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# WEIGHT ELEMENTS OF THE KNOT GROUPS OF SOME THREE-STRAND PRETZEL KNOTS

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**Abstract**

A knot group has weight one, so is normally generated by a single element called a weight element of the knot group. A meridian is a typical weight element, but some knot groups admit other weight elements. We show that for some infinite classes of three-strand pretzel knots and all prime knots with up to eight crossings, the knot groups admit weight elements that are not automorphic images of meridians.

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*Keywords and phrases*: weight element, knot group, pretzel knot, balanced presentation.

## 1. Introduction

Let $G$ be a group. If it is normally generated by a single element, then $G$ is said to have weight one. An element $g$ whose normal closure $\langle\langle g \rangle\rangle$ coincides with $G$ is called a *weight element* of $G$.

For a knot $K$ in the 3-sphere $S^3$, the knot group of $K$ is the fundamental group of its exterior. It is well known that any knot group has weight one, and a meridian is a weight element. Here, we choose a base point $*$ on the boundary of a regular neighbourhood $K \times D^2$, and the curve $* \times \partial D^2$ represents a meridian. Any conjugate of a meridian is also called a meridian. Thus, any meridian gives a weight element of the knot group.

For the knot group of a satellite knot with a certain pattern, Tsau [12] found a weight element that is not the automorphic image of any meridian. Then Silver *et al.* [11] gave infinitely many such weight elements for the knot groups of torus knots, hyperbolic 2-bridge knots and hyperbolic knots with unknotting number one. In fact, they showed more (see Section 2) and conjectured that any nontrivial knot group admits infinitely many nonequivalent weight elements. Recently, Dutra [4] proved this for cable knots and graph knots. He also showed that if any factor of a composite knot admits infinitely many mutually nonconjugate weight elements in its knot group, then so does the knot

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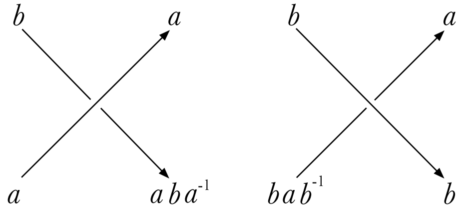


FIGURE 1. Crossing relations.

group of the composite knot. In the literature, the terms 'killer' and 'pseudo-meridian' are used, but we prefer the classical term 'weight element' found in [6].

The purpose of this paper is to find weight elements of the knot groups for some infinite families of three-strand pretzel knots and all prime knots with up to eight crossings. This gives new evidence for the conjecture of Silver *et al.* [11].

THEOREM 1.1. *Let $K$ be a pretzel knot of the form $P(\pm 2, p, q)$ for odd $p$, $q$ with $|p|, |q| > 1$, or $P(\pm 3, 3, m)$ with $|m| > 1$. Then the knot group of $K$ contains infinitely many weight elements that are not automorphic images of a meridian.*

Except for $P(\pm 2, \mp 3, \mp 3)$ and $P(\pm 2, \mp 3, \mp 5)$, which are torus knots, the knots in Theorem 1.1 are hyperbolic [7]. Also, $P(-3, 3, \pm 2)$ is the only one that has unknotting number one among $P(\pm 3, 3, m)$ with $|m| > 1$ [2]. (In fact, $P(-3, 3, \pm 2)$ is expected to be the only three-strand pretzel knot with unknotting number one that is not 2-bridge [2].) Hence, our theorem gives infinitely many new hyperbolic knots that admit infinitely many weight elements. Theorem 1.1 is proved in Sections 3 and 4.

We expect that the knot group of any three-strand pretzel knot admits infinitely many weight elements as above, and that two noncommuting meridians are used to give such elements in the manner described below. In fact, all weight elements that we found are constructed in that way. The difficulty lies in how to prove that a balanced presentation with two generators and two relations gives the trivial group.

THEOREM 1.2. *For any prime knot with at most eight crossings, the knot group contains infinitely many weight elements that are not automorphic images of a meridian.*

Up to seven crossings, all primes knots are 2-bridge, and the case of such knots is covered in [11]. Among 21 prime knots with eight crossings, 12 knots are 2-bridge, one is a torus knot and three are hyperbolic 3-bridge knots with unknotting number one. The case of such knots is also covered by [11]. Thus, only five knots, two of which are not Montesinos knots, remain to be examined. These knots are considered in Section 5.

## 2. Adding twists

For a calculation of a knot group from a diagram of a knot, we use the convention illustrated in Figure 1.

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In general, in a group $G$, a conjugate of an element $b$ by $a$ is defined to be $a^{-1}ba$. Thus, for example, in the right-hand picture of Figure 1, the meridian generator $b$ corresponding to the left top over-arc changes to its conjugate by $a^{-1}$ after passing the under-crossing.

We next explain the modification technique described in [11, Remark 2.1]. Assume that a knot diagram of a knot $K$ contains a crossing as shown in Figure 1. Consider the element $\mu_n = b(ab^{-1})^n$ of the knot group $G(K)$ for any positive integer $n$. Set $A = ba^{-1}$. Then, in the quotient group $G(K)/\langle\langle \mu_n \rangle\rangle$, we have the relations $b = A^n$ and $a = A^{n-1}$. Here, $\langle\langle \mu_n \rangle\rangle$ denotes the normal closure of the element $\mu_n$ in $G(K)$.

Now add any number of vertical full twists below the crossing. If $K'$ is the resulting knot, then the quotient group $G(K')/\langle\langle \mu_n \rangle\rangle$ is isomorphic to $G(K)/\langle\langle \mu_n \rangle\rangle$, because any conjugate of $a$ (respectively $b$) with $b^{\pm}$ (respectively $a^{\pm}$) remains $A^{n-1}$ (respectively $A^n$). This observation gives the following result.

Lemma 2.1 [11]. *With the above notation, if $\mu_n$ is a weight element of $G(K)$, then $\mu_n$ is a weight element of $G(K')$.*

To distinguish weight elements, a parabolic representation of a knot group into $\mathrm{SL}(2, \mathbb{C})$ is used [9], provided that the knot is hyperbolic.

Lemma 2.2 [11]. *Let $K$ be a hyperbolic knot with knot group $G(K)$. If two meridians $a$ and $b$ do not commute in $G(K)$, then the elements $\mu_n = b(ab^{-1})^n$ are pairwise nonconjugate for sufficiently large $n$.*

More precisely, a parabolic (discrete) representation $\rho : G(K) \to \mathrm{SL}(2, \mathbb{C})$ has the form

$$\rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$$

with $\omega \neq 0$. Then the trace of $\rho(\mu_n)$ takes infinitely many distinct values.

As shown in [11], the knot group of any hyperbolic knot with unknotting number one admits infinitely many pairwise nonconjugate weight elements. Let $K$ be the pretzel knot $P(-3, 3, 2)$ as shown in Figure 3. Any crossing of the leftmost twists unties the knot. Hence, the above modification shows that the pretzel knots $P(2k + 1, 3, 2)$ admit infinitely many pairwise nonconjugate weight elements, unless $k = -1$, which yields the unknot. Thus, we should remark that the case of some hyperbolic pretzel knots is covered in [11].

## 3. Pretzel knots $P(\pm 2, p, q)$

In this section, we prove the following result as a part of Theorem 1.1.

Theorem 3.1. *Let $K$ be a pretzel knot $P(\pm 2, p, q)$ for odd $p$, $q$ with $|p|, |q| > 1$. Then the knot group $G(K)$ admits infinitely many weight elements that are not automorphic images of a meridian.*

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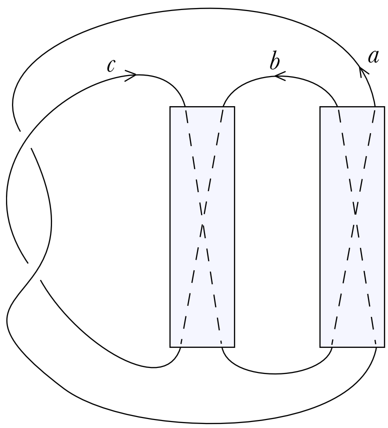


FIGURE 2. The pretzel knot $P(2, p, q)$.

The hardest part of the argument is to confirm that a quotient group is trivial. Fortunately, we can use the next criterion.

THEOREM 3.2 (Miller–Schupp [8]). *Let G be a perfect group generated by two elements a and b that satisfy the equation $a^{-1}b^n a = b^m$, where $m > 0$ and $n > 0$ are relatively prime. Then G is the trivial group.*

Let $K$ be a pretzel knot $P(2, p, q)$ for odd $p, q$ with $|p|, |q| > 1$. Since its mirror image is $P(-2, -p, -q)$, this restriction suffices to prove Theorem 3.1. Figure 2 shows a diagram of $K$ with meridian generators $a$, $b$ and $c$ for its knot group $G(K)$, where each rectangle contains an odd number of vertically arranged half-twists.

For $K$ in this position, $a$, $b$ and $c$ correspond to the three arcs containing the maximal points with respect to the vertical direction. From the three arcs containing the minimal points, we have three relations of $G(K)$, but any one of them can be discarded.

For the right-hand rectangle, the lower right (respectively left) output is represented by a conjugate $b^g$ of $b$ (respectively $a^h$ of $a$). Here, $b^g = g^{-1}bg$ and $a^h = h^{-1}ah$, and the words $g$ and $h$ contain only $a^{\pm 1}$ and $b^{\pm 1}$. Similarly, the lower right output of the middle rectangle is represented by a conjugate $c^k$ of $c$, and $c^k = k^{-1}ck$, where $k$ contains only $b^{\pm 1}$ and $c^{\pm 1}$. Thus, we obtain a presentation

$$G(K) = \langle a, b, c \mid b^g = cac^{-1}, a^h = c^k \rangle. \tag{3.1}$$

LEMMA 3.3. *In $G(K)$, the two meridians a and b do not commute.*

PROOF. If $a$ and $b$ commute in $G(K)$, then the presentation (3.1) reduces to

$$G(K) = \langle a, b, c \mid b = cac^{-1}, a = c^k \rangle.$$

Thus, $G(K)$ is generated by two meridians, which implies that $K$ is 2-bridge [1], which is a contradiction. □

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PROOF OF THEOREM 3.1. Let $\mu_n = b(ab^{-1})^n$ and set $A = ba^{-1}$. In the quotient group $G_n = G(K)/\langle\langle\mu_n\rangle\rangle$, we have the relations $b = A^n$ and $a = A^{n-1}$. Thus, $b^g$ reduces to $A^n$ and $a^h$ reduces to $A^{n-1}$ there. Hence, (3.1) yields

$$G_n = \langle c, A \mid A^n = cA^{n-1}c^{-1}, A^{n-1} = w(c, A^n)\rangle, \tag{3.2}$$

where the word $w(c, A^n)$ comes from $c^k$, so it contains only $c^{\pm 1}$ and $A^{\pm n}$ and it has exponent sum 1 in $c$ and 0 on $A$. By abelianisation of (3.2), we see that $G_n$ is a perfect group. If $n = 1$, then it is straightforward to see that $G_n$ is trivial (in this case, $\mu_1$ is a meridian) and, if $n \geq 2$, then the first relation of (3.2) implies the same conclusion by Theorem 3.2.

By Lemmas 3.3 and 2.2, $G(K)$ contains infinitely many pairwise nonconjugate weight elements, provided that $K$ is hyperbolic. Moreover, any automorphism of a prime knot group is induced by a homeomorphism of the knot exterior [13]. Hence, we can conclude that such $\mu_n$ are not automorphic images of a meridian. □

More generally, for a pretzel knot $P(2l + 2, p, q)$ with $l \geq 0$ and odd $p$, $q$, the same procedure gives the quotient group

$$G_n = \langle c, A \mid A^n = (c^{-1}(A^{n-1}c^{-1})^l)^{-1}A^{n-1}(c^{-1}(A^{n-1}c^{-1})^l), A^{n-1} = w(c, A^n)\rangle,$$

where $w(c, A^n)$ is a conjugate of $c$ as above. It is straightforward to see that $G_n$ is a perfect group.

QUESTION 3.4. *Is this $G_n$ the trivial group?*

For example, we can show that this is true for $P(4, 3, q)$ with odd $q$ as follows. In this case, $w(c, A^n) = (A^n c A^n)^{-1} c (A^n c A^n)$. Hence, the relation $A^{n-1} = w(c, A^n)$ gives $cA^{n-1}c^{-1} = A^{-n}cA^n$, and also $A^n c A^{-(n-1)} = c^{-1}A^n c$. Then these two change the first relation $A^n = (c^{-1}A^{n-1}c^{-1})^{-1}A^{n-1}(c^{-1}A^{n-1}c^{-1})$ of $G_n$ into $A^n c A^{-(n-1)} = A^{-(2n-1)}cA^{2n-1}$, so $A^{3n-1} = cA^{3n-2}c^{-1}$. Thus, Lemma 3.2 implies the desired conclusion.

## 4. Pretzel knots $P(\pm 3, 3, m)$

In this section, we examine the other family of pretzel knots included in Theorem 1.1. The argument is divided into two parts, according to the parity of $m$.

**4.1. Even case.** Let $K$ be the pretzel knot $P(3, 3, 2)$ as illustrated in Figure 3. This knot is the knot $8_5$ in the knot table [10], and it is hyperbolic and has unknotting number two [3]. By the same procedure as in Section 3, we see that the knot group $G(K)$ has a presentation

$$G(K) = \langle a, b, c \mid aba^{-1} = (bcb)c(bcb)^{-1}, (ba^{-1})^{-1}a(ba^{-1}) = (ac)^{-1}c(ac)\rangle, \tag{4.1}$$

where $a, b, c$ are meridians corresponding to arcs as shown in Figure 3.

For any integer $n \geq 1$, let $\mu_n = b(ab^{-1})^n$. If we set $A = ba^{-1}$, then $\mu_n = bA^{-n}$. Let $G_n$ be the quotient group of $G(K)$ by the normal closure $\langle\langle\mu_n\rangle\rangle$ in $G(K)$. Then $b = A^n$ and $a = A^{n-1}$ in $G_n$. Hence, $G_n$ has a presentation

$$\begin{aligned} G_n &= \langle c, A \mid A^n = (A^n c A^n)c(A^n c A^n)^{-1}, A^{n-1} = c^{-1}A^{-(n-1)}cA^{n-1}c\rangle \\ &= \langle c, A \mid A^n c A^n = cA^n c, A^{n-1}cA^{n-1} = cA^{n-1}c\rangle. \end{aligned} \tag{4.2}$$

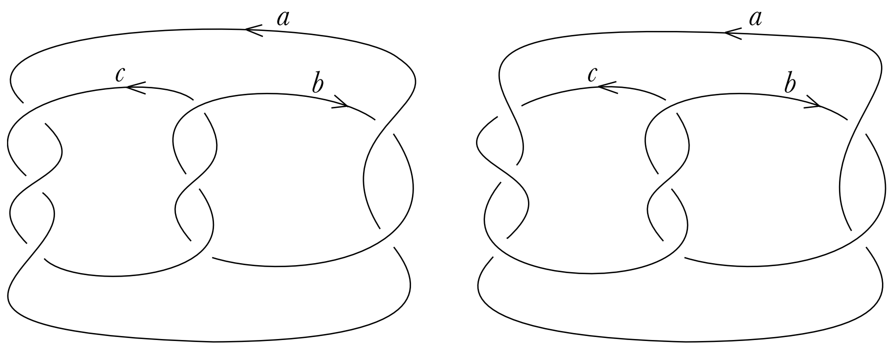


FIGURE 3. The pretzel knots $P(3, 3, 2)$ and $P(-3, 3, 2)$.

LEMMA 4.1. *$G_n$ is the trivial group.*

PROOF. For (4.2), the second relation changes to $c = (cA^{n-1})^{-1}A^{n-1}(cA^{n-1})$, so $c = ((cA^{n-1})^{-1}A(cA^{n-1}))^{n-1}$. Set $B = (cA^{n-1})^{-1}A(cA^{n-1})$. Then the second relation of (4.2) is $c = B^{n-1}$. Also, the first relation of (4.2) changes to $(cA^{n-1})^{-1}A^n(cA^{n-1}) = AcA^{-1}$, which is $B^n = AcA^{-1}$. Thus,

$$\begin{aligned} G_n &= \langle c, A, B \mid B^n = AcA^{-1}, c = B^{n-1}, B = (cA^{n-1})^{-1}A(cA^{n-1})\rangle \\ &= \langle A, B \mid B^n = AB^{n-1}A^{-1}, B = A^{-(n-1)}B^{-(n-1)}AB^{n-1}A^{n-1}\rangle. \end{aligned}$$

By abelianisation, $G_n$ is seen to be a perfect group. If $n = 1$, then it is obvious that $G_n$ is trivial. If $n \geq 2$, then the first relation implies that $G_n$ is the trivial group by Theorem 3.2. □

LEMMA 4.2. *In $G(K)$, the two meridians $a$ and $b$ do not commute.*

PROOF. Assume that $a$ and $b$ commute in $G(K)$. Then the presentation (4.1) of $G(K)$ is equivalent to $\langle a, b, c \mid bcb = cbc, aca = cac\rangle$. This group is isomorphic to the knot group of the granny (or square) knot. This is a contradiction, since the knot group of a prime knot can never be isomorphic to that of a composite knot [5]. □

LEMMA 4.3. *In $G(K)$, the elements $\mu_n$ are pairwise nonconjugate for sufficiently large $n$.*

PROOF. This follows from Lemmas 4.2 and 2.2. □

Similarly, let $K$ be the pretzel knot $P(-3, 3, 2)$, which is the knot $8_{20}$ in the knot table. This is also hyperbolic and has unknotting number one. From Figure 3,

$$G(K) = \langle a, b, c \mid aba^{-1} = (bcb)c(bcb)^{-1}, (ba^{-1})^{-1}a(ba^{-1}) = (aca)c(aca)^{-1}\rangle.$$

It is straightforward to confirm that the quotient group $G_n$ has the same presentation (4.2). Also, Lemmas 4.2 and 4.3 hold *verbatim*.

THEOREM 4.4. *Let $K$ be a pretzel knot $P(\pm 3, 3, m)$ for $m$ even with $|m| > 1$. Then the knot group $G(K)$ admits infinitely many weight elements that are not automorphic images of a meridian.*

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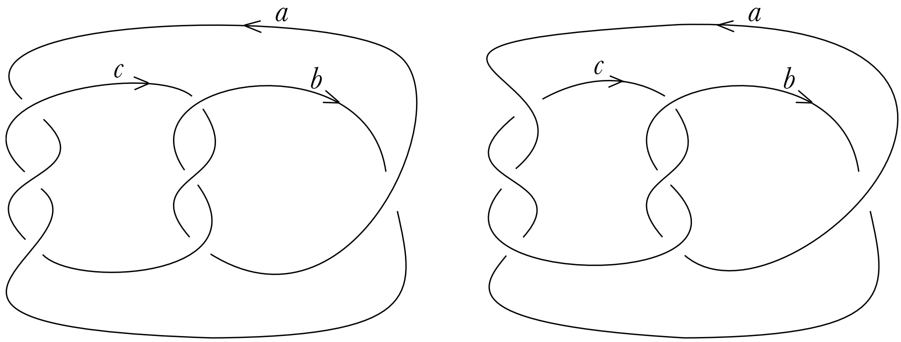


FIGURE 4. The pretzel knots $P(3,3,1)$ and $P(-3,3,1)$.

PROOF. As we have shown above, the pretzel knots $P(\pm 3,3,2)$ satisfy the conclusion. Add any number of full twists in the rightmost twists in the diagram of Figure 3. Let $K$ be the resulting pretzel knot with knot group $G(K)$. Set $\mu_n = b(ab^{-1})^n$ as before. Then $\mu_n$ is a weight element for $G(K)$ by Lemma 2.1. As noted in the introduction, all but $P(3,3,-2)$ are hyperbolic. Hence, we will be done by Lemma 2.2 if we show that two generators $a$ and $b$ do not commute in $G(K)$.

Let $K$ be $P(3,3,2l)$ with $l > 0$. Then $G(K)$ has a presentation with generators $a, b, c$ and two relations

$$\begin{aligned}(a^{-1}(ba^{-1})^{l-1})^{-1}b(a^{-1}(ba^{-1})^{l-1}) &= (bcb)c(bcb)^{-1},\\ (ba^{-1})^{-l}a(ba^{-1})^{l} &= (ac)^{-1}c(ac).\end{aligned}$$

If $a$ and $b$ commute, then these relations collapse into

$$b = (bcb)c(bcb)^{-1}, \quad a = (ac)^{-1}c(ac).$$

Since $K$ is prime, this is a contradiction as in the proof of Lemma 4.2. The case $l < 0$ is similar.

Next, let $K$ be $P(-3,3,2l)$ with $l > 0$. Then $G(K)$ has a presentation with generators $a, b, c$ and two relations

$$\begin{aligned}(a^{-1}(ba^{-1})^{l-1})^{-1}b(a^{-1}(ba^{-1})^{l-1}) &= (bcb)c(bcb)^{-1},\\ (ba^{-1})^{-l}a(ba^{-1})^{l} &= (aca)c(aca)^{-1}.\end{aligned}$$

If $a$ and $b$ commute, then the above argument works again. The case $l < 0$ is also similar. □

**4.2. Odd case.** Let $K$ be $P(3,3,1)$, which is the 2-bridge knot $S(15,4)$ in Schubert's notation, as illustrated in Figure 4. This is hyperbolic and has unknotting number two. The knot group $G(K)$ has a presentation

$$G(K) = \langle a, b, c \mid a = (b^{-1}cb^{-1})^{-1}c(b^{-1}cb^{-1}), aba^{-1} = (ac^{-1})^{-1}c(ac^{-1})\rangle, \tag{4.3}$$

where $a, b, c$ are meridians depicted in Figure 4.

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Again, let $\mu_n = b(ab^{-1})^n$ for an integer $n \geq 1$. If we set $A = ba^{-1}$, then $\mu_n = bA^{-n}$. Let $G_n$ be the quotient group of $G(K)$ by the normal closure $\langle\langle \mu_n \rangle\rangle$ in $G(K)$. Then $b = A^n$ and $a = A^{n-1}$ in $G_n$. Hence, $G_n$ has a presentation

$$\begin{aligned} G_n &= \langle c, A \mid A^{n-1} = (A^{-n}cA^{-n})^{-1}c(A^{-n}cA^{-n}), A^n = (A^{n-1}c^{-1})^{-1}c(A^{n-1}c^{-1})\rangle \\ &= \langle c, A \mid cA^{n-1}c^{-1} = A^ncA^{-n}, c^{-1}A^nc = A^{-(n-1)}cA^{n-1}\rangle. \end{aligned} \tag{4.4}$$

Lemma 4.5. *$G_n$ is the trivial group.*

Proof. If $n = 1$, then the conclusion is clear. Assume that $n \geq 2$. The second relation of (4.4) is equivalent to $(cA^{-(n-1)})^{-1}A^n(cA^{-(n-1)}) = c$. Also, the first relation can be changed to $(cA^{-(n-1)})^{-1}A^nc^{-2}(cA^{-(n-1)}) = c^{-1}A$. Set $B = (cA^{-(n-1)})^{-1}A(cA^{-(n-1)})$. Then the second relation is $B^n = c$ and the first relation changes to

$$(cA^{-(n-1)})^{-1}A^n(cA^{-(n-1)}) \cdot (cA^{-(n-1)})^{-1}c^{-2}(cA^{-(n-1)}) = c^{-1}A$$

and so $B^n(A^{n-1}c^{-2}A^{-(n-1)}) = c^{-1}A$. This further goes to

$$B^{2n}A^{n-1}B^{-2n} = A^n. \tag{4.5}$$

Hence,

$$\begin{aligned} G_n &= \langle c, A, B \mid B^{2n}A^{n-1}B^{-2n} = A^n, B^n = c, B = (cA^{-(n-1)})^{-1}A(cA^{-(n-1)})\rangle \\ &= \langle A, B \mid B^{2n}A^{n-1}B^{-2n} = A^n, A^{-(n-1)}BA^{n-1} = B^{-n}AB^n\rangle. \end{aligned}$$

Taking $n$th and $-(n-1)$st powers of the last relation gives

$$A^{-(n-1)}B^nA^{n-1} = B^{-n}A^nB^n, \tag{4.6}$$

$$A^{-(n-1)}B^{-(n-1)}A^{n-1} = B^{-n}A^{-(n-1)}B^n. \tag{4.7}$$

By applying (4.5) to the right-hand side of (4.6), $A^{-(n-1)}B^nA^{n-1} = B^nA^{n-1}B^{-n}$, which is equivalent to $B^{-n}A^{-(n-1)}B^n = A^{n-1}B^{-n}A^{-(n-1)}$. By (4.7), $A^{-(n-1)}B^{-(n-1)}A^{n-1} = A^{n-1}B^{-n}A^{-(n-1)}$ and so $B^{n-1} = A^{2(n-1)}B^nA^{-2(n-1)}$. Conjugating with $B^{-2n}$ yields

$$B^{n-1} = (B^{2n}A^{2(n-1)}B^{-2n})B^n(B^{2n}A^{-2(n-1)}B^{-2n}).$$

Finally, the square of (4.5) changes this to

$$B^{n-1} = A^{2n}B^nA^{-2n}. \tag{4.8}$$

Now taking the squares of (4.5) and (4.8) gives

$$B^{2n}A^{2(n-1)}B^{-2n} = A^{2n}, \quad A^{2n}B^{2n}A^{-2n} = B^{2(n-1)}. \tag{4.9}$$

Let $x = A^2$ and $y = B^2$. Then (4.9) yields

$$y^nx^{n-1}y^{-n} = x^n, \quad x^ny^nx^{-n} = y^{n-1}. \tag{4.10}$$

Claim 4.6. *Let $M = (n-1)^n$ and $N = n^n$. Then $y^{n^2}x^My^{-n^2} = x^N$.*

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PROOF OF CLAIM 4.6. The $(n-1)$st power of the first relation of (4.10) gives

$$y^n x^{(n-1)^2} y^{-n} = x^{n(n-1)}.$$

Conjugating with $y^{-n}$ yields

$$y^{2n} x^{(n-1)^2} y^{-2n} = y^n x^{n(n-1)} y^{-n}.$$

The right-hand side is equal to $(y^n x^{n-1} y^{-n})^n$, which is $x^{n^2}$ by (4.10). Hence,

$$y^{2n} x^{(n-1)^2} y^{-2n} = x^{n^2}. \tag{4.11}$$

Again, take the $(n-1)$st power of (4.11) and conjugate with $y^{-n}$ to give

$$y^{3n} x^{(n-1)^3} y^{-3n} = x^{n^3}.$$

By repeating this process,

$$y^{(n-1)n} x^{(n-1)^{n-1}} y^{-(n-1)n} = x^{n^{n-1}}, \quad y^{n^2} x^{(n-1)^n} y^{-n^2} = x^{n^n}. \tag{4.12}$$

The second relation is the desired one. (We use the first relation later.) □

Conjugation of the relation of Claim 4.6 by $y^n$ gives

$$y^{n^2-n} x^M y^{-(n^2-n)} = y^{-n} x^N y^n.$$

The right-hand side is $(y^{-n} x^n y^n)^{n^{n-1}}$, which is $(x^{n-1})^{n^{n-1}} = x^{(n-1)n^{n-1}}$ by (4.10). Hence,

$$y^{n^2-n} x^M y^{-(n^2-n)} = x^{(n-1)n^{n-1}}. \tag{4.13}$$

On the other hand, conjugation of the relation of Claim 4.6 by $x^{-n}$ gives

$$(x^n y^{n^2} x^{-n}) x^M (x^n y^{-n^2} x^{-n}) = x^N.$$

From the $\pm n$th power of the second relation of (4.10),

$$y^{(n-1)n} x^M y^{-(n-1)n} = x^N. \tag{4.14}$$

By (4.13) and (4.14), $x^{(n-1)n^{n-1}} = x^{n^n}$ and so $x^{n^{n-1}} = 1$. Then the first relation of (4.12) changes to $y^{(n-1)n} x^{(n-1)^{n-1}} y^{-(n-1)n} = 1$, so $x^{(n-1)^{n-1}} = 1$. Since $n^{n-1}$ and $(n-1)^{n-1}$ are relatively prime, we have $x = 1$. Then $y = 1$ from (4.10). Thus, $A^2 = 1$ and $B^2 = 1$. By (4.5) and (4.8), we obtain $A = 1$ and $B = 1$. □

LEMMA 4.7. *In $G(K)$, the two meridians $a$ and $b$ do not commute.*

PROOF. If $a$ and $b$ commute, then the presentation (4.3) gives

$$\begin{aligned} G(K) &= \langle a, b, c \mid b^{-1}ca = cb^{-1}c, b = ca^{-1}cac^{-1} \rangle \\ &= \langle a, c \mid aca = cac \rangle. \end{aligned}$$

This is isomorphic to the knot group of the trefoil. Since $K$ is not equivalent to the trefoil, this is a contradiction. □

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Hence, $G(K)$ admits infinitely many pairwise nonconjugate weight elements by Lemmas 2.2 and 4.7.

Next, let $K$ be $P(-3, 3, 1)$, which is the hyperbolic 2-bridge knot $S(9, 2)$. Although this knot has unknotting number one, the rightmost crossing in the diagram of Figure 4 does not untie the knot.

As before, the knot group $G(K)$ has a presentation

$$G(K) = \langle a, b, c \mid a = (b^{-1}cb^{-1})^{-1}c(b^{-1}cb^{-1}), aba^{-1} = (a^{-1}ca^{-1})^{-1}c(a^{-1}ca^{-1})\rangle. \quad (4.15)$$

Let $\mu_n = b(ab^{-1})^n$ and $A = ba^{-1}$. Then the quotient group $G_n = G(K)/\langle\langle \mu_n \rangle\rangle$ has a presentation

$$\begin{aligned} G_n &= \langle c, A \mid A^{n-1} = (A^{-n}cA^{-n})^{-1}c(A^{-n}cA^{-n}), \\ &\qquad A^n = (A^{-(n-1)}cA^{-(n-1)})^{-1}c(A^{-(n-1)}cA^{-(n-1)})\rangle \\ &= \langle c, A \mid cA^{n-1}c^{-1} = A^ncA^{-n}, cA^nc^{-1} = A^{n-1}cA^{-(n-1)}\rangle. \end{aligned} \quad (4.16)$$

Lemma 4.8. *$G_n$ is the trivial group.*

Proof. If $n = 1$, the conclusion is straightforward. Now assume that $n \geq 2$ and set $B = (c^{-1}A^{n-1})^{-1}A(c^{-1}A^{n-1})$. Then the second relation of (4.16) gives $B^n = c$, and this changes the first relation to $B^nA^{n-1}B^{-n} = A^nB^nA^{-n}$. Hence,

$$\begin{aligned} G_n &= \langle A, B \mid B^nA^{n-1}B^{-n} = A^nB^nA^{-n}, A^{n-1}BA^{-(n-1)} = B^nAB^{-n}\rangle \\ &= \langle A, B \mid B^{n-1} = AB^nA^{-1}, A^{n-1}BA^{-(n-1)} = B^nAB^{-n}\rangle. \end{aligned}$$

Since this is a perfect group, Theorem 3.2 implies the conclusion. □

For the next lemma, the argument of the proof of Lemma 4.7 does not apply, because the crossing change at the rightmost crossing of the diagram in Figure 4 makes the knot into its mirror image, so the knot group does not change.

Lemma 4.9. *In $G(K)$, two meridians $a$ and $b$ do not commute.*

Proof. If we erase the generator $a$ in the presentation (4.15) by using the first relation,

$$G(K) = \langle b, c \mid b = w^{-1}cw\rangle,$$

where $w = (bc^{-1})^2(b^{-1}c)^2$. Let $\rho : G(K) \to \mathrm{SL}(2, \mathbb{C})$ be a nonabelian parabolic representation given by

$$\rho(b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(c) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix},$$

where $z$ is a root of the polynomial $1 - 2x + 7x^2 - 5x^3 + x^4$, which has no real root. (For such a choice of $z$, the matrix equation $\rho(wb) = \rho(cw)$ holds, so that $\rho$ defines a homomorphism [9].) Then

$$\rho(a) = \begin{pmatrix} 1 + 2z - 3z^2 + z^3 & -4z + 4z^2 - z^3 \\ z - 2z^2 + z^3 & 1 - 2z + 3z^2 - z^3 \end{pmatrix}.$$

This commutes with $\rho(b)$ if and only if $z - 2z^2 + z^3 = 0$, so $z = 1$. Since $z$ is not real, $\rho(a)$ and $\rho(b)$ do not commute. □

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Then Lemma 4.3 works, so $G(K)$ admits infinitely many nonconjugate weight elements.

Theorem 4.10. *Let $K$ be a pretzel knot $P(\pm 3, 3, m)$ for odd $m$ with $|m| > 1$. Then the knot group $G(K)$ admits infinitely many weight elements that are not automorphic images of a meridian.*

Proof. We have already shown that $P(\pm 3, 3, 1)$ satisfies the conclusion. The proof is the same as that of Theorem 4.4. We only show that two generators of $G(K)$ do not commute after adding full twists at the rightmost twist.

First, let $K$ be $P(3, 3, 2l + 1)$ with $l \geq 0$. The knot group $G(K)$ has a presentation with generators $a, b, c$ and two relations $(ba^{-1})^{-l}a(ba^{-1})^{l} = (b^{-1}cb^{-1})^{-1}c(b^{-1}cb^{-1})$ and $(a^{-1}(ba^{-1})^{l})^{-1}b(a^{-1}(ba^{-1})^{l}) = (ac^{-1})^{-1}c(ac^{-1})$. If $a$ and $b$ commute, then the proof of Lemma 4.7 works *verbatim*. The case $l < 0$ is similar. (We remark that if $l = -1$, then $K$ is the trefoil. Then the proof of Lemma 4.7 obviously does not apply.)

Next, let $K$ be $P(-3, 3, 2l + 1)$ with $l \geq 0$. The knot group $G(K)$ has a presentation with generators $a, b, c$ and two relations

$$(ba^{-1})^{-l}a(ba^{-1})^{l} = (b^{-1}cb^{-1})^{-1}c(b^{-1}cb^{-1}),$$
$$(a^{-1}(ba^{-1})^{l})^{-1}b(a^{-1}(ba^{-1})^{l}) = (a^{-1}ca^{-1})^{-1}c(a^{-1}ca^{-1}).$$

If $a$ and $b$ commute in $G(K)$, then these relations collapse into those of (4.15) and so $G(K)$ is isomorphic to the knot group of $P(-3, 3, 1)$. Although $P(-3, 3, 1)$ is 2-bridge, $P(-3, 3, 2l + 1)$ is 3-bridge for $l > 0$. Hence, these knot groups are not isomorphic. Finally, the case $l < 0$ is similar. □

Proof of Theorem 1.1. Theorem 1.1 immediately follows from Theorems 3.1, 4.4 and 4.10. □

## 5. Prime knots with eight crossings

In this section, we examine prime knots with eight crossings to prove Theorem 1.2.

Proof of Theorem 1.2. All prime knots with up to eight crossings are 2-bridge knots or torus knots, except $8_5$, $8_{10}$, $8_{15}$, $8_{16}$, $8_{17}$, $8_{18}$, $8_{20}$ and $8_{21}$. Among these exceptions, $8_{17}$, $8_{20}$ and $8_{21}$ are hyperbolic knots with unknotting number one [3]. The knot $8_5$ is the pretzel knot $P(3, 3, 2)$, which is examined in Section 4. The knot $8_{10}$ is also the pretzel knot $P(3, -3, 1, 2)$, which is obtained from $P(1, -3, 1, 2)$ by adding a full twist on the leftmost strand. This $P(1, -3, 1, 2)$ is the knot $6_3$, a hyperbolic 2-bridge knot, and furthermore $P(-1, -3, 1, 2)$ is trivial. Thus, $8_{10}$ is obtained from $6_3$ by adding a full twist at the unknotting crossing. By Lemmas 2.1 and 2.2, the knot group of $8_{10}$ admits infinitely many nonconjugate weight elements.

The remaining three knots $8_{15}$, $8_{16}$ and $8_{18}$ are examined below. □

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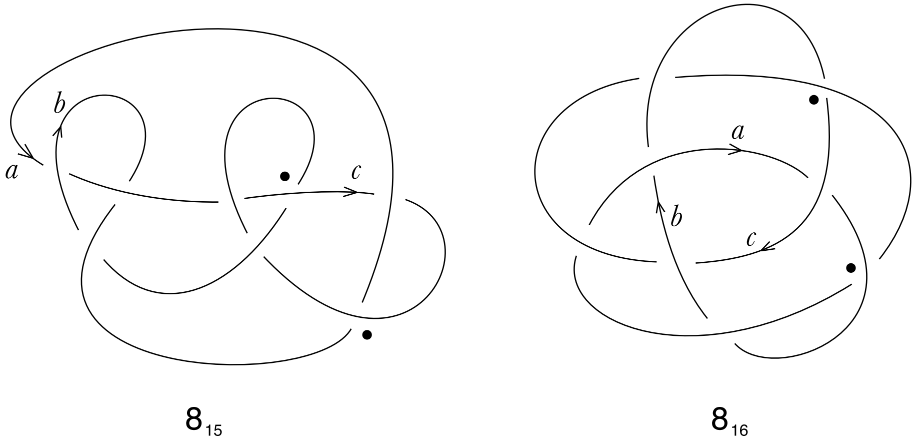


FIGURE 5. The knots $8_{15}$ and $8_{16}$.

**5.1. The knot $8_{15}$.** Let $K$ be the knot $8_{15}$ in the knot table as illustrated in Figure 5. This knot is hyperbolic and has unknotting number two [3]. Also, $K$ is the pretzel knot $P(-3, -3, 2, 1, 1)$.

The knot group $G(K)$ has a presentation with three meridian generators $a, b, c$ and two relations $a(aca^{-1}) = (aca^{-1})(baba^{-1}b^{-1})$ and

$$\begin{aligned} c(a^{-1}bab^{-1}a^{-1}b^{-1}aba^{-1}b^{-1})^{-1}c(a^{-1}bab^{-1}a^{-1}b^{-1}aba^{-1}b^{-1}) \\ = (aba^{-1}b^{-1})^{-1}b(aba^{-1}b^{-1})c. \end{aligned}$$

These relations are read off at the crossings marked by dots in Figure 5.

Let $\mu_n = b(ab^{-1})^n$ and $A = ba^{-1}$. Then $b = A^n$ and $a = A^{n-1}$ in the quotient group $G_n = G(K)/\langle\langle \mu_n \rangle\rangle$. Thus,

$$G_n = \langle c, A \mid c^{-1}A^{n-1}c = A^n, A^{2n-1}cA^{-(2n-1)} = c^{-1}A^n c\rangle.$$

By abelianisation, it is easy to see that $G_n$ is a perfect group. If $n \geq 2$, then the first relation implies that $G_n$ is the trivial group by Theorem 3.2.

If two generators $a$ and $b$ commute in $G(K)$, then the presentation of $G(K)$ reduces to

$$G(K) = \langle a, b, c \mid aca^{-1} = ca^{-1}b, cbaca^{-1}b^{-1} = bc\rangle.$$

By the first relation, the generator $b$ can be eliminated. This means that $G(K)$ is generated by two meridians, which implies that $K$ is 2-bridge [1], which is a contradiction. Therefore, the generators $a$ and $b$ do not commute in $G(K)$ and so Lemma 2.2 shows that $G(K)$ admits infinitely many nonconjugate weight elements.

By Lemma 2.1, the same conclusion holds for pretzel knots $P(m, -3, 2, 1, 1)$ with $m$ odd, because they are hyperbolic [7].

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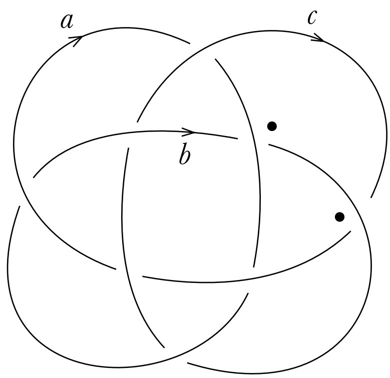


FIGURE 6. The knot $8_{18}$.

**5.2. The knot $8_{16}$.** Let $K$ be the knot $8_{16}$ as shown in Figure 5. The knot group $G(K)$ has a presentation with meridian generators $a, b, c$ indicated there and two relations

$$c \cdot (ba^{-1}b^{-1}a)^{-1}c(ba^{-1}b^{-1}a) = (ba^{-1}b^{-1}a)^{-1}c(ba^{-1}b^{-1}a) \cdot a^{-1}ba,$$
$$(b^{-1}cb)^{-1}a(b^{-1}cb) \cdot c^{-1}ac = c^{-1}ac \cdot (ba^{-1}b^{-1}a)^{-1}c(ba^{-1}b^{-1}a).$$

The relations are read off at the crossings marked by dots.

Let $\mu_n = b(ab^{-1})^n$ and $A = ba^{-1}$. Then $b = A^n$ and $a = A^{n-1}$ in the quotient group $G_n = G(K)/\langle\langle \mu_n \rangle\rangle$. The first relation reduces to $c = A^n$, and then the second relation gives $A = 1$. Hence, $G_n$ is the trivial group.

If the generators $a$ and $b$ commute in $G(K)$, then the first relation implies that $c = b$ and the second gives $aba = bab$. Thus, $G(K)$ is isomorphic to the knot group of the trefoil, which is a a contradiction. Therefore, the generators $a$ and $b$ do not commute in $G(K)$ and Lemma 2.2 applies.

**5.3. The knot $8_{18}$.** Let $K$ be the knot $8_{18}$ as shown in Figure 6. It is a hyperbolic 3-bridge knot with unknotting number two. If we use the meridian generators $a, b, c$ shown there, then the knot group $G(K)$ has a presentation with three generators $a, b, c$ and two relations $cac^{-1} \cdot (b^{-1}a^{-1}ba)^{-1}c(b^{-1}a^{-1}ba) = b \cdot cac^{-1}$ and

$$(bc^{-1}b^{-1})^{-1}a(bc^{-1}b^{-1}) \cdot (b^{-1}a^{-1}ba)^{-1}c(b^{-1}a^{-1}ba) = (b^{-1}a^{-1}ba)^{-1}c(b^{-1}a^{-1}ba) \cdot c.$$

The relations are read off at the crossings marked by dots.

Let $\mu_n = b(ab^{-1})^n$ and $A = ba^{-1}$ again. Then $b = A^n$ and $a = A^{n-1}$ in the quotient group $G_n = G(K)/\langle\langle \mu_n \rangle\rangle$. Thus,

$$G_n = \langle c, A \mid A^{n-1}cA^{-(n-1)} = c^{-1}A^n c, cA^{n-1}c^{-1} = A^{-n}cA^n \rangle.$$

The second relation changes to $cA^{-(n-1)}c^{-1} = A^{-n}c^{-1}A^n$; then $A^n cA^{-(n-1)} = c^{-1}A^n c$. The first relation implies that $A^n cA^{-(n-1)} = A^{n-1}cA^{-(n-1)}$, so $A = 1$. Thus, $c = 1$ and $G_n$ is the trivial group.

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If $a$ and $b$ commute in $G(K)$, then the first relation reduces to $b = caca^{-1}c^{-1}$. This means that $G(K)$ is generated by two meridians $a$ and $c$, which implies that $K$ is 2-bridge [1], which is a a contradiction. Thus, $G(K)$ has infinitely many nonconjugate weight elements by Lemma 2.2.

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