and would then talk them through the way this leads to the power rules (the reverse process to that shown here). This was not, in any sense, a substitute for a formal proof, but rather a pre-calculus taster. With the advent of the Internet and Java it has now become possible to supplement the original worksheet (e-mail the author for a free copy) with dynamic interactive investigations, which can be seen at www.powermaths.org.uk.

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87.40 An elementary single-variable proof of
\[ \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi} \]

‘Do you know what a mathematician is?’, Lord Kelvin once asked a class. He stepped to the blackboard and wrote \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \). Putting his finger on what he had written, he turned to the class. ‘A mathematician is one to whom that is as obvious as that twice two makes four is to you.’ [1].

There are, of course, several proofs in the literature of the result in our title or of obvious equivalents such as \( \int_{0}^{\infty} e^{-x^2} \, dx = \frac{1}{2} \sqrt{\pi} \) and \( \int_{-\infty}^{\infty} e^{-\pi x^2} \, dx = 1 \).

The most familiar (in essence, the original argument of Laplace and his contemporaries) uses a switch to two variables and the evaluation of \( \left( \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx \, dy \) by transforming to polar coordinates. A variation here is to interpret the latter as a volume of revolution which may be evaluated directly either by the ‘shell method’, \( \int_{0}^{\infty} 2\pi x e^{-x^2/2} \, dx \), or as \( \pi \int_{0}^{1} -2 \ln z \, dz \); see [2, 3].

Note that we do not have to transform to polar coordinates here: the transformation \( u = x, v = y/x \), under which \( du \, dv = (1/u) \, dx \, dy \), shows that
\[ \left( \int_{0}^{\infty} e^{-x^2} \, dx \right)^2 = \frac{1}{2} \int_{0}^{\infty} \int_{0}^{\infty} e^{-u^2(1+v^2)} \, d(u^2) \, dv = \frac{1}{2} \int_{0}^{\infty} \frac{dv}{1 + v^2} \]
which evaluates to \( \frac{1}{4} \pi \). A less well-known proof involves contour integration of \( e^{int^2} \, \text{cosec}(\pi z) \) around the parallelogram centred on \( O \) with vertices at \( \pm Re^{i\pi/4} \pm \frac{1}{2} \) and letting \( R \to \infty \) to find \( \int_{-\infty}^{\infty} e^{-\pi x^2} \, dx \), [4].

There are some recondite single-variable proofs. One comes from recognising \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \) as \( \Gamma \left( \frac{1}{2} \right) \) and then either putting \( x = \frac{1}{2} \) into the gamma function identity \( \Gamma(x) \Gamma(1-x) = \pi \, \text{cosec}(\pi x) \), or using the beta function identity \( B \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} / \Gamma \left( 1 \right) \) where \( \Gamma \left( 1 \right) = 1 \) and \( B \left( \frac{1}{2}, \frac{1}{2} \right) = \pi \), on substituting \( x = \sin^2 t \) into its defining integral, [5].

Another springs from an ingenious differentiation under the integral sign to show that
\[ \left( \int_{0}^{x} e^{-t^2} \, dt \right)^2 + \int_{0}^{1} \frac{e^{-x^2/2 - t^2}}{t^2 + 1} \, dt \]
is constant; evaluation as \( x \to 0, \infty \) then yields \( \int_0^\infty e^{-x^2} \, dx = \frac{1}{2} \sqrt{\pi}, \) \([3, 6]\).

The elementary single-variable proof that follows was stimulated by two recent articles on Stirling’s approximation for \( n! \), \([7, 8]\). In both of these, establishing the existence of \( C = \lim_{n \to \infty} \left( \frac{n!}{n^n (1/e)^n} \right) \) follows a standard route. Fowler in \([7]\) then uses Wallis’ product to identify \( C \) as \( \sqrt{2\pi} \); Romik’s novelty in \([8]\) is to evaluate \( C \) via \( \int_0^\infty e^{-x^2} \, dx \). (A probabilistic variation on the latter, using the normal approximation to the binomial distribution, appears in \([9]\).) Our proof proceeds by gluing together the requisite elements of both methods of calculating \( C \).

Consider then the integral \( I_n = \int_{-\sqrt{n}}^{\sqrt{n}} (1 - \frac{x^2}{n})^{n/2} \, dx \). On the one hand, since \( (1 - \frac{x^2}{n})^n \to e^{-x^2} \) as \( n \to \infty \), standard techniques, such as the monotone and dominated convergence theorems of Lebesgue integration, \([10]\), establish that \( \lim_{n \to \infty} I_n = \int_0^\infty e^{-x^2} \, dx \). Here, though, at the suggestion of the referee, we give an ad hoc argument based on inequalities that may be found in \([5, 11]\), that makes our proof self-contained. Since the exponential graph lies above its tangent at \( x = 0 \), we have the inequalities \( e^{-x^2/n} \geq 1 - \frac{x^2}{n} \) and \( e^{x^2/n} \geq 1 + \frac{x^2}{n} \). For \( |x| < \sqrt{n} \), we thus have \( e^{-x^2/2} \geq (1 - \frac{x^2}{n})^{n/2} \) and \( e^{x^2/2} \geq (1 + \frac{x^2}{n})^{n/2} \) from which it follows that:

\[
0 \leq e^{-x^2/2} - \left( 1 - \frac{x^2}{n} \right)^{n/2} = e^{-x^2/2} \left[ 1 - e^{x^2/2} \left( 1 - \frac{x^2}{n} \right)^{n/2} \right] \\
\leq e^{-x^2/2} \left[ 1 - \left( 1 + \frac{x^2}{n} \right) \left( 1 - \frac{x^2}{n} \right)^{n/2} \right] \\
= e^{-x^2/2} \left[ 1 - \left( 1 - \frac{x^4}{n^2} \right) \right] \\
\leq e^{-x^2/2} \left[ 1 - \left( 1 - \frac{x^4}{n^2} \right) \right] \\
= e^{-x^2/2} \left( 1 - \frac{x^4}{n^2} \right) \left[ 1 + \left( 1 - \frac{x^4}{n^2} \right) + \ldots + \left( 1 - \frac{x^4}{n^2} \right)^{n-1} \right] \\
\leq e^{-x^2/2} \frac{x^4}{n^2} \left( 1 + 1 + \ldots + 1 \right) = \frac{1}{n} x^4 e^{-x^2/2}.
\]

This inequality establishes both the pointwise convergence of \( (1 - \frac{x^2}{n})^{n/2} \) to \( e^{-x^2/2} \) and confirms that:

\[
0 \leq \int_{-\infty}^{\infty} e^{-x^2/2} \, dx - I_n = \int_{|x| > \sqrt{n}} e^{-x^2/2} \, dx + \int_{-\sqrt{n}}^{\sqrt{n}} e^{-x^2/2} - \left( 1 - \frac{x^2}{n} \right)^{n/2} \, dx \\
\leq \int_{|x| > \sqrt{n}} e^{-x^2/2} \, dx + \frac{1}{n} \int_{-\infty}^{\infty} x^4 e^{-x^2/2} \, dx
\]
which tends to 0 as \( n \to \infty \) as claimed.
On the other hand, the change of variable \(x = \sqrt{n} \cos t\) shows that

\[ I_n = \sqrt{n} \int_{-\pi/2}^{\pi/2} \sin^{n+1} t \, dt = \sqrt{n} S_{n+1}, \text{ say.} \]

The reduction formula \((n + 1)S_{n+1} = nS_n - 1\) means that

\[ (n + 1)S_{n+1}S_n = nS_nS_{n-1} = \ldots = 1, S_1S_0 = 2\pi, \]

from which

\[ (n + 1)S_{n+1}^2 < (n + 1)S_{n+1}S_n = 2\pi = nS_nS_{n-1} < nS_{n-1}^2 = n \frac{(n + 1)^2}{n^2} S_{n+1}^2 \]

which in turn unwraps to give

\[ \left( \frac{n}{n + 1} \right)^2 2\pi < \frac{nS_{n+1}^2}{n + 1} < n 2\pi. \]

Thus \(\lim_{n \to \infty} I_n = \lim_{n \to \infty} \sqrt{n} S_{n+1} = \sqrt{2\pi}\), as required.

This proof is, in fact, a variation on one that featured in the Gazette, [12, 13], where the authors also reduce the evaluation of \(\int_1^1 \frac{du}{\sqrt{-2 \ln u}}\), which is equivalent to \(\int_0^\infty e^{-x^2} \, dx\), to the calculation of \(\lim_{m \to \infty} \sqrt{m} \int_0^{\pi/2} \sin^m x \, dx\).

Finally, notwithstanding the many proofs that we have described, it is surely the case that Kelvin in our opening quotation employed a large measure of tongue-in-cheek hyperbole!

References

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### 87.41 Yet another look at $n^{1/n}$

In [1], the inequality $n^{1/n} < 1 + 1/\sqrt{n}$ was derived using an essentially graphical approach.

Subsequently, the result $n^{1/n} < 1 + \sqrt{2/n}$ was obtained by Hirschhorn ([2]) who deftly exploited the inequality $(1 + x)^n > 1 + \frac{1}{2}n(n - 1)x^2$ for $x > 0$, with $x = \sqrt{2/n}$.

Now (albeit with a little more effort) we can obtain an even sharper result using the binomial theorem.

First note that the inequality

$$(1 + x)^n + (1 - x)^n = 2 \sum_{r=0}^{[n/2]} \binom{n}{2r} x^{2r} \geq 2 + n(n - 1)x^2$$

holds for all $n, x$. Putting $x = 1/\sqrt{n}$ yields

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n + \left(1 - \frac{1}{\sqrt{n}}\right)^n \geq 2 + \frac{n(n - 1)}{n} = 1 + n.$$

Replacing the second binomial term by 1, we get $(1 + 1/\sqrt{n})^n > n$, so $n^{1/n} < 1 + 1/\sqrt{n}$.

This problem has recently appeared in another journal ([3]).

**References**

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### 87.42 Evaluating $T_{n,r}(\varphi) = \sum_{p=0}^{n-1} \tan^r \left( \varphi + \frac{p\pi}{n} \right)$

Let $C_n(\varphi) = \sum_{p=0}^{n-1} \cos \left( \varphi + \frac{p\pi}{n} \right)$ and $S_n(\varphi) = \sum_{p=0}^{n-1} \sin \left( \varphi + \frac{p\pi}{n} \right)$

where $n$ is a positive integer and $\varphi$ an arbitrary angle. Then the series $C_n(\varphi)$ and $S_n(\varphi)$ may be readily summed by the standard technique of considering $C_n(\varphi) + iS_n(\varphi) = \sum_{p=0}^{n-1} \exp \left[ i \left( \varphi + \frac{p\pi}{n} \right) \right]$. The latter is a geometric series...