SECTORIAL COVERS FOR CURVES OF CONSTANT LENGTH

BY

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1. In answer to a question raised by Leo Moser, A. Meir proved some years ago that every plane arc of unit length lies in some closed semidisk of radius \( \frac{1}{2} \). His elegant, unpublished argument is reproduced here with his kind permission.

**Theorem 1 (A. Meir).** Every plane arc of length \( L \) lies in some closed semidisk of radius \( \frac{L}{2} \).

**Proof.** The assertion is clear for closed curves, for such a curve plainly lies in a semidisk of radius \( \frac{L}{2} \) centered at a point of contact of any support line of the curve. Let \( \Gamma \) be an arc of length \( L \) having distinct endpoints \( P \) and \( Q \), let \( l \) be a line of support parallel to the line \( PQ \) and touching \( \Gamma \) at a point \( R \), and let \( P' \) and \( Q' \) be the points symmetric to \( P \) and \( Q \) in \( l \) (Figure 1). Let \( O \) be the point in which the lines \( PQ' \) and \( QP' \) meet \( l \). Each point \( X \) on \( \Gamma \) lies between \( R \) and \( P \) or between \( R \) and \( Q \) along \( \Gamma \), and we may suppose that \( X \) lies between \( R \) and \( P \). Because the median of a triangle is shorter than the average of the lengths of the two adjacent sides,

\[
OX \leq \frac{1}{2}(XP + XQ') \leq \frac{1}{3}(PX + XR + RQ) \leq \frac{1}{2}L.
\]

Thus \( \Gamma \) lies in the semidisk of radius \( \frac{L}{2} \) and edge \( l \) centered at the point \( O \).

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In §3 of this note we generalize this result to circular sectors and show that there is a sector of area less than $0.3451L^2$ that can accommodate every arc of length $L$. Meir’s semidisk has area $\pi L^2/8 \approx 0.3927L^2$.

Section 2 is devoted to a characterization of circular sectors that contain a translate of every closed curve of length $L$.

In §4 we show that the least area of a convex set that contains a translate of every closed curve of length $L$ lies between $0.15544L^2$ and $0.15900L^2$, and in §5 we show that the least area of a convex set that contains a displacement of every arc of length $L$ lies between $0.21946L^2$ and $0.34423L^2$.

2. A circular sector is circumscribed about a curve if the curve lies in the sector and has a point on the circular boundary arc and a point on each of the boundary radii. We begin with a result about circular sectors that are circumscribed about a closed curve of length $L$.

Let $\text{Csc } x = \csc x$ when $0 < x < \pi/2$ and $\text{Csc } x = 1$ when $\pi/2 < x < \pi$. For $r > 0$ and $0 < \theta \leq \pi$, we denote the circular sector with radius $r$ and vertex angle $\theta$ by $S(r, \theta)$.

**Lemma 2.** If a circular sector $S(r, \theta)$ is circumscribed about a closed curve of length $L$, then $r < (L/2) \text{Csc } \theta$.

**Proof.** Let the sector $S(r, \theta) = \triangle ABC$ be circumscribed about a closed curve $\Gamma$ of length $L$, and let $X$, $Y$, and $Z$ be points of $\Gamma$ on the circular arc $BC$ and radial segments $AB$ and $AC$, respectively. The perimeter $p$ of $\triangle XYZ$ is at most $L$, and $p$ equals $L$ precisely when the curve $\Gamma$ coincides with $\triangle XYZ$. Let $X'$ and $X''$ be the points symmetric to $X$ in the lines $AB$ and $AC$ respectively. If $0 < \theta < \pi/2$, then

$$p = X'Y + YZ + ZX'' \geq X'X'' = 2r \sin \theta.$$ 

If $\pi/2 \leq \theta \leq \pi$, then

$$p = X'Y + YZ + ZX'' \geq X'Z + ZX'' \geq X'A + AX'' = 2r.$$ 

In either case,

$$r \leq \frac{1}{2}p \text{Csc } \theta \leq \frac{1}{2}L \text{Csc } \theta.$$ 

When $\theta$ is acute, the equality occurs precisely when $\Gamma$ coincides with $\triangle XYZ$ and the points $X'$, $Z$, $Y$, and $X''$ are collinear. When $\theta$ is not acute, the equality occurs precisely when $\Gamma$ is a radial segment (traversed twice).

A compact, convex set in the plane is a translation cover for a family of plane arcs if for each arc in the family there is a translation that carries the arc into the set. We can use Lemma 2 to characterize sectorial translation covers for the family $\mathcal{C}_L$ of all closed curves of length $L$.

**Theorem 3.** A sector $S(r, \theta)$ is a translation cover for $\mathcal{C}_L$ if and only if $r \geq (L/2) \text{Csc } \theta$. 

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Proof. Suppose that \( S(r, \theta) = \langle BAC \rangle \) is a sector satisfying \( r \geq (L/2) \csc \theta \), and suppose that \( \Gamma \) is a given closed curve of length \( L \). By a translation we can suppose that \( \Gamma \) lies in \( \angle BAC \) and touches each of the rays \( AB \) and \( AC \). Let \( r_0 \) be the maximum distance from the vertex \( A \) to any point on the (translated) curve. Then the sector with vertex \( A \) and radius \( r_0 \) is circumscribed about the curve, and according to Lemma 2, \( r_0 \leq (L/2) \csc \theta \leq r \). It follows that the (translated) curve lies in \( S(r, \theta) \). Conversely, it is plain that no sector with angle \( \theta \) and smaller radius can be a translation cover for all closed curves of length \( L \), because the width \( r \sin \theta \) of any covering sector \( S(r, \theta) \) in the direction perpendicular to a boundary ray must be at least \( L/2 \).

**Corollary 4.** The circular sector of least area that is a translation cover for \( \mathcal{C}_L \) has angle \( \theta_0 \) and radius \( (L/2) \csc \theta_0 \), where \( \theta_0 \approx 1.16556 \) is the least positive root of the equation \( \tan \theta = 2\theta \). The area of this sector is approximately \( 0.1725L^2 \).

**Proof.** For each \( \theta \), the smallest sector \( S(r, \theta) \) that is a translation cover for \( \mathcal{C}_L \) has radius \( r = (L/2) \csc \theta \) and area \( f(\theta) = \frac{1}{2}L^2 \theta \csc^2 \theta \). This function has a unique minimum on the interval \( (0, \pi) \) at the least positive root \( \theta_0 \) of the equation \( \tan \theta = 2\theta \).

3. A compact, convex (plane) set is a displacement cover for a family of (plane) arcs if for each arc in the family there is a displacement (i.e., a map of the plane that can be factored as a product of a translation and a rotation) that carries the arc into the set. By combining Lemma 2 with a reflection of J. Ralph Alexander's, we obtain a result on sectorial displacement covers for the family \( \mathcal{A}_L \) of arbitrary arcs of length \( L \) that generalizes Meir's semidisk result.

**Theorem 5.** If \( r \geq (L/2) \csc \theta \), then the circular sector \( S(r, 2\theta) \) is a displacement cover for \( \mathcal{A}_L \); and conversely when \( \theta \geq \pi/6 \).

**Proof.** Suppose that \( r \geq (L/2) \csc \theta \), and let \( \Gamma \) be an arc of length \( L \). The assertion follows from Theorem 3 if \( \Gamma \) is closed, so suppose that \( \Gamma \) has distinct endpoints \( P \) and \( Q \). Let \( m \) be the perpendicular bisector of the segment \( PQ \), and let \( \Gamma' \) be the closed curve of length \( L \) that results from reflecting the points of \( \Gamma \) lying on one side of \( m \) through \( m \) (Figure 2). Let \( l \) be a support line of \( \Gamma' \) that makes an angle \( \theta \) with \( m \). Then the circular sector with radius \( r \), sides on \( l \) and \( m \), and center at the point \( A \) in which \( l \) and \( m \) intersect surrounds \( \Gamma' \); and it is evident that the sector \( \langle BAC \rangle \) with radius \( r \) and vertex angle \( 2\theta \) covers \( \Gamma \). Every displacement cover must have diameter at least \( L \), so when \( \theta \geq \pi/6 \) no smaller sector is a cover.

Whether a sector \( S(r, 2\theta) \) with radius smaller than \( (L/2) \csc \theta \) can accommodate each arc of length \( L \) when \( \theta < \pi/6 \) is not known.
COROLLARY 6. There is a circular sector with area less than \(0.3451L^2\) that is a displacement cover for \(\mathcal{A}_L\).

Proof. For each \(\theta\) in \((0, \pi/2]\), the sector \(S(r, 2\theta)\) with radius \(r=(L/2) \csc \theta\) is a displacement cover for \(\mathcal{A}_L\) and has area \(f(\theta)=\frac{1}{2}L^2\theta \csc^2 \theta\). This function has a unique minimum value \(f(\theta_0)\approx 0.34501L^2\) at the least positive root \(\theta_0 \approx 1.16556\) of the equation \(\tan \theta = 2\theta\).

4. Problems of finding sets of certain kinds that can accommodate in a specified way each arc from a specified family are called "worm" problems, and a great variety of such problems, mostly unsolved, can be found in the literature. (For examples and further references, see [4] and the lists of research problems compiled by Croft [2], [3] and Moser [6].)

The smallest triangular translation cover for the family \(\mathcal{C}_L\) of all closed curves of length \(L\) is the equilateral triangle of side \(2L/3\) (see [9]). The smallest sectorial translation cover, determined in Corollary 4, is a little smaller than this smallest triangle.

But we can do a bit better. Once a given closed curve has been translated into a covering sector, a further translation will produce a point of contact with the circular boundary arc without taking the curve outside the sector. Then the curve surely cannot enter the small sector \(\Delta DAE\) having the same vertex and boundary rays and radius \((L/2)(\csc \theta_0-1)\), or its length would be greater than \(L\) (Figure 3). It follows that the truncated sector obtained by clipping the small isosceles triangle \(\Delta DAE\) with \(AD=AE=(L/2)(\csc \theta_0-1)\) from a covering sector \(S((L/2) \csc \theta, \theta)\) is
again a translation cover. This truncated sector has area
\[ f(\theta) = \frac{L^2}{2} \left[ \theta \csc^2 \theta - \sin \theta (\csc \theta - 1)^2 \right], \]
and minimizing this function on \((0, \pi]\) shows that there is such a truncated sector having area less than \(0.15900 L^2\); the minimum truncated sector has angle equal to the least positive root \(\theta_1 \approx 1.12120\) of the equation
\[ \tan \theta = 2 \theta - \sin \theta \cos^3 \theta. \]
This minimal truncated sector is the one pictured in Figure 3.

In 1921, Pál [7] proved that every convex set having minimal width \(t\) has area at least \(t^2/\sqrt{3}\) (see also [10, pp. 60, 221–222]). It follows that every translation cover for \(\mathcal{C}_L\) has area at least \(L^2/(4\sqrt{3}) \approx 0.144L^2\), because every such set obviously has minimal width \(t \geq L/2\). By modifying Pál’s argument, we can strengthen this lower bound to approximately \(0.15544L^2\).

For the following discussion, let \(T\) be a translation cover for \(\mathcal{C}_L\). We say that a triangle \(\Delta ABC\) is embedded in \(T\) if \(A, B,\) and \(C\) lie on the boundary of \(T\) and if there are support lines at \(A, B,\) and \(C\) that form a triangle enclosing \(T\).

**Lemma 7.** If \(\Delta ABC\) is embedded in \(T\) and if \(B'\) and \(C'\) are points of \(T\) across the line \(BC\) from \(A\) so that the segments \(B'C'\) and \(BC\) are parallel, then \(B'C' < BC\).

**Proof.** If \(B'C' \geq BC\), then (by the parallel postulate) every support line at \(B\) meets
LEMMA 8. Every triangle embedded in $T$ has perimeter at least $L$.

Proof. Suppose $\triangle ABC$, with perimeter $p$, is embedded in $T$. By hypothesis there are points $A'$, $B'$, and $C'$ in $T$ so that $\triangle A'B'C'$ is (positively) homothetic to $\triangle ABC$ and has perimeter $p' = L$. If $\triangle A'B'C'$ is a translate of $\triangle ABC$, then $p = p' = L$. Otherwise, let $X$ be the center of the homothety.

We show first that $X$ cannot lie in one of the (open) angular regions formed by the sides of $\triangle ABC$ off the vertices. Suppose $X$ lies in such a region, say in the interior of $\angle DAE$ (Figure 4a). Since this set is disjoint from $T$ (otherwise $A$ would be an interior point of $T$), $A$ must lie on the segment $A'X$. Suppose $A' \neq A$. Then since the segments $B'C'$ and $BC$ are parallel and $\triangle ABC$ is embedded in $T$, $B'C' < BC$, an obvious contradiction.

Consequently $X$ lies in one of the closed angular regions bounded by an angle of $\triangle ABC$, say in the closed region bounded by $\angle BAC$ (Figure 4b). Then $A'$ lies on the segment $AX$, and it follows at once that $p \geq p' = L$.

It was proved by Blaschke [1, pp. 370-371] that if $S$ is an incircle of a compact, convex set $T$ with boundary $\partial T$, then either $S \cap \partial T$ contains two points that are the ends of a diameter of $S$, or $S \cap \partial T$ contains three points that are the vertices of an acute triangle (see also [10, pp. 59, 215-216]).

COROLLARY 9. The inradius of $T$ is at least $L\sqrt{3}/9$.

Proof. Let $r$ be the inradius of $T$, and let $S$ be an incircle. If $S \cap \partial T$ contains two points $P$ and $Q$ that are the ends of a diameter of $S$, then $PQ = 2r \geq L/2$, and so $r \geq L/4 \geq L\sqrt{3}/9$. If on the other hand $S \cap \partial T$ contains three points $A$, $B$, and $C$
that form an acute triangle, then $\Delta ABC$ is embedded in $T$ (because the unique support lines to $T$ at $A$, $B$, and $C$ are perpendicular to the radii of $S$ to these points, and the center of $S$ lies inside $\Delta ABC$); and it follows from the lemma and Jensen’s inequality [5, pp. 23–25, 28] that

$$
\frac{r}{2 \sin \alpha} = \frac{b}{2 \sin \beta} = \frac{c}{2 \sin \gamma} = \frac{p}{2(\sin \alpha + \sin \beta + \sin \gamma)} \geq \frac{L}{2(\sin \alpha + \sin \beta + \sin \gamma)} \geq \frac{L}{6 \sin \frac{\pi}{3}} = \frac{L\sqrt{3}}{9},
$$

proving the assertion.

For the sake of completeness, we include a sketch of the relevant portions of Pál’s argument (from [7, pp. 313–314]). For each $r$ in $[L/6, L/4]$, let $\Phi_r$ be the convex hull of a circle of radius $r$ and three points $X$, $Y$, and $Z$ at distance $(L/2) - r$ from the center of the circle, arranged so that the “caps” that are added to the circle do not overlap (see Figure 5). The area $f(r)$ of a figure $\Phi_r$ is given by

$$
f(r) = \pi r^2 + \frac{3r}{2} (L^2 - 4rL)^{1/2} - 3r^2 \arccos \frac{2r}{L - 2r};
$$

and since $f'(r) > 0$ for $L/6 \leq r < L/4$, the area $f(r)$ is an increasing function of $r$. We claim that $T$ contains a figure $\Phi_r$ for some $r \geq L\sqrt{3}/9$ and that consequently the area of $T$ is at least $f(L\sqrt{3}/9)$.

Let $r$ be the inradius of $T$, and let $S$ be an incircle with center $0$. If $S \cap \partial T$ contains two points that are the ends of a diameter of $S$, then $r \geq L/4$ (as observed before), and the circle $\Phi_{L/4}$ is a subset of $T$. Otherwise $S \cap \partial T$ contains three points $A$, $B$, and $C$ that form an acute triangle. The support lines $l_A$, $l_B$, and $l_C$ to $T$ at $A$, $B$, and $C$ are tangent to $S$. Let $D$, $E$, and $F$ be points on $\partial T$ at which the support lines $l_D$, $l_E$, and $l_F$ are parallel to $l_A$, $l_B$, and $l_C$ respectively (Figure 5).

The distance between $l_A$ and $l_D$ is at least $L/2$, so $AD \geq L/2$; and $DO \geq (L/2) - r$ since $AO = r$. Let $X$ be the point on the segment $OD$ so that $OX = (L/2) - r$. The tangent lines from $X$ to $S$ form a cap that lies entirely in $T$ and on the opposite side of the line $BC$ from $A$.

Similarly we find points $Y$ on the segment $OE$ and $Z$ on the segment $OF$; and the circle $S$ and points $X$, $Y$, and $Z$ determine a figure $\Phi_r \subseteq T$, where $r$ is the inradius of $T$. It follows that the area of $T$ must be at least the area $f(r)$ of $\Phi_r$, which, since $f(r)$ is increasing, must be at least $f(L\sqrt{3}/9)$. In summary, we have the following theorem.
THEOREM 10. Every translation cover for the family of closed curves of length $L$ has area at least $f(L\sqrt{3}/9) \approx 0.15544L^2$, where $f(r)$ is given by (1); and there exists a translation cover for this family having area less than $0.15900L^2$.

The circle $\Phi_{L/4}$ is the only figure $\Phi_r$ that is a translation cover for $\mathcal{C}_L$, because the minimal width of each $\Phi_r$ for $r<L/4$ is less than $L/2$.

5. By truncating a sectorial displacement cover, we can produce a smaller displacement cover for the family $\mathcal{A}_L$ of all arcs of length $L$. Indeed, the region produced by clipping the small isosceles triangle with vertex angle $2\theta$ and sides of length $(L/2)(\csc \theta - 1)$ from the vertex of a covering sector $\mathcal{S}((L/2)\csc \theta, 2\theta)$ is again a displacement cover for $\mathcal{A}_L$, as can easily be seen by applying the reflection argument employed in the proof of Theorem 5. Its area,

$$f(\theta) = \frac{L^2}{8} [2\theta \csc^2 \theta - \sin 2\theta (\csc \theta - 1)^2],$$

has a unique minimum value $f(\theta_2) \approx 0.34423L^2$ at the least positive root $\theta_2 \approx 1.14687$ of the equation

$$\tan \theta = 2\theta - \tan \theta (\cos^2 \theta - 2 \sin^3 \theta + \sin^4 \theta).$$

The best lower bound we know for the area of such covers is $0.21946L^2$, which arises as follows. Schaer [8] showed that the arc of length 1 that has maximum thickness, i.e., whose minimum width is as large as possible, has thickness $b_0 \approx 0.43893$. Every displacement cover for $\mathcal{A}_L$ must have diameter $d$ at least $L$ and width $w$ in the direction perpendicular to a diameter at least $b_0L$. Consequently its area must be at least $wd/2 \geq b_0L^2/2 \approx 0.21946L^2$. In summary, we have the following theorem.
Theorem 11. Every displacement cover for the family of all arcs of length $L$ has area at least $0.21946L^2$; and there exists a displacement cover for this family with area less than $0.34423L^2$.

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References

6. Leo Moser, Poorly formulated unsolved problems of combinatorial geometry. (Mimeographed.)