This leads to an extra factor in Z of $e^{i\alpha(\mathbf{0})} e^{-i\alpha(\mathbf{R})}$:

$$Z(\beta, \mathbf{R}) = Z(\beta) \left\langle e^{i\alpha(\mathbf{0})} e^{-i\alpha(\mathbf{R})} \right\rangle \quad (10.20)$$

The free energy of this configuration is

$$\Delta F(\beta, \mathbf{R}) = -[T \ln Z(\beta, \mathbf{R}) - T \ln Z(\beta)] = -T \ln \left\langle e^{i\alpha(\mathbf{0})} e^{-i\alpha(\mathbf{R})} \right\rangle \tag{10.21}$$

First consider the low-temperature limit, $\beta g^2/2a \gg 1$. Then only the eigenvalues $E = 0, \pm 1$ matter and

$$\begin{aligned} & \left\langle e^{i\alpha(\mathbf{0})} e^{-i\alpha(\mathbf{R})} \right\rangle \\ &= \prod_{\mathbf{x}} \int_{-\pi}^{\pi} \frac{d\alpha(\mathbf{x})}{2\pi} \prod_{\text{links at } \mathbf{x}} \left\{ e^{i[\alpha(\mathbf{0}) - \alpha(\mathbf{R})]} + e^{-\beta g^2/2a} e^{i[\alpha(\mathbf{x}) - \alpha(\mathbf{x}+\mathbf{n}) + \alpha(\mathbf{0}) - \alpha(\mathbf{R})]} \right. \\ & \quad \left. + e^{-\beta g^2/2a} e^{i[-\alpha(\mathbf{x}) + \alpha(\mathbf{x}+\mathbf{n}) + \alpha(\mathbf{0}) - \alpha(\mathbf{R})]} \right\} \end{aligned} \tag{10.22}$$

The first of the three exponentials integrates to zero. So do the second and third, except for those paths that connect the two charges. For simplicity, choose \mathbf{R} to lie on an axis running through the origin. Then

$$\left\langle e^{i\alpha(\mathbf{0})} e^{-i\alpha(\mathbf{R})} \right\rangle = 2 \left(e^{-\beta g^2/2a} \right)^{N_{\text{links}}(0, R)} = 2e^{-\beta g^2 R/2a^2} \tag{10.23}$$

where $N_{\text{links}}(0, R) = R/a$ is the number of links connecting the charges. Therefore

$$\Delta F = \frac{g^2}{2a^2} R \tag{10.24}$$

The potential energy is linear and thus confining.

The high-temperature limit, $\beta g^2/2a \ll 1$, is left as an exercise. It should be no surprise that the answer is a Coulombic potential,

$$\Delta F = -\frac{g^2}{R} \tag{10.25}$$

Since the low- and high-temperature limits have completely opposite behavior, one should expect a phase transition separating them. The critical temperature is estimated as $\beta_c g^2/2a \approx 1$ or

$$T_c \approx \frac{g^2}{2a} \tag{10.26}$$

This depends very strongly on the lattice spacing. In fact, since this is a quantum field theory, quantum corrections will cause the effective coupling constant to depend on a , namely, $g^2(a)$ will replace g^2 in the estimate for the critical temperature. This being an Abelian theory it does not have the property of asymptotic freedom. Therefore $g^2(a)$ will grow with decreasing a . Hence T_c will grow without bound as $a \rightarrow 0$. The

low-temperature confining phase does not smoothly extrapolate to the continuum limit but is separated from it by a phase transition; confinement exists only in the discretized lattice version of the theory. This is well and good since we know that QED is not a confining theory.

10.2 Nonabelian gauge theory

Both QCD and electroweak theory involve nonabelian gauge groups, SU(3) in the former case and SU(2) in the latter. Essentially all modern numerical calculations in these theories use the Lagrangian formulation, not the Hamiltonian one. Calculations are done on a finite discrete lattice of volume $V = L^3$, with

$$L = N_s a \quad (10.27)$$

where a is the lattice spacing and N_s is the number of sites in each of the three spatial directions. The imaginary time variable is also discrete: $0 \leq \tau \leq \beta$ as usual with

$$\beta = \frac{1}{T} = N_\tau a \quad (10.28)$$

where N_τ is the number of sites in the imaginary time direction. The unit directional vectors \mathbf{n} in three spatial dimensions must be extended to unit directional vectors n_α in four Euclidean dimensions. It is then convenient to define $x_4 = \tau$. The lattice spacings in the space and time directions need not be the same, and sometimes they are chosen differently, but equal spacing is the norm.

The notions of site, link, and plaquette all carry over from the lattice version of the Abelian theory. The generalization of the link variable from U(1), as given in (10.1), to SU(N) is straightforward, but for definiteness we specialize to SU(2) for the rest of this section:

$$U(x; n_\alpha) = \exp[ia\sigma_j A_\alpha^j(x)] = u_4 I + \boldsymbol{\sigma} \cdot \mathbf{u} \quad (10.29)$$

Here the link begins at the site $x = (\mathbf{x}, x_4)$ and goes in the direction n_α . The σ_j with $j = 1, 2, 3$ are the Pauli matrices while I is the identity matrix. These link variables are elements of the group SU(2). In the continuum limit the A_α^j will be identified as $1/g$ times the four-vector potential. (It is conventional to factor out the coupling constant.) Compare with (10.9). By the definition of the link variables there is a constraint

$$u_4^2 + \mathbf{u}^2 = 1 \quad (10.30)$$

This is a compact gauge group.

The action should be defined so that (i) it reduces to the continuum expression, (ii) it is gauge invariant even on the lattice, and (iii) it is as

simple as possible. Requirement (iii) means that an infinite number of extra terms that all vanish in the continuum limit $a \rightarrow 0$ could be added. Actually it could be advantageous to add such extra terms if it means that the continuum limit is approached more rapidly and hence more efficiently in terms of computer time and memory. This goes under the title of improved actions, and will be discussed in Section 10.4. Motivated by the lattice action for the Abelian theory, the simplest possible action for SU(2) is

$$S(U) = \frac{4}{g^2} \sum_{\text{plaquettes } abcd} \left(1 - \frac{1}{2} \text{Tr } U_{ab} U_{bc} U_{cd} U_{da}\right) \quad (10.31)$$

That this reduces to the proper continuum action is left as an exercise. This action is invariant under the gauge transformation

$$U(x; n_\alpha) \rightarrow V(x) U(x; n_\alpha) V^{-1}(x + a n_\alpha) \quad (10.32)$$

where

$$V(x) = \exp[i a \sigma_j \Lambda_j(x)] \quad (10.33)$$

This invariance is obvious at a glance.

The functional integral expression for the partition function involves integration over all possible field configurations:

$$Z = \int \prod_{\text{links } ab} dU_{ab} \exp[-S(U)] \quad (10.34)$$

When integrating over the link variables U it must be remembered that they are unitary matrices in the group SU(2) and therefore one must use the appropriate Haar measure. One could integrate over the u_0 and \mathbf{u} subject to the constraint (10.30), or one could integrate over three angles in four-dimensional Euclidean space.

At this point perturbation theory could be used to compute physical observables on the finite lattice at finite temperature. However, it is much more interesting to attempt to evaluate the large but finite-dimensional integral for Z using Monte Carlo techniques. The results of such numerical work are the subject of Sections 10.4 and 10.5.

10.3 Fermions

Introducing fermionic fields on a lattice has been a challenge. The most used techniques result in a multiplication of the number of fermion species in the continuum limit and/or the breaking of chiral symmetry on the lattice when the fermions are massless. Much technical work has been

done to overcome these problems. In this section we introduce the reader to the most commonly used techniques, as originally formulated.

The staggered-fermion approach was invented by Kogut and Susskind [7]. In order to define Dirac fields ψ with finite derivatives in the continuum limit, they introduced two separate two-component spinors residing on alternate lattice sites. In the Hamiltonian formalism on a cubic lattice, a lattice site \mathbf{x} is defined to be even or odd according to whether $s \equiv (x_1 + x_2 + x_3)/a$ is an even or odd integer. The upper two components of a four-component Dirac field, ψ_{upper} , reside on even lattice sites while the lower two components, ψ_{lower} , reside on odd lattice sites. The Hamiltonian for free fermions is taken to be

$$H = \frac{1}{ia} \sum_{\mathbf{x}, \mathbf{n}} \psi^\dagger(\mathbf{x}) \boldsymbol{\sigma} \cdot \mathbf{n} \psi(\mathbf{x} + a\mathbf{n}) + m \sum_{\mathbf{x}} (-1)^s \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) \quad (10.35)$$

Imposition of the canonical commutation relations

$$\left\{ \psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{x}') \right\} = \delta_{\alpha, \beta} \delta_{\mathbf{x}, \mathbf{x}'} \quad (10.36)$$

leads to the equation of motion

$$\begin{aligned} i \frac{\partial \psi(\mathbf{x})}{\partial t} &= [\psi(\mathbf{x}), H] \\ &= \frac{1}{ia} \sum_{\mathbf{n}} \boldsymbol{\sigma} \cdot \mathbf{n} \psi(\mathbf{x} + a\mathbf{n}) + m(-1)^s \psi(\mathbf{x}) \\ &= \frac{1}{2ia} \sum_{\mathbf{n}} \boldsymbol{\sigma} \cdot \mathbf{n} [\psi(\mathbf{x} + a\mathbf{n}) - \psi(\mathbf{x} - a\mathbf{n})] \\ &\quad + m(-1)^s \psi(\mathbf{x}) \end{aligned} \quad (10.37)$$

In the continuum limit the finite differences become derivatives. Remembering that the upper and lower components of ψ reside on even and odd lattice sites, the equation of motion becomes

$$\begin{aligned} i \frac{\partial \psi_{\text{upper}}}{\partial t} &= -i \boldsymbol{\sigma} \cdot \nabla \psi_{\text{lower}} + m \psi_{\text{upper}} \\ i \frac{\partial \psi_{\text{lower}}}{\partial t} &= -i \boldsymbol{\sigma} \cdot \nabla \psi_{\text{upper}} - m \psi_{\text{lower}} \end{aligned} \quad (10.38)$$

This is the Dirac equation for free fermions.

Coupling to the gauge field can be done in such a way as to render the Hamiltonian gauge invariant. Inspection of (10.35) suggests that we rotate the Dirac field according to

$$\psi(\mathbf{x}) \rightarrow V(\mathbf{x}) \psi(\mathbf{x}) \quad (10.39)$$

The kinetic energy term in the free Hamiltonian that involves neighboring sites x and $x + an_\alpha$ requires us to use the parallel-transporter or link

variable U to connect them in a gauge-invariant way:

$$H = \frac{1}{ia} \sum_{\mathbf{x}, \mathbf{n}} \psi^\dagger(\mathbf{x}) \boldsymbol{\sigma} \cdot \mathbf{n} U(\mathbf{x}; \mathbf{n}) \psi(\mathbf{x} + \mathbf{an}) + m \sum_{\mathbf{x}} (-1)^s \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) \tag{10.40}$$

Recalling (10.32) we see immediately that this Hamiltonian is minimally coupled and gauge invariant. Taking the zero-lattice-spacing limit in the usual way reproduces the correct continuum equations.

The staggered-fermion approach of Kogut and Susskind can also be expressed in Lagrangian form and on a lattice. After some work one finds the action

$$S_{\text{fermion}}^{\text{KS}} = \frac{1}{a} \sum_{xx'} \bar{\psi}(x) [D^{\text{KS}}(x, x') + am\delta(x, x')] \psi(x') \tag{10.41}$$

where in this expression ψ has only one Dirac component. The matrix is

$$\begin{aligned} D^{\text{KS}}(x, x') &= \frac{1}{2} \sum_{j=1}^3 \text{sign}(x, j) \left[\delta(x + an_j, x') U(x; n_j) - \delta(x, x' + an_j) U^\dagger(x'; n_j) \right] \\ &+ \frac{1}{2} \left[\delta(x + an_4, x') U(x; n_4) e^{a\mu} - \delta(x, x' + an_4) U^\dagger(x'; n_4) e^{-a\mu} \right] \end{aligned} \tag{10.42}$$

and the sign factor is given by

$$\text{sign}(x, j) = (-1)^{x_4/a} \begin{cases} 1 & \text{if } j = 1 \\ (-1)^{x_1/a} & \text{if } j = 2 \\ (-1)^{(x_1+x_2)/a} & \text{if } j = 3 \end{cases} \tag{10.43}$$

The delta functions appearing in (10.42) are Kronecker not Dirac. A chemical potential μ has also been added. Integrating over the fermion field gives the usual determinant of the operator:

$$Z = \int \prod_{\text{links } ab} dU_{ab} \exp[-S(U)] \det[D^{\text{KS}}(U) + am] \tag{10.44}$$

where the action $S(U)$ is due to the gauge fields alone. If one takes the continuum limit with zero mass one finds not one but four species of fermion. This is an illustration of the fermion doubling (better to say multiplication) problem on the lattice. Sometimes the $(N_f/4)$ th root of the fermion determinant is taken to represent N_f species of fermion; for example, $N_f = 1$ for one species.

Wilson [1] introduced fermions on the lattice in a different way. Every lattice site is associated with a four-component Dirac field. The action is

taken to be

$$S_{\text{fermion}}^{\text{W}} = -\frac{1}{2a} \sum_{x, n_\alpha} \bar{\psi}(x) \gamma_\alpha [U(x; n_\alpha) \psi(x + an_\alpha) - U^\dagger(x - an_\alpha; n_\alpha) \psi(x - an_\alpha)] + m \sum_x \bar{\psi}(x) \psi(x) \quad (10.45)$$

This is gauge invariant and reduces to the correct action in the continuum limit. For example, for free fermions $U = 1$ and the term

$$\frac{1}{2a} [\psi(x + an_\alpha) - \psi(x - an_\alpha)] \rightarrow \partial_\alpha \psi(x) \quad (10.46)$$

in the continuum limit. There is a corresponding matrix $D^{\text{W}}(U)$ that replaces $D^{\text{KS}}(U)$ in (10.44). In this case one finds that the number of species increases by 16 in the continuum limit and when the fermion is massless. The reason is easy to see for free fermions. Instead of the expression (2.94) for $\ln Z$ one finds the replacement ($m = 0$ and $\mu = 0$)

$$\omega_n^2 + \mathbf{p}^2 \rightarrow \frac{1}{a^2} \sin^2(ap_4) + \frac{1}{a^2} \sum_{i=1}^3 \sin^2(ap_i) \quad (10.47)$$

The lattice propagator has poles not only at zero momentum but also at all the corners of the Brillouin zone, namely, $p_j = \pm\pi/a$, $p_4 = \pm\pi/a$. The way out of this is to introduce another term in the action proportional to $a\bar{\psi}\partial_\alpha^2\psi$ that vanishes in the $a \rightarrow 0$ limit. However, with any finite lattice spacing chiral symmetry is broken, and so the chiral condensate cannot serve as an order parameter on the lattice.

Specific calculations with quarks will be reviewed in Section 10.5.

10.4 Phase transitions in pure gauge theory

The best-understood lattice gauge theories are the pure gauge theories without quarks. Extensive numerical calculations have been done for SU(2) and SU(3). Results for the equation of state in the vicinity of T_c for SU(3) are shown in Figure 10.3. The SU(3) theory undergoes a first-order phase transition. It has been found that any SU(N) theory, with N equal to or greater than 3, undergoes a first-order transition [8], while SU(2) undergoes a second-order transition. This was predicted on the basis of universality arguments [9]. The essential degrees of freedom below T_c may be thought of as glueballs, while above T_c they may be thought of as gluons. In either region the degrees of freedom certainly do interact amongst themselves to a greater or lesser extent.

One must ask just how big a lattice ought to be in order to obtain results that are truly representative of the continuum limit for temperatures of the order of one to several hundred MeV. The necessary size

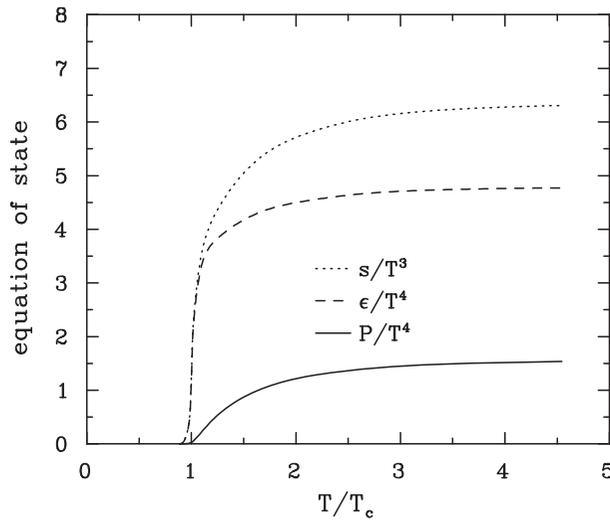


Fig. 10.3. The equation of state of pure SU(3) gauge theory with no quarks. The results shown are extrapolations to the continuum limit, with an estimated uncertainty of order ± 0.1 . The latent heat is about $1.5T_c^4$. The data were taken from [10].

may be estimated as follows. Hadrons, including glueballs, have a spatial extent of the order of 1 fm. Thus the size of the system should be at least 5 to 10 fm on a side in order to contain enough particles that the thermodynamic limit is approximately attained. Because the boundary conditions are usually chosen to be periodic in space, the effective size of the box is somewhat reduced due to surface effects. Therefore we should be conservative and require a box with sides of length 10 fm. At the other end of the scale, a hadron has internal structure characterized by a length of 0.1 fm. If changes in the hadronic structure, such as deconfinement, due to finite temperature are to be seen then the lattice spacing should be no larger than about 0.05 fm. Taken together, this implies that the lattice should be at least 100 to 200 sites per spatial dimension. The temporal dimension has length $\beta = 1/T$. Taking a lattice spacing of 0.05 fm and a temperature of 200 MeV requires about 20 sites in the temporal direction. Numerical calculations with lattices of size up to $N_s = 64$ or 128 and $N_\tau = 16$ or 32 have been done. Extensive work on scaling with system size shows that this is probably large enough to obtain reasonable results.

Rather than going to larger lattices, it can be advantageous to add additional terms to the simplest actions described in the previous sections. These terms vanish in the continuum limit, being higher order in a , but can noticeably improve the approach to the continuum. For example, suppose that the thermal average of some observable has the Taylor series

expansion in the lattice spacing

$$\frac{\langle \mathcal{O} \rangle_{\text{lattice}}}{\langle \mathcal{O} \rangle_{\text{continuum}}} = 1 + \sum_{n=1}^{\infty} c_{2n} a^{2n} \tag{10.48}$$

for the simplest lattice action. By the addition of judicious terms to the action it is possible to cancel the term $c_2 a^2$, bringing about a faster convergence to the continuum limit. Of course, the remaining coefficients are likely to be modified as a result, $c_{2n} \rightarrow c'_{2n}$. The modified coefficients may be larger or smaller or even of opposite sign. One approach is to add six planar link terms to the action, which is then called a tree-level improved 1×2 action. Another approach is to change certain coefficients in the action to correspond to the renormalization group; the action is then termed RG-improved. Yet another approach is to recognize that the coefficients c_{2n} can be expanded in powers of g^2 . The coefficients in the action can be adjusted to make c_{2n} equal to zero to some order in g^2 ; this is the Symanzik improvement program [11].

If there is a deconfinement phase transition then the free energy of a heavy quark–antiquark pair should grow linearly with separation below T_c and be Debye screened above. This free energy can be calculated using the Wilson line

$$W(\mathbf{x}, \beta) = T_\tau \exp \left(i \int_0^\beta d\tau \lambda_a A_4^a(\mathbf{x}, \tau) \right) \tag{10.49}$$

and Polyakov loop

$$L(\mathbf{x}) = \frac{1}{N} \text{Tr} W(\mathbf{x}, \beta) \tag{10.50}$$

Here the λ_a are the Gell-Mann matrices for SU(3) (for SU(2) they would be the Pauli matrices), Tr is the trace with respect to the indices of those matrices, and T_τ denotes time ordering. This is useful because a static, immovable, quark field evolves in imaginary time according to

$$\psi(\mathbf{x}, \tau) = W(\mathbf{x}, \tau) \psi(\mathbf{x}, 0) \tag{10.51}$$

which solves the Dirac equation. The free energy of a system that contains one quark with color index c located at \mathbf{x} and one antiquark with color index c' located at \mathbf{x}' is then determined by

$$\begin{aligned} \exp(-\beta F_{q\bar{q}}) &= \frac{1}{N^2} \sum_{a,a'} \sum_s \langle s | \psi_a(\mathbf{x}, 0) \psi_{a'}^c(\mathbf{x}', 0) \\ &\quad \times \exp(-\beta H) \psi_a^\dagger(\mathbf{x}, 0) \psi_{a'}^{c\dagger}(\mathbf{x}', 0) | s \rangle \\ &= \frac{1}{N^2} \sum_{a,a'} \sum_s \langle s | \exp(-\beta H) \psi_a(\mathbf{x}, \beta) \psi_a^\dagger(\mathbf{x}, 0) \\ &\quad \times \psi_{a'}^{c\dagger}(\mathbf{x}', 0) \psi_{a'}^c(\mathbf{x}', 0) | s \rangle \end{aligned} \tag{10.52}$$

where the superscript c indicates the operation of charge conjugation. The states $|s\rangle$ do not include any quarks; these must be created by the field operators acting on $|s\rangle$. This can be expressed in terms of the Polyakov loop

$$\exp(-\beta F_{q\bar{q}}) = \text{Tr} \left[\exp(-\beta H) L(\mathbf{x}) L^\dagger(\mathbf{x}') \right] \quad (10.53)$$

This is the free energy of the entire system of gluons plus quark and antiquark. To obtain the free energy $\Delta F_{q\bar{q}}$ associated with the quark and antiquark only we must divide by the partition function $Z = \text{Tr} \exp(-\beta H)$ for a system of gluons only, obtaining

$$\exp(-\beta \Delta F_{q\bar{q}}) = \langle L(\mathbf{x}) L^\dagger(\mathbf{x}') \rangle \quad (10.54)$$

The generalization to an assembly of N_q quarks and $N_{\bar{q}}$ antiquarks is straightforward:

$$\exp(-\beta \Delta F_{N_q N_{\bar{q}}}) = \langle L(\mathbf{x}_1) \cdots L(\mathbf{x}_{N_q}) L^\dagger(\mathbf{x}'_1) \cdots L^\dagger(\mathbf{x}'_{N_{\bar{q}}}) \rangle \quad (10.55)$$

The transcription of the Polyakov loop to the lattice is

$$L(\mathbf{x}) = \frac{1}{N} \text{Tr} \prod_{j=0}^{N_\tau-1} U(\mathbf{x}, ja; n_4) \quad (10.56)$$

which is the trace of the product of the U matrices along the time axis.

The link variables U are required to be periodic in time, but the class of allowable gauge transformations is not restricted to those that are periodic. Among them is a special set of gauge transformations that obey

$$V(\mathbf{x}, \beta) = V(\mathbf{x}, 0) e^{i2\pi n/N} \quad (10.57)$$

where n is an integer. The action of the pure gauge theory is invariant too. This is a global $Z(N)$ symmetry. However, the Polyakov loop is changed:

$$L(\mathbf{x}) \rightarrow L(\mathbf{x}) e^{i2\pi n/N} \quad (10.58)$$

and so is the free energy of a system of static quarks and antiquarks:

$$\exp(-\beta \Delta F_{N_q N_{\bar{q}}}) \rightarrow \exp(-\beta \Delta F_{N_q N_{\bar{q}}}) e^{i2\pi n(N_q - N_{\bar{q}})/N} \quad (10.59)$$

Unless $N_q - N_{\bar{q}}$ is an integral multiple of N , the free energy of this assembly of quarks and antiquarks is infinite. This is one manifestation of quark confinement. For $SU(3)$ this means that the number of quarks minus antiquarks must be an integer multiple of 3, whereas for $SU(2)$ it must be an integer multiple of 2.

If the $Z(N)$ symmetry of the pure gauge theory is spontaneously broken then there ought to be N distinct possible values of $\langle L \rangle$, with

$$\langle L \rangle = e^{i2\pi n/N} L_0 \quad n = 0, 1, \dots, N-1 \quad (10.60)$$

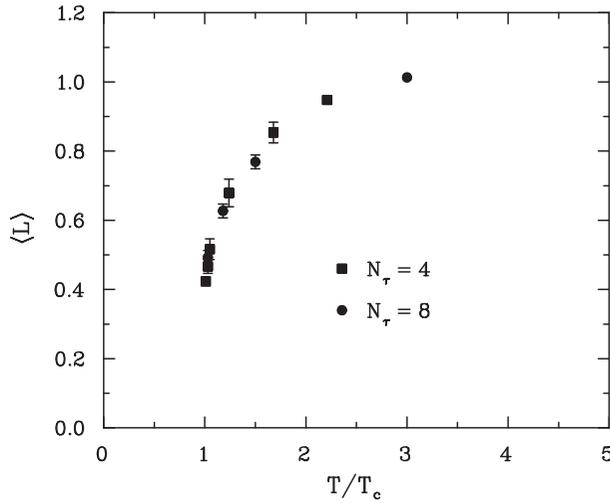


Fig. 10.4. The average value of the renormalized Polyakov loop as a function of temperature on lattices of spatial size $N_s = 32$. It is zero below a critical temperature. Systematic errors are not included. It can go above unity because it is normalized to the short-distance perturbative result on the lattice. The data are from [12].

Therefore $\langle L \rangle$ is an order parameter analogous to the magnetization in a $Z(N)$ spin system. Numerical calculations with the latter systems show a second-order phase transition for $N = 2$ and a first-order transition for $N \geq 3$, in agreement with explicit calculations for $SU(N)$ gauge theories. Calculation of the mean value of L as a function of temperature for $SU(3)$ does indeed show the expected behavior of an order parameter, as may be seen in Figure 10.4.

The static quark–antiquark free energy has contributions from the color singlet and octet potentials:

$$\exp(-\beta\Delta F_{1,1}(r, T)) = \frac{1}{9} \exp[-\beta\Delta F_1(r, T)] + \frac{8}{9} \exp[-\beta\Delta F_8(r, T)] \quad (10.61)$$

The octet is repulsive and the singlet is attractive. The latter is usually of most interest. It can be separated out via

$$\exp[-\beta\Delta F_1(r, T)] = \frac{1}{3} \text{Tr}(W(\mathbf{x}, \beta)W^\dagger(\mathbf{0}, \beta)) \quad (10.62)$$

This requires us to fix the gauge in order to obtain a physically relevant observable. Monte Carlo calculations for the color-singlet potential at various temperatures near T_c are shown in Figure 10.5. At large values of the separation r the free energy is independent of separation, indicating that the linear confining potential characteristic of the low-temperature phase is screened.

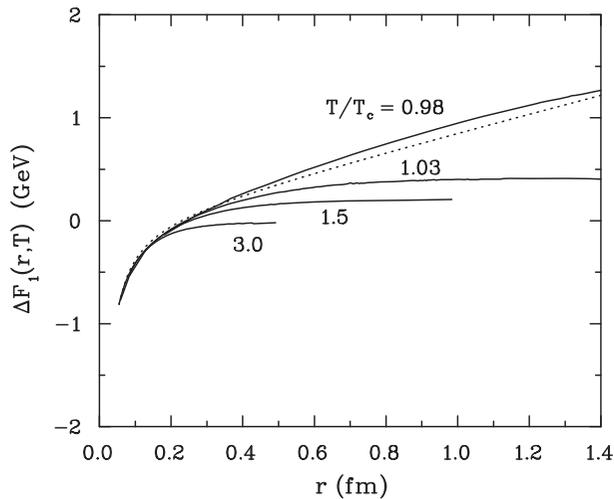


Fig. 10.5. The free energy of a heavy quark–antiquark pair in the color-singlet state as a function of separation for various temperatures. The calculations were done for $N_s = 32$ and $N_\tau = 4, 8,$ and 16 . The data are from [13]. The dotted line represents a zero-temperature potential $V(r) = -4\alpha_s/3r + \sigma r$ with $\alpha_s = 0.18$.

In the pure gauge theories all physical observables are expressed in terms of the lattice spacing. The renormalization group relates the coupling, the lattice spacing, and the scale parameter, Λ_L . To two-loop order the relationship is

$$a\Lambda_L = \frac{1}{[\beta_0 g^2(a)]^{\beta_1/2\beta_0^2}} \exp\left(\frac{-1}{2\beta_0 g^2(a)}\right) \quad (10.63)$$

where the coefficients β_0 and β_1 were given in (8.39). The scale parameter Λ_L can be related to the scale parameters in other schemes, such as $\bar{\Lambda}_{\text{MS}}$.

The best approach to obtaining physically relevant numbers from lattice calculations is to express the results as dimensionless ratios. Then the explicit dependence on the lattice spacing drops out, and hopefully the sensitivity to nonzero a is reduced. For example, pure SU(3) gauge theory at $T = 0$ does not have the usual assortment of hadrons, such as pions and nucleons. It only has glueballs. A relevant ratio then would be $T_c/\sqrt{\sigma}$, where σ is the string tension, which may be obtained from the asymptotic part of the color singlet potential at $T = 0$. An average of several existing calculations yields $T_c/\sqrt{\sigma} = 0.632 \pm 0.002$. Now, there is no absolute argument that says that the string tension in quarkless SU(3) must be the same as in the real world with its six flavors of quarks with various masses. Still, one needs some scale to compare with and it might as well be the string tension. From the phenomenology of heavy quark systems and from the observed linear Regge trajectories it is known that

$\sqrt{\sigma} \approx 420$ MeV. In this case $T_c \approx 265$ MeV. If one chose a different comparison scheme then one would undoubtedly get a different number for the critical temperature in MeV because the relationship among observables depends on the number of quarks and their masses.

10.5 Lattice QCD

Inclusion of quarks on the lattice intensifies the numerical difficulty tremendously. A straightforward evaluation of the fermion determinant involves $(N_s^3 N_\tau)!$ terms. Since most entries in the fermion matrix are zero, clever techniques can be used to reduce this number significantly, but the task is still formidable. As in the pure gauge theory, additional terms can be added to the fermion part of the action that quicken the approach to the continuum, albeit at the expense of computational time.

The quark masses used in lattice calculations to date are not constants but scale with either the lattice spacing, $am = \text{constant}$, or with the temperature, $m/T = \text{constant}$. In principle the values of the up, down, and strange quark masses should be adjusted to yield the experimentally observed values of the pion and kaon masses. Then all other hadron masses, and all calculations done at finite temperature, would be absolute predictions of the theory. However, it turns out that the computational time grows quickly with decreasing quark mass, so that this goal has not been achieved yet.

Figure 10.6 shows the energy density versus temperature for two flavors of light quarks, two flavors of light quarks and one heavier quark, and three flavors of light quarks. Light and heavy mean $m_{\text{light}} = 0.4T$ and $m_{\text{heavy}} = T$, roughly corresponding to up and down quarks and strange quarks. There is a big jump in the energy density centered at a temperature defined to be T_c . Finite size and lattice spacing ($N_\tau = 4$) prevent one from concluding whether there is a first- or second-order phase transition or only a very rapid crossover. Extrapolation to the physical value of the ratios of pion to rho and omega vector meson masses suggests that $T_c = 172 \pm 9$ MeV for two light quarks, where the quoted uncertainty is statistical only. The systematic uncertainty is comparable in magnitude. The value of T_c is reduced by 15 to 20 MeV for a world of three light quarks.

The Polyakov loop cannot be used as an order parameter when dynamical quarks are included because the quark part of the action is not invariant under $Z(3)$ transformations. This can be understood intuitively by remembering that the Polyakov loop is used to measure the free energy of any configuration of static quarks and antiquarks. When a quark and antiquark are pulled apart the potential ceases to be linear in the separation,

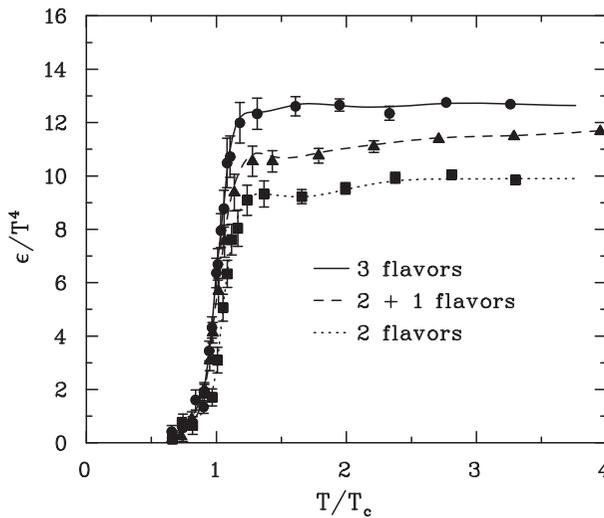


Fig. 10.6. Energy density in units of T^4 as a function of T/T_c for two flavors of light quarks, two flavors of light quarks and one flavor of heavier quark, and three flavors of light quarks. The lattice size is $16^3 \times 4$. The data are from [14].

because the string can break owing to the creation of a light dynamical quark and antiquark. So what then could the order parameter be? If some of the quarks have zero mass then the theory has chiral symmetry, and the quark condensate $\langle \bar{\psi}\psi \rangle$ serves as an order parameter. Figure 10.7 shows the temperature dependence of this condensate for a sequence of ever decreasing light quark masses, for two light flavors and one heavier flavor. The quark condensate goes to zero if the light quark mass is light enough, but only decreases monotonically without ever reaching zero for more massive light quarks. This is also reflected in the hysteresis behavior when the system is numerically cooled and heated and cooled again by Monte Carlo.

To study the effects of varying the number of quark flavors and their masses, Pisarski and Wilczek [16] constructed an effective Lagrangian for an order-parameter field taken to be $\Phi_{ij} = f\bar{q}_i(1 + \gamma_5)q_j$, where f is a constant. The Lagrangian should reflect the symmetries of the QCD Lagrangian. For N_f flavors of massless quarks the symmetry group is

$$G_f = U(1)_A \times SU(N_f) \times SU(N_f) \rightarrow G'_f = Z(N_f)_A \times SU(N_f) \times SU(N_f)$$

Here the classical axial $U(1)_A$ symmetry is broken to $Z(N_f)_A$ symmetry, owing to the quantum axial anomaly. The form of the effective

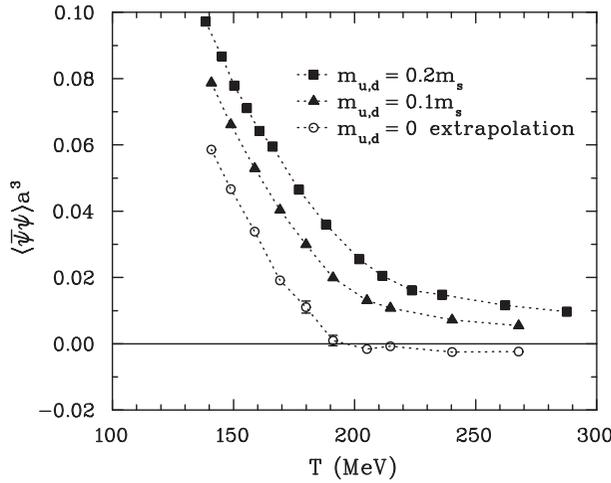


Fig. 10.7. The quark condensate, measured in lattice units, versus temperature. The strange quark mass is fixed in such a way that the mass of the ϕ meson takes its physical value as calculated on the lattice. The lattice size is $16^3 \times 8$. The data are from [15].

(renormalizable) Lagrangian is

$$\begin{aligned}
 \mathcal{L}_{\text{eff}} = & \frac{1}{2} \text{Tr} \left(\partial_\mu \Phi^\dagger \partial^\mu \Phi \right) - \frac{1}{2} m_\Phi^2 \text{Tr} \left(\Phi^\dagger \Phi \right) \\
 & - \frac{1}{3} \pi^2 g_1 \left[\text{Tr} \left(\Phi^\dagger \Phi \right) \right]^2 - \frac{1}{3} \pi^2 g_2 \text{Tr} \left[\left(\Phi^\dagger \Phi \right)^2 \right] \\
 & + c \left(\det \Phi^\dagger + \det \Phi \right) + \text{Tr} \left[M \left(\Phi^\dagger + \Phi \right) \right] \quad (10.64)
 \end{aligned}$$

The determinants originate in the anomaly and are sometimes associated with instanton effects. In hadronic phenomenology they are necessary to give the η' its large observed mass. The last term involving the matrix M represents the effect of nonzero quark masses. Pisarski and Wilczek then used universality to infer the behavior of the QCD system from studies of simpler systems with the same symmetry. Those systems were studied using an ϵ expansion in $4 - \epsilon$ dimensions. Assuming that all quarks are massless, they found that for $N_f = 1$ there is no true phase transition, for $N_f = 2$ the phase transition may be first or second order depending on the strength of the anomaly at T_c as reflected in the coefficient c , and for $N_f \geq 3$ the phase transition is first order.

The likely phase diagram in the m_s versus $m_u = m_d$ plane is shown in Figure 10.8. When all quarks are infinitely heavy it is as if they do not exist at the temperatures of interest, and there is a first-order deconfining phase transition as in the pure $SU(3)$ gauge theory. When all three

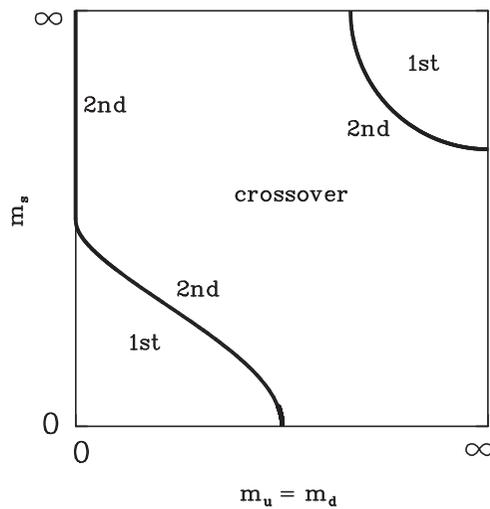


Fig. 10.8. A possible phase diagram for QCD in the strange quark mass versus light quark mass plane. The lower left-hand corner exhibits a first-order chiral-symmetry phase transition, the upper right-hand corner a deconfinement phase transition. These are separated from an intermediate region of rapid crossover between phases.

flavors are massless there is a first-order chiral-symmetry-restoring phase transition. When the strange quark is heavy and the up and down quarks have zero mass there is a second-order chiral-symmetry-restoring phase transition. When all three flavors have masses of the order of several hundred MeV, there is no true thermodynamic phase transition but only a sharp crossover with a jump in the energy density over a small range of temperatures. A variety of lattice calculations seem to support this general picture, but it should still be considered merely as a reasonable conjecture.

The application of lattice QCD to the study of finite-density matter is in its infancy. The difficulty lies in the fact that the chemical potential acts as a constant imaginary time component of the vector potential; see (5.18), (5.19), and (10.42). As such the fermion determinant is complex. This should not lead to a complex partition function; the imaginary part must average to zero. However, it does mean that straightforward Monte Carlo sampling techniques cannot be applied. Two interesting approaches are (i) a Taylor series expansion in powers of $(\mu_B/T)^2$ and (ii) calculation with an imaginary chemical potential followed by analytic continuation to a real baryon chemical potential μ_B . The latter approach was followed by de Forcrand and Philipsen. For two flavors of light quarks they found

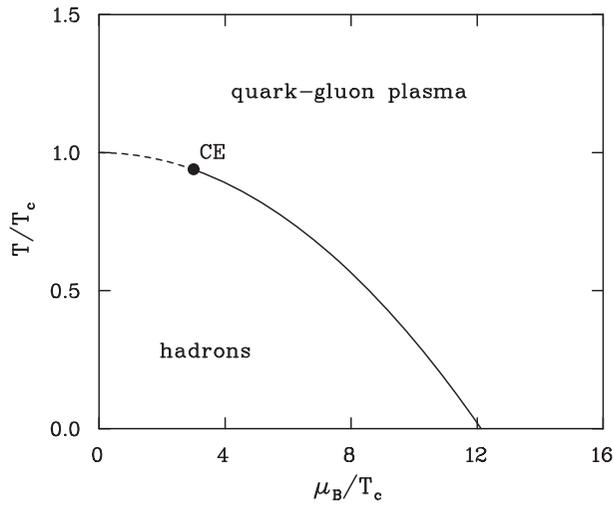


Fig. 10.9. A possible critical line for QCD. The solid curve represents a first-order phase transition terminating at a second-order phase transition at the critical endpoint labeled CE. The broken line represents a rapid crossover.

the critical line [17]

$$\frac{T}{T_c} = 1 - 0.0056 \pm 0.0004 \left(\frac{\mu_B}{T_c} \right)^2 \quad (10.65)$$

and for three flavors of light quarks [18]

$$\frac{T}{T_c} = 1 - 0.0068 \pm 0.0001 \left(\frac{\mu_B}{T_c} \right)^2 \quad (10.66)$$

The stated range of validity is $|\mu_B| < 500$ MeV; the systematic uncertainties are at least as large as the quoted statistical uncertainties. If these are optimistically extrapolated to zero temperature, then taking $T_c \approx 160$ MeV one gets $\mu_B \approx 2$ GeV. This is quite reasonable. The above relations were shown not to be sensitive to the precise numerical value of the quark mass. The critical line for $N_f = 3$ is shown in Figure 10.9. The lattice calculations did not establish conclusively whether the critical line represents a true phase transition or just a rapid crossover between hadronic matter and quark-gluon plasma. If the quark masses are not small enough to yield a first-order phase transition at zero baryon density but finite temperature, there may exist a critical point along the critical line. At lower baryon density there is a rapid crossover and at higher baryon density there is a first-order phase transition. This would connect nicely with the color superconductivity analysis; see Figure 8.4. At the present time the existence of a critical point is not well established.

10.6 Exercises

- 10.1 Show that (10.5) leads to the correct commutation relation between the vector potential and the electric field in the continuum limit.
- 10.2 Derive the Coulomb potential (10.25) in the high-temperature limit of the lattice gauge theory.
- 10.3 Show that (10.31) is gauge invariant.
- 10.4 Calculate the continuum limit of (10.31).
- 10.5 Consider a free neutral boson on a $N_s^3 N_\tau$ lattice of spacing a in all four directions in the limit $N_s \gg N_\tau \gg 1$. Since the action is quadratic, the functional integral expression for the partition function can be evaluated exactly. Following the analysis in Chapter 2 show that the propagator can be written as

$$\mathcal{D}^{-1}(\mathbf{p}, p_4) = \frac{4}{a^2} \sin^2\left(\frac{ap_4}{2}\right) + \frac{4}{a^2} \sum_{i=1}^3 \sin^2\left(\frac{ap_i}{2}\right) + m^2$$

where $-\pi/a \leq p_4 \leq \pi/a$ and $-\pi/a \leq p_i \leq \pi/a$, which defines the Brillouin zone. This reduces to the usual scalar propagator in the $a \rightarrow 0$ limit. This propagator has only one minimum, which is located at $\mathbf{p} = \mathbf{0}$, $p_4 = 0$.

- 10.6 Using periodic boundary conditions in the spatial directions, and antiperiodic boundary conditions in the temporal direction, compute the partition function for massless staggered fermions in the limit of large N_s and N_τ .
- 10.7 Repeat Exercise 10.5 for Wilson fermions.
- 10.8 Show formally that the thermodynamic identities are obeyed with the action (10.42). This implies that the chemical potential in (10.42) has been implemented correctly.
- 10.9 Work out the details leading from (10.51) to (10.52).
- 10.10 Derive the result (10.60), which holds when $Z(N)$ symmetry is spontaneously broken.
- 10.11 Construct a function $P_{\text{fit}}(T)$ that parametrizes the results of the pure SU(3) lattice results. Be sure to compare with entropy and energy densities too.
- 10.12 Are there other terms that could be added to (10.64) for two flavors of massless quarks? If so, what are they?

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