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Periods of automorphic forms: the case of $(\mathrm{GL}_{n+1} \times \mathrm{GL}_n, \mathrm{GL}_n)$

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Abstract

Following Jacquet, Lapid and Rogawski, we define a regularized period of an automorphic form on $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$ along the diagonal subgroup GL_n and express it in terms of the Rankin–Selberg integral of Jacquet, Piatetski-Shapiro and Shalika. This extends the theory of Rankin–Selberg integrals to all automorphic forms on $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$.

1. Introduction

Let F be a number field and $\mathbb{A}$ the ring of adeles of F. Let G be a connected reductive algebraic group over F and G' a closed subgroup of G over F. Let $\mathscr{A}(G)$ and $\mathscr{A}(G')$ denote the spaces of automorphic forms on $G(\mathbb{A})$ and $G'(\mathbb{A})$, respectively. For $\varphi \in \mathscr{A}(G)$ and $\varphi' \in \mathscr{A}(G')$, we will consider the integral

$$\int_{G'(F)\backslash G'(\mathbb{A})} \varphi(g)\varphi'(g)\, dg. \tag{1.1}$$

When this integral is convergent, it is called the period of $\varphi \otimes \varphi'$ along G', and often plays a significant role in the theory of automorphic representations and L-functions.

For example, in the case of a special orthogonal group $G = \mathrm{SO}_{n+1}$ and its subgroup $G' = \mathrm{SO}_n$, Gross and Prasad [GP92] proposed a conjecture on the nonvanishing of the period in terms of the central value of a certain automorphic L-function. Further, a more precise conjecture in [II10] gives an exact formula for the square of the period in terms of L-values and endoscopy. Also, the Gross–Prasad conjecture has been generalized to other classical groups by Gan, Gross and Prasad [GGP12]. Their conjecture includes the case of a unitary group $G = \mathrm{U}_{n+1}$ and its subgroup $G' = \mathrm{U}_n$, a substantial part of which has been proven by Wei Zhang [Zha14a, Zha14b].

In his proof of the Gan–Gross–Prasad conjecture, Wei Zhang developed the theory of the relative trace formula of Jacquet and Rallis [JR11], which compares

$$\mathrm{U}_n\backslash(\mathrm{U}_{n+1} \times \mathrm{U}_n)/\mathrm{U}_n$$

with

$$\mathrm{Res}_{E/F}(\mathrm{GL}_n)\backslash\mathrm{Res}_{E/F}(\mathrm{GL}_{n+1} \times \mathrm{GL}_n)/(\mathrm{GL}_{n+1} \times \mathrm{GL}_n).$$

Here E is the quadratic extension of F which splits the unitary groups. On the other hand, since Zhang employed a simple trace formula, he could treat only automorphic representations satisfying certain local conditions. To prove a relation of the period to endoscopy, it is necessary

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to remove these local conditions, which requires one to develop the relative trace formula in general.

Towards establishing the spectral expansion of the relative trace formula of Jacquet and Rallis, we will study the period in the case of a general linear group $G = \mathrm{GL}_{n+1}$ and its subgroup $G' = \mathrm{GL}_n$. If φ and φ' are cusp forms, then the integral (1.1) is absolutely convergent and can be expressed in terms of the central value of the tensor product L-function via the integral representation of Jacquet, Piatetski-Shapiro and Shalika. The purpose of this paper is to make sense of (1.1) even when it may not be convergent, and to prove its expression in terms of the L-value.

We regularize the integral (1.1) following Jacquet, Lapid and Rogawski [JLR99], but unlike their case, none of the Siegel sets of G' is contained in any Siegel set of G, so our case is more complicated. More precisely, we define a mixed truncation operator Λ^T_m, which carries smooth functions on $G(F)\backslash G(\mathbb{A})$ of uniform moderate growth to functions on $G'(F)\backslash G'(\mathbb{A})$ of rapid decay, for a sufficiently regular $T \in \mathfrak{a}_0^G$, by setting

$$\Lambda^T_m\varphi(g) = \sum_P (-1)^{\dim \mathfrak{a}_P^G} \sum_{\gamma \in P(F)\backslash P(F)WG'(F)} \varphi_P(\gamma g)\hat{\tau}_P(H_P(\gamma g) - T).$$

Here P runs over standard parabolic subgroups of G, W is the Weyl group of G, φ_P is the constant term of φ along P, and we refer to the notation section below for unexplained terms. This operator is a variant of Arthur's truncation operator [Art80], which is suitable for studying the integral (1.1); in the $n = 1$ case, it was introduced by Jacquet and Chen (see [JC01, § 8.1]). For $\varphi \in \mathscr{A}(G)$ and $\varphi' \in \mathscr{A}(G')$, the integral

$$\int_{G'(F)\backslash G'(\mathbb{A})} \Lambda^T_m\varphi(g)\varphi'(g)\, dg$$

is absolutely convergent and defines a function in T of the form

$$\sum_\lambda p_\lambda(T)e^{\langle \lambda, T\rangle}.$$

Here λ runs over a finite subset of $(\mathfrak{a}^G_{0,\mathbb{C}})^*$ and $p_\lambda(T)$ is a polynomial in T. When the exponents of φ and φ' satisfy a certain mild restriction (see Definition 3.2 for details), we can define a regularized period $\mathbf{P}^{G'}(\varphi \otimes \varphi')$ such that

$$\mathbf{P}^{G'}(\varphi \otimes \varphi') = p_0(T),$$

where the polynomial $p_0(T)$ attached to $\lambda = 0$ turns out to be constant. If φ is a cusp form, then since $\Lambda^T_m\varphi = \varphi$, the identity

$$\mathbf{P}^{G'}(\varphi \otimes \varphi') = \int_{G'(F)\backslash G'(\mathbb{A})} \varphi(g)\varphi'(g)\, dg$$

is evident. It turns out that this identity always holds when the right-hand side is absolutely convergent (see Corollary 3.10). What is special about the general linear groups is that one can easily construct from a single automorphic form $\varphi' \in \mathscr{A}(G')$ a holomorphic family φ'_s over $\mathbb{C}$ of automorphic forms on $G'(\mathbb{A})$ by setting $\varphi'_s(g) := \varphi'(g)|\det g|^s$. It is important to note that the regularized period $\mathbf{P}^{G'}(\varphi \otimes \varphi'_s)$ is well-defined for $s \in \mathbb{C}$ in general position.

We can compute the meromorphic function $s \mapsto \mathbf{P}^{G'}(\varphi \otimes \varphi'_s)$ explicitly. Let $W^{\psi}(\varphi)$ and $W^{\bar{\psi}}(\varphi')$ be the Whittaker functions of φ and φ' defined by

$$W^{\psi}(g, \varphi) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ug)\overline{\psi(u)}\, du, \quad W^{\bar{\psi}}(g, \varphi') = \int_{N'(F)\backslash N'(\mathbb{A})} \varphi'(ug)\psi(u)\, du,$$

where N and N' are the subgroups of upper triangular unipotent matrices in G and G', respectively, and ψ is a nontrivial character of $F\backslash\mathbb{A}$, which we regard as a generic character of $N(F)\backslash N(\mathbb{A})$ and $N'(F)\backslash N'(\mathbb{A})$. Consider the zeta integral

$$I(s, \varphi, \varphi') = \int_{N'(\mathbb{A})\backslash G'(\mathbb{A})} W^{\psi}(g, \varphi) W^{\bar{\psi}}(g, \varphi')\, |\det g|^s\, dg,$$

which is studied by Jacquet, Piatetski-Shapiro and Shalika in [JPS83, JS90, Jac09]. It can be shown to converge absolutely for the real part of s sufficiently large and uniformly for s in a compact set by a gauge estimate combined with Franke's theorem (see [JPS79, §§ 13 and 4.6]). Then our main result is the following theorem.

Theorem 1.1. *For $\varphi \in \mathscr{A}(G)$ and $\varphi' \in \mathscr{A}(G')$, we have the identity*

$$\mathbf{P}^{G'}(\varphi \otimes \varphi'_s) = I(s, \varphi, \varphi').$$

Theorem 1.1 shows that the functional $\mathbf{P}^{G'}$ is $G'(\mathbb{A})$-invariant. The regularized period $\mathbf{P}^{G'}(\varphi \otimes \varphi')$ turns out to be zero unless both φ and φ' are generic, in which case it can be expressed in terms of the central value of the tensor product L-function. More precisely, assume that φ and φ' belong to automorphic representations π and π' of $G(\mathbb{A})$ and $G'(\mathbb{A})$, respectively, induced from irreducible cuspidal automorphic representations of Levi subgroups, and that they are decomposable. Then we deduce that

$$\mathbf{P}^{G'}(\varphi \otimes \varphi'_s) = L\biggl(s + \frac{1}{2}, \pi \times \pi'\biggr) \cdot \prod_v \frac{I(s, W^{\psi_v}_{\varphi_v}, W^{\bar{\psi}_v}_{\varphi'_v})}{L(s + \frac{1}{2}, \pi_v \times \pi'_v)}.$$

Here $L(s, \pi \times \pi')$ is the tensor product L-function and $I(s, W^{\psi_v}_{\varphi_v}, W^{\bar{\psi}_v}_{\varphi'_v})$ is the local zeta integral of Jacquet, Piatetski-Shapiro and Shalika, defined by

$$I(s, W^{\psi_v}_{\varphi_v}, W^{\bar{\psi}_v}_{\varphi'_v}) = \int_{N'_v\backslash G'_v} W^{\psi_v}_{\varphi_v}(g) W^{\bar{\psi}_v}_{\varphi'_v}(g)\, |\det g|^s\, dg.$$

Corollary 5.7 shows that the functional $\mathbf{P}^{G'}$ does not vanish on $\pi \otimes \pi'$ if and only if $L(1/2, \pi \times \pi') \neq 0$.

Let π and π' be irreducible residual automorphic representations of $G(\mathbb{A})$ and $G'(\mathbb{A})$, respectively. Assume that π is not one-dimensional. Corollary 5.8 shows that for $\varphi \in \pi$ and $\varphi' \in \pi'$, the integral (1.1) is absolutely convergent and equal to zero.

Theorem 1.1 looks straightforward at first glance, but the proof is rather indirect and relies on a series of reduction steps. An important ingredient is the fact that any automorphic form is a linear combination of derivatives of cuspidal Eisenstein series, which allows us to assume that φ is a derivative of a cuspidal Eisenstein series. Franke proved this deep result in [Fra98]. We use Cauchy's integral formula and Fubini's theorem to reduce the computation to the case where φ is a cuspidal Eisenstein series.

Formally, an automorphic form φ has a Fourier expansion of the form

$$\varphi(g) = \sum_{i=0}^{n} \sum_{\gamma \in \mathscr{P}'_i(F)\backslash G'(F)} W^{\psi}_{\mathcal{Q}_i}(\gamma g, \varphi_{\mathcal{Q}_i}),$$

where $\mathscr{P}'_i$ is the subgroup of the standard parabolic subgroup of G' of type $(i, n-i)$ consisting of matrices whose GL_{n-i} part is upper triangular unipotent, $\mathcal{Q}_i$ is the standard parabolic subgroup of G of type $(i, n+1-i)$, and $W^{\psi}_{\mathcal{Q}_i}(\varphi_{\mathcal{Q}_i})$ is the Whittaker function of the GL_{n+1-i} part of the constant term $\varphi_{\mathcal{Q}_i}$. This is simply an inductive abelian Fourier expansion beginning along the last column of N (see [Sha74] and Proposition 4.2). If we ignore convergence issues, then a formal computation yields

$$\int_{G'(F)\backslash G'(\mathbb{A})} \varphi(g)\varphi'_s(g)\, dg \text{ `='} \sum_{i=0}^{n} \int_{\mathscr{U}'_i(\mathbb{A})\mathrm{GL}_i(F)\backslash G'(\mathbb{A})} W^{\psi}_{\mathcal{Q}_i}(g, \varphi_{\mathcal{Q}_i}) W^{\bar{\psi}}_{\mathcal{Q}'_i}(g, \varphi'_{\mathcal{Q}'_i})\, |\det g|^s\, dg,$$

where $\mathcal{Q}'_i$ is the standard parabolic subgroup of G' of type $(i, n-i)$ and $\mathscr{U}'_i$ is the unipotent radical of the standard parabolic subgroup of G' of type $(i, 1, 1, \ldots, 1)$. The zeroth term in the right-hand side is $I(s, \varphi, \varphi')$. For $i > 0$, the ith term involves an integral of an exponential function over the multiplicative group of positive real numbers, and hence it is divergent. An integral of this type should be interpreted as zero for the reason explained by Lapid and Rogawski in [LR03] (see also [Cas93]). Following [LR03], we compute the absolutely convergent integral

$$\int_{G'(F)\backslash G'(\mathbb{A})} \theta(g)\varphi'_s(g)\, dg \tag{1.2}$$

in two ways to circumvent the convergence problems, where θ is a pseudo-Eisenstein series on $G(\mathbb{A})$ and $\varphi' \in \mathscr{A}(G')$. We may suppose that

$$\theta(g) = \int_{\lambda \in (\mathfrak{a}^G_{P,\mathbb{C}})^*,\ \Re\lambda = \kappa} \beta(\lambda) E(g, \phi, \lambda)\, d\lambda,$$

where $\kappa \in (\mathfrak{a}^G_P)^*$ is positive enough, β is a Paley–Wiener function on $(\mathfrak{a}^G_{P,\mathbb{C}})^*$, and $E(\phi, \lambda)$ is a cuspidal Eisenstein series induced from a parabolic subgroup P of G. On one hand, we transform (1.2), under some mild restriction on β, into the integral

$$\int_{\Re\lambda = \kappa} \beta(\lambda) \mathbf{P}^{G'}(E(\phi, \lambda) \otimes \varphi'_s)\, d\lambda$$

of the regularized periods. On the other hand, we transform (1.2), under another mild restriction on β, into the integral

$$\int_{\Re\lambda = \kappa} \beta(\lambda) I(s, E(\phi, \lambda), \varphi')\, d\lambda$$

of the zeta integrals by inserting the Fourier expansion of θ. To justify the manipulation, we use uniform estimates for archimedean Whittaker functions due to Jacquet [Jac04]. Strictly speaking, we need to consider the convolution $f * \theta$ by $f \in C_c^\infty(G_\infty)$ to apply Jacquet's estimates. There are sufficiently many β, which allows us to extract the desired identity

$$\mathbf{P}^{G'}(E(\phi, \lambda) \otimes \varphi'_s) = I(s, E(\phi, \lambda), \varphi').$$

This paper is organized as follows. In § 2 we define the mixed truncation operator Λ^T_m and its analogue for parabolic subgroups (and Weyl elements). In § 3 we define the regularized

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period $\mathbf{P}^{G'}(\varphi \otimes \varphi')$, which exploits these mixed truncation operators, and show that it equals the coefficient $p_0(T)$ of the zero exponent of the polynomial exponential function defined by the period of the truncated automorphic form. In § 4 we prove Theorem 1.1, and in § 5 we discuss some simple consequences thereof.

Notation

For two integers $a \leqslant b$, we denote the set $\{a, a+1, \dots, b\}$ by $[a, b]$. When $a > b$, we understand that $[a, b] = \emptyset$, $\sum_{i=a}^{b} = 0$ and $\prod_{i=a}^{b} = 1$. For a complex number z, $\Re z$ denotes the real part of z. The same notation will be used for elements of the complexification of a real vector space. For a finite-dimensional real vector space V, we denote by V^* the space of real linear forms on V, by $V^*_{\mathbb{C}}$ the space of complex linear forms on V, and by $\mathbb{C}[V]$ the space of polynomial functions on V.

Let F be a number field with adele ring $\mathbb{A}$. Let G be a connected reductive algebraic group over F. We write G_v for the localization of G at a place v of F and G_∞ for the product of all the archimedean localizations of G. Throughout this paper, the letters P and Q are reserved for parabolic subgroups of G defined over F, the letters M and L for their Levi subgroups, and the letters U and V for their unipotent radicals. Thus

$$P = MU, \quad Q = LV.$$

If M appears as a subscript or a superscript, we shall often write P instead of M for the subscript or superscript. Let $\mathrm{Rat}(M)$ be the group of algebraic characters of M defined over F. Put

$$\mathfrak{a}_P^* = \mathfrak{a}_M^* = \mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \mathfrak{a}_P = \mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{R}).$$

The canonical pairing on $\mathfrak{a}_P^* \times \mathfrak{a}_P$ is denoted by $\langle \ , \ \rangle$. We define a function $H_P : M(\mathbb{A}) \to \mathfrak{a}_P$ by the requirement that

$$e^{\langle \chi, H_P(m) \rangle} = |\chi(m)|$$

for all $\chi \in \mathrm{Rat}(M)$ and $m \in M(\mathbb{A})$. Let $M(\mathbb{A})^1$ be the intersection of the kernels of the homomorphisms $|\chi|$, where χ ranges over $\mathrm{Rat}(M)$. Let Z_M be the maximal split torus in the center of M. Choose an isomorphism $Z_M \simeq \mathbb{G}_m^l$, and let A_P be the image of $(\mathbb{R}_+^\times)^l$ in $Z_{M,\infty}$, where $l = \dim \mathfrak{a}_P$ and $\mathbb{R} \hookrightarrow F \otimes_{\mathbb{Q}} \mathbb{R}$ is given by $x \mapsto 1 \otimes x$. Note that $M(\mathbb{A}) = A_P \times M(\mathbb{A})^1$ and H_P induces an isomorphism $A_P \simeq \mathfrak{a}_P$. We denote by e^X the element in A_P such that $H_P(e^X) = X$. Using an Iwasawa decomposition, we extend H_P to the left $U(\mathbb{A})$-invariant, right K-invariant function on $G(\mathbb{A})$, where K is a fixed good maximal compact subgroup of $G(\mathbb{A})$. Let W^M be the Weyl group of M. Then W^M acts naturally on $\mathfrak{a}_P$ and $\mathfrak{a}_P^*$.

Discrete groups are equipped with the counting measures, unipotent groups U with the Haar measures giving $U(F)\backslash U(\mathbb{A})$ volume 1, and K with the Haar measure of total volume 1. We choose Haar measures on $M(\mathbb{A})$ for all Levi subgroups M of G compatibly with respect to the Iwasawa decomposition. We also have a Haar measure on A_P through its isomorphism with $\mathfrak{a}_P$ once we fix a Haar measure on $\mathfrak{a}_P$.

Let $\mathscr{A}_P(G)$ be the space of automorphic forms on $U(\mathbb{A})P(F)\backslash G(\mathbb{A})$, i.e. smooth, K-finite and $\mathfrak{z}$-finite functions of moderate growth, where $\mathfrak{z}$ is the center of the universal enveloping algebra of the complexified Lie algebra of G_∞. We write $\mathscr{A}_P^1(G)$ for the subspace of those $\phi \in \mathscr{A}_P(G)$ such that $\phi(ag) = e^{\langle \rho_P, H_P(a) \rangle} \phi(g)$ for all $a \in A_P$ and $g \in G(\mathbb{A})$, where the function $e^{\langle \rho_P, H_P(\cdot) \rangle}$ is the square root of the modulus function of $P(\mathbb{A})$. Let $\mathscr{A}_P^c(G)$ be the space of cusp forms in $\mathscr{A}_P^1(G)$.

When $P = G$, we will omit the subscript P. For any smooth function ϕ on $P(F)\backslash G(\mathbb{A})$ and any parabolic subgroup $Q \subset P$, the constant term of ϕ along Q is defined by

$$\phi_Q(g) = \int_{V(F)\backslash V(\mathbb{A})} \phi(vg)\, dv.$$

The map $\phi \mapsto \phi_Q$ sends $\mathscr{A}_P(G)$ to $\mathscr{A}_Q(G)$. According to [MW95, § I.3.2], an automorphic form $\phi \in \mathscr{A}_P(G)$ admits a finite decomposition

$$\phi(uamk) = \sum_i Q_i(H_P(a))\phi_i(mk)e^{\langle \lambda_i + \rho_P, H_P(a)\rangle} \tag{1.3}$$

for $u \in U(\mathbb{A})$, $a \in A_P$, $m \in M(\mathbb{A})^1$ and $k \in K$, where $\lambda_i \in \mathfrak{a}^*_{P,\mathbb{C}}$, $Q_i \in \mathbb{C}[\mathfrak{a}_P]$ and $\phi_i \in \mathscr{A}_P(G)$ such that $\phi_i(ag) = \phi_i(g)$ for all $a \in A_P$ and $g \in G(\mathbb{A})$. The set of distinct exponents λ_i occurring in (1.3) is uniquely determined by ϕ and is called the set of exponents of ϕ. When $Q \subset P$, the exponents of ϕ along Q are, by definition, the exponents of ϕ_Q, and the cuspidal exponents of ϕ along Q are the exponents of the cuspidal component of ϕ along Q; they are denoted by $\mathscr{E}_Q(\phi)$ and $\mathscr{E}_Q^{\mathrm{cusp}}(\phi)$, respectively.

For each positive integer m, we denote by G_m the general linear group GL_m, by T_m the subgroup of diagonal matrices, by B_m the subgroup of upper triangular matrices, and by N_m the subgroup of upper triangular matrices with unit diagonal. A parabolic subgroup of G_m is said to be standard if it contains B_m. A standard Levi subgroup of G_m is the unique Levi factor containing T_m of a standard parabolic subgroup of G_m. By parabolic and Levi subgroups we shall always mean standard parabolic and Levi subgroups. All these groups are regarded as algebraic groups over F. A composition of m is a sequence of positive integers whose sum is m. There is a bijection between the set of compositions of m and the set of standard parabolic subgroups of G_m, namely, for each composition $\mathbf{n} = (n_1, \dots, n_t)$ of m, the standard parabolic subgroup $P_{\mathbf{n}} = M_{\mathbf{n}}U_{\mathbf{n}}$ of G_m is given by

$$M_{\mathbf{n}} = \left\{ \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_t \end{pmatrix} \,\middle|\, g_i \in G_{n_i} \right\}, \quad U_{\mathbf{n}} = \left\{ \begin{pmatrix} \mathbf{1}_{n_1} & * & * \\ & \ddots & * \\ & & \mathbf{1}_{n_t} \end{pmatrix} \right\}.$$

When $P = P_{\mathbf{n}}$, we set

$$\mathbf{I}_P = \{n_1 + \dots + n_k \mid k \in [1, t]\}.$$

Let K_m be the standard maximal compact subgroup of $G_m(\mathbb{A})$. The Weyl group $W^m = W^{G_m}$ is identified with the symmetric group $\mathfrak{S}_m$. We take permutation matrices in G_m as representatives of elements in W^m. In particular, we have $W^m \subset K_m$. We identify the spaces $\mathfrak{a}_{B_m}$ and $\mathfrak{a}^*_{B_m}$ with $\mathbb{R}^m$. We fix a positive-definite W^m-invariant scalar product on $\mathfrak{a}_{B_m}$. This defines a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}_{B_m}$, which in turn determines Haar measures on $\mathfrak{a}_{B_m}$ and its subspaces. We define a height $\|\cdot\|$ on $G_m(\mathbb{A})$ by

$$\|g\| = \prod_v \|g_v\|, \quad \|g_v\| = \max_{i,j\in[1,m]} \{|(g_v)_{i,j}|,\, |(g_v^{-1})_{i,j}|\}.$$

For any smooth function ϕ on $G_m(\mathbb{A})$, $s \in \mathbb{C}$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, we define functions ϕ_s and ϕ_λ on $G_m(\mathbb{A})$ by

$$\phi_s(g) = \phi(g)|\det g|^s, \quad \phi_\lambda(g) = \phi(g)e^{\langle \lambda, H_P(g)\rangle}, \quad g \in G_m(\mathbb{A}).$$

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We fix a positive integer n and write $G=G_{n+1}$ and $G'=G_n$. When $G_m=G$, we will omit the subscript or superscript m, and we adopt the same notation, adding a prime $'$ for G'. Thus $B=TN$ and $B'=T'N'$ are the Borel subgroups of G and G', and K and K' are the maximal compact subgroups of G and G', respectively. For convenience, we will write 0 for any subscript where our notation would normally call for B. Thus $\mathfrak{a}_0=\mathfrak{a}_B$, $\mathfrak{a}'_0=\mathfrak{a}_{B'}$, $\mathfrak{a}_0^*=\mathfrak{a}_B^*$, $(\mathfrak{a}'_0)^*=\mathfrak{a}_{B'}^*$, $H_0=H_B$, $H'_0=H_{B'}$, $A_0=A_B$, $A'_0=A_{B'}$ and so on. We embed G' into G via the map

$$g\mapsto\begin{pmatrix} g & \\ & 1\end{pmatrix}.$$

Then the associated embedding $\mathfrak{a}'_0\hookrightarrow\mathfrak{a}_0$ is given by

$$(\lambda_1,\dots,\lambda_n)\mapsto(\lambda_1,\dots,\lambda_n,0).$$

Let Δ_0^P be the set of simple roots of T in M. If Q is a parabolic subgroup contained in P, then we have canonical direct sum decompositions

$$\mathfrak{a}_Q=\mathfrak{a}_Q^P\oplus\mathfrak{a}_P,\quad \mathfrak{a}_Q^*=(\mathfrak{a}_Q^P)^*\oplus\mathfrak{a}_P^*.$$

Let Δ_Q^P be the set of linear forms on $\mathfrak{a}_Q$ obtained by restriction of elements in the complement of Δ_0^Q in Δ_0^P. Then $\mathfrak{a}_P$ is the subspace of $\mathfrak{a}_Q$ annihilated by Δ_Q^P. For each $\alpha\in\Delta_Q^P$, let $\alpha^\vee$ be the projection of $\beta^\vee$ to $\mathfrak{a}_Q^P$, where β is the root in Δ_0 whose restriction to $\mathfrak{a}_Q^P$ is α. Set $(\Delta^\vee)_Q^P=\{\alpha^\vee\mid \alpha\in\Delta_Q^P\}$. Let $\hat{\Delta}_Q^P$ be the dual basis of $(\Delta^\vee)_Q^P$ in $(\mathfrak{a}_Q^P)^*$. We define $(\hat{\Delta}^\vee)_Q^P$ to be the basis of $\mathfrak{a}_Q^P$ dual to Δ_Q^P. When $P=G$, we will omit the superscript G. For example, we will write $\Delta_0=\Delta_0^G$, $\Delta_P=\Delta_P^G$, $\Delta_P^\vee=(\Delta^\vee)_P^G$, $\hat{\Delta}_P=\hat{\Delta}_P^G$, $\hat{\Delta}_P^\vee=(\hat{\Delta}^\vee)_P^G$ and $W=W^G$. If we put

$$\varpi_j^\vee=\frac{1}{n+1}(\underbrace{n+1-j,\dots,n+1-j}_{j},\underbrace{-j,\dots,-j}_{n+1-j}),$$

then $\hat{\Delta}_0^\vee=\{\varpi_j^\vee\mid j\in[1,n]\}$ and

$$\hat{\Delta}_P^\vee=\{\varpi_j^\vee\mid j\in\mathbf{I}_P\smallsetminus\{n+1\}\}.$$

We denote by X^Q (respectively X_Q^P or X_P) the canonical projection of $X\in\mathfrak{a}_0$ onto $\mathfrak{a}_0^Q$ (respectively $\mathfrak{a}_Q^P$ or $\mathfrak{a}_P$) given by the decomposition $\mathfrak{a}_0=\mathfrak{a}_0^Q\oplus\mathfrak{a}_Q^P\oplus\mathfrak{a}_P$, and similarly for $\mathfrak{a}_0^*$. We extend the linear functionals in Δ_Q^P (respectively $(\hat{\Delta}^\vee)_Q^P$) to elements of $\mathfrak{a}_0^*$ (respectively $\mathfrak{a}_0$) by means of these projections. We write $\rho_0\in\mathfrak{a}_0^*$ for half the sum of the positive roots of T in G and denote by ρ_0^Q, ρ_Q^P and ρ_P its projections onto $(\mathfrak{a}_0^Q)^*$, $(\mathfrak{a}_Q^P)^*$ and $\mathfrak{a}_P^*$, respectively.

2. Mixed truncation

For any pair of parabolic subgroups $Q\subset P$, we write τ_Q^P for the characteristic function of the subset

$$\{X\in\mathfrak{a}_0\mid\langle\alpha,X\rangle>0\text{ for all }\alpha\in\Delta_Q^P\}$$

and $\hat{\tau}_Q^P$ for the characteristic function of the subset

$$\{X\in\mathfrak{a}_0\mid\langle\varpi,X\rangle>0\text{ for all }\varpi\in\hat{\Delta}_Q^P\}.$$

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When $P = G$, we will omit the superscript G. Put

$$\mathfrak{a}_0^+ = \{X \in \mathfrak{a}_0 \mid \langle \alpha, X \rangle > 0 \text{ for all } \alpha \in \Delta_0\}$$

and

$$(\mathfrak{a}_P^*)^+ = \{\Lambda \in \mathfrak{a}_P^* \mid \langle \Lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha^\vee \in \Delta_P^\vee\}.$$

Fix a composition $\mathbf{n} = (n_1, \dots, n_t)$ of $n+1$ and put $P = P_{\mathbf{n}}$. For $i \in [1, n+1]$ we define $w_i \in W$ to be the cyclic permutation

$$w_i = (i, i+1, \dots, n+1) = \begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \mathbf{1}_{n+1-i} & 0 \end{pmatrix}.$$

For $w = w_i$ we define a standard parabolic subgroup $P_w = M_w U_w$ of G' by

$$P_w = w^{-1} P w \cap G', \quad M_w = w^{-1} M w \cap G', \quad U_w = w^{-1} U w \cap G'.$$

If $i \in [n_1 + \dots + n_{j-1} + 1, n_1 + \dots + n_j]$, then P_w is the standard parabolic subgroup of G' attached to the composition $(n_1, \dots, n_{j-1}, n_j - 1, n_{j+1}, \dots, n_t)$. We write

$$M_w = \mathbf{M}_w \times \mathcal{M}_w, \quad M = w \mathbf{M}_w w^{-1} \times \mathfrak{M}_w,$$
$$\mathfrak{a}_{P_w} = \mathfrak{a}_{\mathbf{M}_w} \oplus \mathfrak{a}_{\mathcal{M}_w}, \quad \mathfrak{a}_P = w \mathfrak{a}_{\mathbf{M}_w} \oplus \mathfrak{a}_{\mathfrak{M}_w},$$

where

$$\mathbf{M}_w \simeq \prod_{k \neq j} G_{n_k}, \quad G_{n_j - 1} \simeq \mathcal{M}_w \subset \mathfrak{M}_w \simeq G_{n_j}.$$

The set ${}_M W_{G'}^G$ of reduced representatives for $W^M \backslash W^G / W^{G'}$ is given by

$${}_M W_{G'}^G = \{w_i \mid i \in \mathbf{I}_P\}.$$

Let Q be a parabolic subgroup contained in P. When $w \in {}_M W_{G'}^G$, we can identify $W^L \backslash W^M / w W^{M_w} w^{-1}$ with the set ${}_L W_{M_w}^M$ of reduced representatives. We can also identify ${}_L W_{M_w}^M$ with ${}_{L \cap \mathfrak{M}_w} W_{\mathcal{M}_w}^{\mathfrak{M}_w}$. More precisely, if $w = w_i$ with $i = n_1 + \dots + n_j$ and $Q = P_{\mathbf{m}}$ with $\mathbf{m} = (m_1, \dots, m_r)$ such that

$$m_1 + \dots + m_{a-1} = n_1 + \dots + n_{j-1}, \quad m_1 + \dots + m_b = n_1 + \dots + n_j,$$

then

$${}_L W_{M_w}^M = \{(m_1 + \dots + m_k, m_1 + \dots + m_k + 1, \dots, n_1 + \dots + n_j) \in \mathfrak{S}_{n+1} \mid k \in [a, b]\}.$$

We write ${}_0 W_{M_w}^M$ in place of ${}_T W_{M_w}^M$. Note that

$${}_L W_{G'}^G = \bigsqcup_{w \in {}_M W_{G'}^G} {}_L W_{M_w}^M w. \tag{2.1}$$

Let $w \in {}_L W_{G'}^G$. Then $Q_M^w = w^{-1} Q w \cap M_w$ is a parabolic subgroup of M_w. We can view $\mathfrak{a}_{\mathbf{M}_w}$ and $\mathfrak{a}_{\mathbf{M}_w}^*$ as subspaces of $\mathfrak{a}_{\mathbf{L}_w}$ and $\mathfrak{a}_{\mathbf{L}_w}^*$, respectively. Put

$$\mathfrak{a}_{\mathbf{L}_w}^{\mathbf{M}_w} = \{X \in \mathfrak{a}_{\mathbf{L}_w} \mid \langle \lambda, X \rangle = 0 \text{ for } \lambda \in \mathfrak{a}_{\mathbf{M}_w}^*\}.$$

Then $\mathfrak{a}_{\mathbf{L}_w} = \mathfrak{a}_{\mathbf{L}_w}^{\mathbf{M}_w} \oplus \mathfrak{a}_{\mathbf{M}_w}$. Notice that $w \mathfrak{a}_{\mathbf{L}_w}^{\mathbf{M}_w}$ is not a subspace of $\mathfrak{a}_Q^P$ in general.

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Let $T \in \mathfrak{a}_0^+$ be a truncation parameter. Recall that for any smooth function ϕ on $P(F)\backslash G(\mathbb{A})$, Arthur's truncation of ϕ was defined in [Art80] to be

$$\Lambda^{T,P}\phi(g) = \sum_{Q\subset P}(-1)^{\dim \mathfrak{a}_Q^P} \sum_{\gamma\in Q(F)\backslash P(F)} \phi_Q(\gamma g)\hat{\tau}_Q^P(H_Q(\gamma g) - T).$$

We define a mixed truncation of a smooth function φ on $G(F)\backslash G(\mathbb{A})$, which is a function on $G'(F)\backslash G'(\mathbb{A})$, by

$$\Lambda_m^T\varphi(g) = \sum_P (-1)^{\dim \mathfrak{a}_P^G} \sum_{w\in {}_MW_{G'}^G} \sum_{\gamma\in P_w(F)\backslash G'(F)} \varphi_P(w\gamma g)\hat{\tau}_P(H_P(w\gamma g) - T)$$

for $g \in G'(\mathbb{A})$. It is noteworthy that

$$\Lambda_m^T\varphi(g) = \sum_P (-1)^{\dim \mathfrak{a}_P^G} \sum_{\gamma\in P(F)\backslash P(F)WG'(F)} \varphi_P(\gamma g)\hat{\tau}_P(H_P(\gamma g) - T).$$

Note that $\Lambda_m^T\varphi$ depends only on the image of T in the intersection $(\mathfrak{a}_0^G)^+$ of $\mathfrak{a}_0^+$ with $\mathfrak{a}_0^G$. For $w \in {}_MW_{G'}^G$, a wth mixed truncation of a smooth function ϕ on $P(F)\backslash G(\mathbb{A})$ is a function on $U_w(\mathbb{A})M_w(F)\backslash G'(\mathbb{A})$ defined by

$$\Lambda_{m,w}^{T,P}\phi(g) = \sum_{Q\subset P}(-1)^{\dim \mathfrak{a}_Q^P} \sum_{\xi\in {}_LW_{M_w}^M} \sum_{\delta\in Q_M^{\xi w}(F)\backslash M_w(F)} \phi_Q(\xi w\delta g)\hat{\tau}_Q^P(H_Q(\xi w\delta g) - T).$$

Remark 2.1. In the formula for the wth mixed truncation operator, the sum over δ is really over a finite set which depends on g but is independent of ϕ (see [Art78, Lemma 5.1]).

Langlands' combinatorial lemma asserts that

$$\sum_{Q\subset R\subset P} (-1)^{\dim \mathfrak{a}_Q^R}\hat{\tau}_Q^R\tau_R^P = \begin{cases} 1 & \text{if } P = Q,\\ 0 & \text{otherwise}\end{cases} \tag{2.2}$$

for any pair of parabolic subgroups $Q \subset P$. For any $H, X \in \mathfrak{a}_0$ put

$$\Gamma_Q^P(H, X) = \sum_{Q\subset R\subset P} (-1)^{\dim \mathfrak{a}_R^P}\tau_Q^R(H)\hat{\tau}_R^P(H - X).$$

Langlands' lemma gives the formulae

$$\hat{\tau}_Q^P(H - X) = \sum_{Q\subset R\subset P} (-1)^{\dim \mathfrak{a}_R^P}\hat{\tau}_Q^R(H)\Gamma_R^P(H, X), \tag{2.3}$$

$$\tau_Q^P(H - X) = \sum_{Q\subset R\subset P} \Gamma_Q^R(H - X, -X)\tau_R^P(H). \tag{2.4}$$

LEMMA 2.2.

(i) *If φ is a smooth function on $G(F)\backslash G(\mathbb{A})$, then for $g \in G'(\mathbb{A})$,*

$$\varphi(g) = \sum_P \sum_{w\in {}_MW_{G'}^G} \sum_{\gamma\in P_w(F)\backslash G'(F)} \Lambda_{m,w}^{T,P}\varphi(\gamma g)\tau_P(H_P(w\gamma g) - T).$$

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(ii) *If ϕ is a smooth function on $P(F)\backslash G(\mathbb{A})$ and $w \in {}_M W^G_{G'}$, then for $g \in G'(\mathbb{A})$,*

$$\Lambda^{T+T',P}_{m,w}\phi(g) = \sum_{Q\subset P} \sum_{\xi\in {}_L W^M_{M_w}} \sum_{\delta\in Q^{\xi w}_M(F)\backslash M_w(F)} \Lambda^{T,Q}_{m,\xi w}\phi(\delta g)\Gamma^P_Q(H_Q(\xi w\delta g) - T, T').$$

Proof. Substituting the definition of $\Lambda^{T,P}_{m,w}\varphi$ into the right-hand side of the formula in (i), we see that it is equal to

$$\sum_P \sum_{w\in {}_M W^G_{G'}} \sum_{\gamma\in P_w(F)\backslash G'(F)} \sum_{Q\subset P} (-1)^{\dim \mathfrak{a}^P_Q} \sum_{\xi\in {}_L W^M_{M_w}} \sum_{\delta\in Q^{\xi w}_M(F)\backslash M_w(F)} \varphi_Q(\xi w\delta\gamma g)\hat{\tau}^P_Q(H_Q(\xi w\delta\gamma g) - T)\tau_P(H_P(w\gamma g) - T).$$

Using (2.1) and observing that

$$Q^{\xi w}_M = Q_{\xi w} \cap M_w, \quad \tau_P(H_P(w\gamma g) - T) = \tau_P(H_Q(\xi w\delta\gamma g) - T),$$

we combine the sum over w, γ, ξ and δ into the double sum over ${}_L W^G_{G'}$ and $Q_{\xi w}(F)\backslash G'(F)$ to write this as the sum over Q, $w \in {}_L W^G_{G'}$ and $\gamma \in Q_w(F)\backslash G'(F)$ of

$$\sum_{P\supset Q} (-1)^{\dim \mathfrak{a}^P_Q}\varphi_Q(w\gamma g)\hat{\tau}^P_Q(H_Q(w\gamma g) - T)\tau_P(H_P(w\gamma g) - T).$$

Assertion (i) now follows from (2.2).

Assertion (ii) is a formal consequence of (2.3). □

Let $\mathfrak{S}^P = \omega A^P_0(t_0)K$ be a Siegel set of $G(\mathbb{A})$ relative to a parabolic subgroup P, where $\omega \subset N(\mathbb{A})T(\mathbb{A})^1$ is compact, $t_0 \in \mathfrak{a}_0$ is negative enough and

$$A^P_0(t_0) = \{a \in A_0 \mid \langle\alpha, H_0(a) - t_0\rangle > 0 \text{ for all } \alpha \in \Delta^P_0\}.$$

We also take a Siegel set $\mathfrak{S}^{P'} = \omega' A^{P'}_0(t'_0)K'$ of $G'(\mathbb{A})$. We fix ω, ω', t_0 and t'_0 so that $G(\mathbb{A}) = P(F)\mathfrak{S}^P$ and $G'(\mathbb{A}) = P'(F)\mathfrak{S}^{P'}$ for all P and P'. We write

$$\mathfrak{S} = \mathfrak{S}^G, \quad \mathfrak{S}' = \mathfrak{S}^{G'}, \quad A_0(t_0) = A^G_0(t_0), \quad A'_0(t'_0) = A^{G'}_0(t'_0).$$

Since $w_i N' w_i^{-1} \subset N$, we may take ω so that $w_i\omega' w_i^{-1} \subset \omega$ for all $i \in [1, n+1]$. Put

$$A'_0(t'_0)^{(i)} = \{a \in A'_0(t'_0) \mid w_i a w_i^{-1} \in A_0(t_0)\}.$$

For a suitable choice of t_0 and t'_0, we have

$$A'_0(t'_0) = \bigcup_{i=1}^{n+1} A'_0(t'_0)^{(i)}. \tag{2.5}$$

From now on, we require that T be a suitably regular point in $\mathfrak{a}^+_0$. Put

$$\mathfrak{S}^P_{\leqslant T} = \{g \in \mathfrak{S}^P \mid \langle\varpi, H_0(g) - T\rangle \leqslant 0 \text{ for all } \varpi \in \hat{\Delta}^P_0\}.$$

Let $F^P(\,\cdot\,, T)$ be the characteristic function of $P(F)\mathfrak{S}^P_{\leqslant T}$.

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Lemma 2.3. *For a parabolic subgroup P of G, $w\in {}_MW^G_{G'}$ and $g\in G'(\mathbb{A})$,*

$$\sum_{Q\subset P}\sum_{\xi\in {}_LW^M_{M_w}}\sum_{\delta\in Q^{\xi w}_M(F)\backslash M_w(F)} F^Q(\xi w\delta g,T)\tau^P_Q(H_Q(\xi w\delta g)-T)=1.$$

Proof. Using the decomposition $P_w=M_wU_w$, we may write the inner sum as the sum over $Q_{\xi w}(F)\backslash P_w(F)$. There is $\delta\in P_w(F)$ such that $\delta g\in\mathfrak{S}^{P_w}$. From (2.5) one can find an element $\xi\in {}_0W^M_{M_w}$ such that $\xi w\delta g\in\mathfrak{S}^P$. We apply [Art78, Lemma 6.3] with $Q=B$, $\Lambda\in(\mathfrak{a}_0^*)^+$ and $H=H_0(\xi w\delta g)-T$ to find a parabolic subgroup $Q\subset P$ satisfying the following conditions:

- $\langle\varpi,H_0(\xi w\delta g)-T\rangle\leqslant 0$ for all $\varpi\in\hat{\Delta}^Q_0$;
- $\langle\alpha,H_0(\xi w\delta g)-T\rangle>0$ for all $\alpha\in\Delta^P_Q$.

It follows that

$$F^Q(\xi w\delta g,T)\tau^P_Q(H_Q(\xi w\delta g)-T)=1.$$

Since F^Q and H_Q are left $Q(F)$-invariant, we may assume that $\xi\in {}_LW^M_{M_w}$ after translating ξ by an element in W^L if necessary. Thus the given sum is at least 1. Lemma 6.4 of [Art78] asserts that for any $x\in G(\mathbb{A})$,

$$\sum_{Q\subset P}\sum_{\delta\in Q_M(F)\backslash M(F)} F^Q(\delta x,T)\tau^P_Q(H_Q(\delta x)-T)=1,$$

where we have put $Q_M=Q\cap M$. The double sum over ξ and δ can be combined into a single sum over $Q_M(F)\backslash Q_M(F)\,{}_LW^M_{M_w}wM_w(F)$, so the given sum is at most 1. □

To simplify notation, we put

$$M_w(\mathbb{A})'=\mathbf{M}_w(\mathbb{A})^1\times\mathcal{M}_w(\mathbb{A}).$$

Lemma 2.4. *If ϕ is a smooth function on $P(F)\backslash G(\mathbb{A})$ such that it and its derivatives have uniform moderate growth, then for any $\lambda\in(\mathfrak{a}_0')^*$, there exists a constant $C>0$ such that for all $g\in\mathfrak{S}^{P_w}\cap M_w(\mathbb{A})'$ and $k\in K'$,*

$$|\Lambda^{T,P}_{m,w}\phi(gk)|\leqslant Ce^{\langle\lambda,H_0'(g)\rangle}.$$

Proof. If we set

$$\psi(m_1,m_2)=\phi(m_1wm_2),\quad m_1\in\mathcal{M}_w(\mathbb{A}),\ m_2\in\mathbf{M}_w(\mathbb{A}),$$

then for $m_1\in\mathcal{M}_w(\mathbb{A})$ and $m_2\in\mathbf{M}_w(\mathbb{A})$,

$$\Lambda^{T,P}_{m,w}\phi(m_1m_2)=(\Lambda^{T,\mathcal{M}_w}_m\otimes\Lambda^{T,\mathbf{M}_w})\psi(m_1,m_2),$$

where we apply the mixed truncation to the first variable and Arthur's truncation to the second variable. We may therefore assume that $P=G$. Although it only remains to check that each step of the argument in [Art80] applies, we shall go over the proof, keeping track of the dependence on T in view of our later application.

Let $g \in \mathfrak{S}'$. We multiply the summand corresponding to P and w in the definition of $\Lambda_m^T\phi(g)$ by the left-hand side of Lemma 2.3. Then $\Lambda_m^T\phi(g)$ equals

$$\begin{aligned}&\sum_P (-1)^{\dim \mathfrak{a}_P^G} \sum_{w\in {}_MW_{G'}^G} \sum_{\gamma\in P_w(F)\backslash G'(F)} \sum_{Q\subset P} \sum_{\xi\in {}_LW_{M_w}^M} \sum_{\delta\in Q_M^{\xi w}(F)\backslash M_w(F)} \\ &\qquad\qquad F^Q(\xi w\delta\gamma g, T)\tau_Q^P(H_Q(\xi w\delta\gamma g) - T)\phi_P(w\gamma g)\hat{\tau}_P(H_P(w\gamma g) - T) \\ &= \sum_P \sum_{Q\subset P} (-1)^{\dim \mathfrak{a}_P^G} \sum_{w\in {}_LW_{G'}^G} \sum_{\gamma\in Q_w(F)\backslash G'(F)} \\ &\qquad\qquad F^Q(w\gamma g, T)\phi_P(w\gamma g)\tau_Q^P(H_Q(w\gamma g) - T)\hat{\tau}_P(H_P(w\gamma g) - T).\end{aligned}$$

For a given pair of parabolic subgroups $Q \subset P$, we can write

$$\tau_Q^P\hat{\tau}_P = \sum_{R\supset P} \sigma_Q^R,$$

where

$$\sigma_Q^R = \sum_{S\supset R} (-1)^{\dim \mathfrak{a}_R^S}\tau_Q^S\hat{\tau}_S.$$

We apply this identity to the product of the functions τ_Q^P and $\hat{\tau}_P$ which occurs in the expansion above. The function $\Lambda_m^T\phi(g)$ becomes the sum over pairs $Q \subset P$, elements $w \in {}_LW_{G'}^G$ and $\gamma \in Q_w(F)\backslash G'(F)$ of the product

$$F^Q(w\gamma g, T)\sigma_Q^P(H_Q(w\gamma g) - T)\phi_{Q,P}(w\gamma g),$$

where we put

$$\phi_{Q,P}(x) = \sum_{Q\subset R\subset P} (-1)^{\dim \mathfrak{a}_R^G}\phi_R(x), \quad x \in G(\mathbb{A}).$$

For the moment, we fix $Q \subset P$, w, γ and g. We regard γ as an element in $G'(F)$ which we are free to left-multiply by an element in $Q_w(F)$. Then we can assume that

$$\gamma g = n^*n_*mak,$$

where $k \in K'$, n^*, n_* and m belong to fixed compact fundamental domains in $U_w(\mathbb{A})$, $(M_w \cap N')(\mathbb{A})$ and $T'(\mathbb{A})^1$, and $a \in A_0'$ with

$$\sigma_Q^P(H_0(wa) - T) \neq 0, \quad \langle\beta, H_0(wa) - t_0\rangle > 0, \quad \langle\varpi, H_0(wa) - T\rangle \leqslant 0$$

for all $\beta \in \Delta_0^Q$ and all $\varpi \in \hat{\Delta}_0^Q$. Let $\{\varpi_\alpha^\vee \mid \alpha \in \Delta_Q^P\}$ (respectively $\{\varpi_\beta \mid \beta \in \Delta_0^Q\}$) stand for the basis of $\mathfrak{a}_Q^P$ (respectively of $(\mathfrak{a}_0^Q)^*$) which is dual to Δ_Q^P (respectively to $(\Delta^\vee)_0^Q$). We can decompose the vector $H_0(wa)^G$ as

$$H_0(wa)^G = \sum_{\alpha\in\Delta_Q^P} t_\alpha\varpi_\alpha^\vee + H^* - \sum_{\beta\in\Delta_0^Q} r_\beta\beta^\vee + T,$$

where t_α and r_β are real numbers and H^* is a vector in $\mathfrak{a}_P^G$. Note that $r_\beta = -\langle\varpi_\beta, H_0(wa) - T\rangle$ is nonnegative for all $\beta \in \Delta_0^Q$. By [Art78, Corollary 6.2], $t_\alpha > 0$ for all $\alpha \in \Delta_Q^P$, and H^* belongs to a compact subset whose volume can be bounded by some polynomial, say $\prod_{\alpha\in\Delta_Q^P} p_\alpha(t_\alpha)$. Recall that $\langle\alpha, \beta^\vee\rangle \leqslant 0$ for all $\alpha \neq \beta$ in Δ_0. Each root $\delta \in \Delta_0^P \smallsetminus \Delta_0^Q$ satisfies

$$\langle\delta, H_0(wa)\rangle = t_\alpha - \sum_{\beta\in\Delta_0^Q} r_\beta\langle\delta, \beta^\vee\rangle + \langle\delta, T\rangle > 0,$$

where α is the projection of δ onto $\mathfrak{a}_Q^*$. Thus the projection of $H_0(wa)$ onto $\mathfrak{a}_0^P$ belongs to a translate of the positive chamber, and hence $a^{-1}n_*a$ remains in a fixed compact subset independently of T. Then

$$\phi_{Q,P}(w\gamma g) = \phi_{Q,P}(wn_*mak) = \phi_{Q,P}(wac),$$

where $c = a^{-1}n_*mak$ belongs to a fixed compact subset of $G'(\mathbb{A})$ independent of T. We write $waw^{-1} = a_0a_1a_2$, where $a_0 \in A_G$, $a_1 \in A_Q \cap G(\mathbb{A})^1$ and $a_2 \in A_0 \cap L(\mathbb{A})^1$. Note that $\|a_0\| \leqslant \|a_1a_2\|$. For each positive integer m, the argument of [Art80, pp. 93–95] gives a constant $c_m(\phi)$ such that $|\phi_{Q,P}(wac)|$ is bounded by

$$c_m(\phi)\sum_I e^{-3m\langle\beta_I, H_0(a_1)\rangle}\int_{N_I(F)\backslash N_I(\mathbb{A})} |R(\mathrm{Ad}(wc)^{-1}\mathrm{Ad}(a_2)^{-1}Y_I^{3m})\phi(uwac)|\,du,$$

where β_I is a positive sum of the roots in Δ_Q^P, N_I is a subgroup of V, Y_I is a left invariant differential operator on $(M\cap V)_\infty$ and R is the regular representation of $G(\mathbb{A})$. We can choose a finite set of left invariant differential operators $\{X_i\}$ on G_∞ such that $\mathrm{Ad}(wc)^{-1}\mathrm{Ad}(a_2)^{-1}Y_I^{3m}$ is a linear combination of $\{X_i\}$ for any $Q \subset P$, w, I, c and a_2. Since the projection of $H_0(a_2)$ onto $\mathfrak{a}_0^P$ belongs to a translate of the positive chamber, the coefficients are bounded independently of c, a_2 and T.

Set $d(T) = \min\{\langle\alpha, T\rangle \mid \alpha \in \Delta_0\}$. We shall let T vary over suitably regular points such that $d(T) \geqslant \epsilon_0\|T\|$ for some fixed positive number ϵ_0. Corollary 6.2 of [Art78], referred to above, concludes that $\|a_1\|$ is bounded by a fixed power of $e^{\langle\beta_I, H_0(a_1)\rangle}$. Note that

$$e^{\epsilon_0\|T\|} \leqslant e^{d(T)} \leqslant e^{\langle\beta_I, T\rangle} \leqslant e^{\langle\beta_I, H_0(a_1)\rangle}.$$

Therefore $\|\gamma g\| = \|n^*ac\|$ is bounded by a constant multiple of a fixed power of $e^{\langle\beta_I, H_0(a_1)\rangle}$. By the assumption on ϕ, we can take m so that

$$\sum_I\sum_i e^{-m\langle\beta_I, H_0(a_1)\rangle}\int_{N_I(F)\backslash N_I(\mathbb{A})} |R(X_i)\phi(uwac)|\,du$$

is bounded for all $Q \subset P$ and w. There is a constant c such that $\|x\| \leqslant c\|\gamma x\|$ for all $\gamma \in G'(F)$ and $x \in \mathfrak{S}'$ (see [MW95, § I.2.2]). It follows that for any $N_1 > 0$ there exists $C_1 > 0$ such that

$$|\phi_{Q,P}(w\gamma g)| \leqslant C_1\|g\|^{-N_1}$$

for all $\gamma \in G'(F)$ and $g \in \mathfrak{S}'$ with $F^Q(w\gamma g, T)\sigma_Q^P(H_Q(w\gamma g) - T) = 1$.

On the other hand,

$$\sum_{\delta\in Q(F)\backslash G(F)} F^Q(\delta x, T)\sigma_Q^P(H_Q(\delta x) - T) \leqslant C_2\|x\|^{N_2}, \quad x \in G(\mathbb{A})^1,$$

for some constants $C_2 > 0$ and $N_2 > 0$ (see [Art80, pp. 96–97]). Since the summand takes values 0 or 1, a similar estimate holds for the sum over ${}_LW_{G'}^G$ and $Q_w(F)\backslash G'(F)$. There exist $c', t, t' > 0$

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such that $c'\|a\|^t\|x\|^{t'} \leqslant \|ax\|$ for all $a \in A_G$ and $x \in G(\mathbb{A})^1$. Therefore, for any $N_3 > 0$ there exists $C_3 > 0$ such that

$$|\Lambda_m^T \phi(g)| \leqslant C_3 \|g\|^{-N_3}.$$

For a given $\lambda \in (\mathfrak{a}_0')^*$, we can take $C_4 > 0$ and $N_4 > 0$ such that $e^{-\langle \lambda, H_0'(g)\rangle} \leqslant C_4 \|g\|^{N_4}$. We obtain the desired estimate by taking $N_3 = N_4$. □

Remark 2.5. If ϕ is an automorphic form, then we have proven that the rate of rapid decrease of $\Lambda_{m,w}^{T,P}\phi$ is majorized in terms of the rate of slow increase of finitely many derivatives of ϕ and hence in terms of the exponents of finitely many derivatives of ϕ. If $\phi(\lambda)$ is an analytic family of automorphic forms, then its exponents vary analytically, and hence, for any $\mu \in (\mathfrak{a}_0')^*$, there is a locally bounded function $c(\lambda)$ such that for all $g \in \mathfrak{S}^{P_w} \cap M_w(\mathbb{A})'$ and $k \in K'$,

$$|\Lambda_{m,w}^{T,P}\phi(\lambda)(gk)| \leqslant c(\lambda) e^{\langle \mu, H_0'(g)\rangle}.$$

3. Periods of automorphic forms on $\mathbf{GL}_{n+1} \times \mathbf{GL}_n$

3.1 Integrals over cones

The regularization of the period integral is based on a regularization of integrals of polynomial exponential functions over cones in vector spaces. We recall its basic properties and refer the reader to [JLR99, § II] for additional explanation.

Let V be a finite-dimensional real vector space. A polynomial exponential function on V is a function of the form

$$f(X) = \sum_{i=1}^{r} p_i(X) e^{\langle \lambda_i, X\rangle},$$

where $\lambda_i \in V_{\mathbb{C}}^*$ and $p_i \in \mathbb{C}[V]$. The decomposition above is unique if the λ_i are distinct and $p_i \neq 0$ for all i. We call λ_i the exponents of f. We denote the characteristic function of a subset $\mathcal{Y}$ of V by $\tau^{\mathcal{Y}}$.

For $w = w_i \in {}_0W_{G'}^G$, we define a $\mathbb{C}$-linear map $\mathrm{pr}_w : \mathfrak{a}_{0,\mathbb{C}}^* \to (\mathfrak{a}_{0,\mathbb{C}}^G)^*$ by

$$\mathrm{pr}_w(\lambda_1, \dots, \lambda_{n+1}) = \left(\lambda_1, \dots, \lambda_{i-1}, \lambda_i - \sum_{j=1}^{n+1} \lambda_j, \lambda_{i+1}, \dots, \lambda_{n+1}\right).$$

Let P be a parabolic subgroup of G. For $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$, we denote the restriction of $\mathrm{pr}_w(\lambda)$ to $\mathfrak{a}_P$ by $\eta_P^w(\lambda)$. Note that the map $\lambda \mapsto \eta_P^w(w\lambda)$ restricts to an isomorphism $\mathfrak{a}_{\mathbf{M}_w,\mathbb{C}}^* \simeq (\mathfrak{a}_{P,\mathbb{C}}^G)^*$. Fix $c \in \mathfrak{a}_0'$. For any polynomial exponential function f on $\mathfrak{a}_0'$, we define a polynomial exponential function f_c^w on $\mathfrak{a}_P^G$ by

$$f_c^w((wX)^G) = f(X + c)$$

for $X \in \mathfrak{a}_{\mathbf{M}_w}$. Since

$$\langle \eta_P^w(w\lambda), (wX)^G\rangle = \langle \lambda, X\rangle, \quad \lambda \in \mathfrak{a}_{0,\mathbb{C}}^*,\ X \in \mathfrak{a}_{\mathbf{M}_w},$$

if the exponents of the restriction of f to $\mathfrak{a}_{\mathbf{M}_w}$ are λ_i, then the exponents of f_c^w are $\eta_P^w(w\lambda_i)$.

Put $t_P = \dim \mathfrak{a}_P^G$. Let $\mathcal{C}$ be a subset of $\mathfrak{a}_0$ of the form

$$\mathcal{C} = \{X \in \mathfrak{a}_0 \mid \langle \mu_j, X\rangle > 0 \text{ for all } j \in [1, t_P]\},$$

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where $\{\mu_j\}_{j=1}^{t_P}$ is a basis of $(\mathfrak{a}_P^G)^*$. Let $\{e_j\}_{j=1}^{t_P}$ be the corresponding dual basis of $\mathfrak{a}_P^G$. Fix $T \in \mathfrak{a}_0$. The #-integral

$$\int_{\mathfrak{a}_{\mathbf{M}_w}+c}^{\#} f(X)\tau^{\mathcal{C}}(wX-T)\,dX = \int_{\mathfrak{a}_P^G}^{\#} f_c^w(X)\tau^{\mathcal{C}}(X-T+wc)\,dX$$

is discussed in [JLR99]. It exists if and only if $\langle \eta_P^w(w\lambda_i), e_j\rangle \neq 0$ for all i,j. The ordinary integral

$$\int_{\mathfrak{a}_{\mathbf{M}_w}+c} f(X)\tau^{\mathcal{C}}(wX-T)\,dX$$

converges if and only if $\Re\langle \eta_P^w(w\lambda_i), e_j\rangle < 0$ for all i,j. In this case it coincides with the #-integral. The function

$$T \mapsto \int_{\mathfrak{a}_{\mathbf{M}_w}+c}^{\#} f(X)\tau^{\mathcal{C}}(wX-T)\,dX$$

is a polynomial exponential function on $\mathfrak{a}_0$ whose exponents are $\eta_P^w(w\lambda_i)$. The function

$$c \mapsto \int_{\mathfrak{a}_{\mathbf{M}_w}+c}^{\#} f(X)\tau^{\mathcal{C}}(wX-T)\,dX$$

is a polynomial exponential function on $\mathfrak{a}_0'$ whose exponents are given by

$$c \mapsto \langle w\lambda_i - \eta_P^w(w\lambda_i), wc\rangle, \quad c \in \mathfrak{a}_0'.$$

For $\lambda \in \mathfrak{a}_{0,\mathbb{C}}^*$ we deduce the explicit formula

$$\int_{\mathfrak{a}_{\mathbf{M}_w}+c}^{\#} e^{\langle\lambda,X\rangle}\tau^{\mathcal{C}}(wX-T)\,dX = (-1)^{t_P} v(e_1,\ldots,e_{t_P}) \frac{e^{\langle\lambda,c\rangle}e^{\langle\eta_P^w(w\lambda),T-wc\rangle}}{\prod_{j=1}^{t_P}\langle\eta_P^w(w\lambda),e_j\rangle} \tag{3.1}$$

from [JLR99, (15)], where $v(e_1,\ldots,e_{t_P})$ is the volume of the parallelotope formed by $\{e_j\}_{j=1}^{t_P}$.

Let Q be a parabolic subgroup contained in P and g a compactly supported function on $\mathfrak{a}_Q^P$. In [JLR99] the domain of the #-integral is extended to functions of the form

$$h(Y) = g(Y_Q^P)\tau^{\mathcal{C}}(Y), \quad Y \in \mathfrak{a}_0,$$

which we call functions of type (C). If f is a polynomial exponential function on $\mathfrak{a}_0'$ and $f(Y)h(wY-T)$ is #-integrable over $\mathfrak{a}_{\mathbf{L}_w}+c$, then

$$\int_{\mathfrak{a}_{\mathbf{L}_w}+c}^{\#} f(Y)h(wY-T)\,dY = \int_{\mathfrak{a}_{\mathbf{L}_w}^{\mathbf{M}_w}+c} g((wY-T)_Q^P)\int_{\mathfrak{a}_{\mathbf{M}_w}}^{\#} f(X+Y)\tau^{\mathcal{C}}(w(X+Y)-T)\,dX\,dY. \tag{3.2}$$

This identity follows by definition and analytic continuation. Note that the map $Y \mapsto (wY)_Q^P$ restricts to an isomorphism of $\mathfrak{a}_{\mathbf{L}_w}^{\mathbf{M}_w}$ onto $\mathfrak{a}_Q^P$.

LEMMA 3.1. *Let Q be a parabolic subgroup of G, $w \in {}_LW_{G'}^G$, $c \in \mathfrak{a}_0'$, f a polynomial exponential function on $\mathfrak{a}_0'$ and $\mathcal{C}$ a cone in $\mathfrak{a}_Q^G$. For each parabolic subgroup P with $P \supset Q$, let g_P be a function on $\mathfrak{a}_Q^P$ and $\mathcal{C}_P$ a cone in $\mathfrak{a}_P^G$, and set $h_P(X) = g_P(X_Q^P)\tau^{\mathcal{C}_P}(X)$ for $X \in \mathfrak{a}_0$. Assume that*

$$\tau^{\mathcal{C}}(X) = \sum_{P\supset Q} a_P h_P(X)$$

for some constants a_P and that either

– g_P *is compactly supported and* $\mathcal{C}_P \subset \mathcal{C}$ *for every* P*; or*
– g_P *is the characteristic function of a cone* $\mathcal{C}_Q^P$ *in* $\mathfrak{a}_Q^P$ *and* $\mathcal{C}_Q^P \times \mathcal{C}_P \subset \mathcal{C}$ *for every* P.

Assume further that $f(X)h_P(wX - T)$ *is* $\#$*-integrable over* $\mathfrak{a}_{\mathbf{L}_w} + c$ *for every* P. *Then* $f(X)\tau^{\mathcal{C}}(wX - T)$ *is* $\#$*-integrable over* $\mathfrak{a}_{\mathbf{L}_w} + c$ *and*

$$\int^{\#}_{\mathfrak{a}_{\mathbf{L}_w}+c} f(X)\tau^{\mathcal{C}}(wX-T)\,dX = \sum_{P\supset Q} a_P \int^{\#}_{\mathfrak{a}_{\mathbf{L}_w}+c} f(X)h_P(wX-T)\,dX.$$

Proof. The proof is the same as that of Lemma 6 in [JLR99]. □

3.2 Regularization of the period integral

A regularization of the period integral was introduced in [JLR99, LR03] for the Galois case. We will observe that the construction carries over to the context of this paper.

Let P be a parabolic subgroup of G and $w \in {}_MW^G_{G'}$. Automorphic forms $\phi \in \mathscr{A}_P(G)$ and $\phi' \in \mathscr{A}_{P_w}(G')$ have decompositions of type (1.3), namely,

$$\begin{aligned}\phi(uamk) &= \sum_i Q_i(H_P(a))\phi_i(mk)e^{\langle \lambda_i+\rho_P, H_P(a)\rangle},\\ \phi'(u'a'm'k') &= \sum_j Q'_j(H_{P_w}(a'))\phi'_j(m'k')e^{\langle \lambda'_j+\rho_{P_w}, H_{P_w}(a')\rangle}\end{aligned} \tag{3.3}$$

for

$$\begin{gathered} u\in U(\mathbb{A}),\quad a\in A_P,\quad m\in M(\mathbb{A})^1,\quad k\in K,\\ u'\in U_w(\mathbb{A}),\quad a'\in A_{P_w},\quad m'\in M_w(\mathbb{A})^1,\quad k'\in K'.\end{gathered}$$

We define $Q_{ij} \in \mathbb{C}[\mathfrak{a}'_0]$ by

$$Q_{ij}(X) = Q_i((wX)_P)Q'_j(X_{P_w}),\quad X\in\mathfrak{a}'_0.$$

Let $g = ue^X mk$ be an Iwasawa decomposition of $g \in G'(\mathbb{A})$ relative to P_w with $X \in \mathfrak{a}_{\mathbf{M}_w}$ and $m \in M_w(\mathbb{A})'$. Since $H_P(wm) \in \mathfrak{a}_{\mathcal{M}_w}$ is the projection of $H_{P_w}(m) = wH_{P_w}(m) \in \mathfrak{a}_{\mathcal{M}_w}$ onto $\mathfrak{a}_P$ for any $m \in M_w(\mathbb{A})'$, if we set

$$X(m) = X + H'_0(m),\quad \Lambda_{ij} = \lambda_i + w\lambda'_j,$$

then

$$\Lambda^{T,P}_{m,w}\phi(g)\phi'(g) = \sum_{i,j} Q_{ij}(X(m))\Lambda^{T,P}_{m,w}\phi_i(mk)\phi'_j(mk)e^{\langle \Lambda_{ij}+\rho_P+w\rho_{P_w}, wX(m)\rangle}.$$

Let τ_k be a function on $\mathfrak{a}_P^G$ of type (C) which depends continuously on $k \in K'$.

For parabolic subgroups $Q \subset P$ of G and $w \in {}_LW^G_{G'}$, we put

$$\varrho^P_{Q,w} = \rho^P_Q - w\rho^{P_w}_{Q_w}.$$

When $P = G$, we write $\varrho_{Q,w}$ in place of $\varrho^G_{Q,w}$. Note that when $P = P_{\mathbf{n}}$ with $\mathbf{n} = (n_1, \dots, n_t)$ and $w = w_i$ with $i = n_1 + \dots + n_j$,

$$\begin{aligned}\varrho_{P,w} &= \rho_0 - \rho_0^P - w(\rho'_0 - \rho_0^{P_w})\\ &= (\underbrace{\tfrac12,\dots,\tfrac12}_{n_1+\dots+n_{j-1}}, \underbrace{0,\dots,0}_{n_j-1}, \tfrac12(n_{j+1}+\dots+n_t-n_1-\dots-n_{j-1}), \underbrace{-\tfrac12,\dots,-\tfrac12}_{n_{j+1}+\dots+n_t}) \in \mathfrak{a}_0^*.\end{aligned}$$

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For $\lambda\in\mathfrak{a}_{0,\mathbb{C}}^*$ and $\lambda'\in(\mathfrak{a}'_{0,\mathbb{C}})^*$, we define $\kappa_{P,w}(\lambda,\lambda')\in\mathfrak{a}^*_{\mathcal{M}_w,\mathbb{C}}$ by demanding that

$$\langle\kappa_{P,w}(\lambda,\lambda'),Y\rangle=\langle\lambda+w\lambda'+\varrho_{P,w}-\eta_P^w(\lambda+w\lambda'+\varrho_{P,w}),Y\rangle$$

for $Y\in\mathfrak{a}_{\mathcal{M}_w}$. Assume that the #-integrals

$$\int^{\#}_{\mathfrak{a}_{\mathbf{M}_w}+H_{P_w}(m)}Q_{ij}(X)e^{\langle\Lambda_{ij}+\varrho_{P,w},wX\rangle}\tau_k(wX-T)\,dX$$

exist for all i, j and k. Then this integral is equal to a function of the form

$$e^{\langle\eta_P^w(\Lambda_{ij}+\varrho_{P,w}),T\rangle}e^{\langle\kappa_{P,w}(\lambda_i,\lambda'_j),H_{P_w}(m)\rangle}p_{ijk}(H_{P_w}(m),T),$$

where $p_{ijk}\in\mathbb{C}[\mathfrak{a}_{\mathcal{M}_w}\oplus\mathfrak{a}_0]$ depends continuously on $k\in K'$. We define the #-integral

$$\int^{\#}_{P_w(F)\backslash G'(\mathbb{A})}\Lambda^{T,P}_{m,w}\phi(g)\phi'(g)\tau_k(H_P(wg)-T)\,dg$$

as

$$\sum_{i,j}e^{\langle\eta_P^w(\Lambda_{ij}+\varrho_{P,w}),T\rangle}\int_{K'}\int_{M_w(F)\backslash M_w(\mathbb{A})'}p_{ijk}(H_{P_w}(m),T)e^{\langle\kappa_{P,w}(\lambda_i,\lambda'_j),H_{P_w}(m)\rangle}\times\Lambda^{T,P}_{m,w}\phi_i(mk)\phi'_j(mk)\,dm\,dk.$$

Lemma 2.4 and Remark 3.5 guarantee the convergence of this integral.

DEFINITION 3.2. Let $\mathscr{A}(G\times G')^*$ be the space of pairs $(\varphi,\varphi')\in\mathscr{A}(G)\oplus\mathscr{A}(G')$ which satisfy

$$\langle\mathrm{pr}_w(\lambda+w\lambda')+\varrho_{P,w},\varpi^\vee\rangle\neq 0$$

for any proper parabolic subgroup P of G and

$$w\in{}_MW^G_{G'},\quad\lambda\in\mathscr{E}_P(\varphi),\quad\lambda'\in\mathscr{E}_{P_w}(\varphi'),\quad\varpi^\vee\in\hat{\Delta}^\vee_P.$$

When $(\varphi,\varphi')\in\mathscr{A}(G\times G')^*$, all the #-integrals

$$\mathbf{P}^{G',T}_{P,w}(\varphi\otimes\varphi')=\int^{\#}_{P_w(F)\backslash G'(\mathbb{A})}\Lambda^{T,P}_{m,w}\varphi(g)\varphi'_{P_w}(g)\tau_P(H_P(wg)-T)\,dg$$

exist and a regularized period $\mathbf{P}^{G'}(\varphi\otimes\varphi')$ is defined as the sum

$$\sum_P\sum_{w\in{}_MW^G_{G'}}\mathbf{P}^{G',T}_{P,w}(\varphi\otimes\varphi').$$

It is worth emphasizing that this definition is based on Lemma 2.2(i).

PROPOSITION 3.3. *$\mathbf{P}^{G'}$ is well-defined and is independent of T.*

Proof. We use Lemma 2.2(ii) to write $\mathbf{P}^{G',T+T'}_{P,w}(\varphi\otimes\varphi')$ as

$$\int^{\#}_{P_w(F)\backslash G'(\mathbb{A})}\sum_{Q\subset P}\sum_{\xi\in{}_LW^M_{M_w}}\sum_{\delta\in Q^{\xi w}_M(F)\backslash M_w(F)}\Lambda^{T,Q}_{m,\xi w}\varphi(\delta g)\varphi'_{P_w}(\delta g)\times\Gamma^P_Q(H_Q(\xi w\delta g)-T,T')\tau_P(H_P(wg)-T-T')\,dg.$$

We take the sum over Q and ξ outside the integral, which will be justified by the absolute convergence of each integral, as explained below. Expand φ_P and φ'_{P_w} as in (3.3). Let $g=ue^Xmk'$ be an Iwasawa decomposition of $g\in G'(\mathbb{A})$ relative to P_w with $X\in\mathfrak{a}_{\mathbf{M}_w}$ and $m\in M_w(\mathbb{A})'$.

Since $\Gamma_Q^P(Y,Z)$ depend only on the projections of Y and Z onto $\mathfrak{a}_Q^P$, we may write the $\#$-integral of the sum over δ as the integral over $k' \in K'$ and sum over i,j of

$$\int_{Q_M^{\xi w}(F)\backslash M_w(\mathbb{A})'} \Lambda^{T,Q}_{m,\xi w}\phi_i(mk')\phi'_j(mk')\Gamma_Q^P(H_Q(\xi wm)-T,T') \\ \times \int^{\#}_{\mathfrak{a}_{\mathbf{M}_w}+H'_0(m)} Q_{ij}(X)e^{\langle \Lambda_{ij}+\varrho_{P,w},wX\rangle}\tau_P(wX-T-T')\,dX\,dm.$$

Since $\Gamma_Q^P(X-T,T')$ is a compactly supported function of $X\in\mathfrak{a}_Q^P$, each term is separately integrable over $Q_M^{\xi w}(F)\backslash M_w(\mathbb{A})'$ by Lemma 2.4; hence we may take the sum over Q and ξ outside the integral. Choose a decomposition of type (1.3) for $(\phi_i)_Q$ and $(\phi'_j)_{Q_{\xi w}}$:

$$(\phi_i)_Q(uamk') = \sum_k q_{ik}(H_Q(a))\phi_{ik}(mk')e^{\langle \lambda_{ik}+\rho_Q^P,H_Q(a)\rangle},$$
$$(\phi'_j)_{Q_{\xi w}}(u'a'm'k'') = \sum_l q'_{jl}(H_{Q_{\xi w}}(a'))\phi'_{jl}(m'k'')e^{\langle \lambda'_{jl}+\rho^{P_w}_{Q_{\xi w}},H_{Q_{\xi w}}(a')\rangle} \tag{3.4}$$

for

$$u\in V(\mathbb{A}),\quad a\in A_Q,\quad m\in L(\mathbb{A})^1,\quad k'\in K,$$
$$u'\in V_{\xi w}(\mathbb{A}),\quad a'\in A_{Q_{\xi w}},\quad m'\in L_{\xi w}(\mathbb{A})^1,\quad k''\in K',$$

where

$$\lambda_{ik}\in(\mathfrak{a}^P_{Q,\mathbb{C}})^*,\quad \lambda'_{jl}\in(\mathfrak{a}^{P_w}_{Q_{\xi w},\mathbb{C}})^*,\quad q_{ik}\in\mathbb{C}[\mathfrak{a}_Q^P],\quad q'_{jl}\in\mathbb{C}[\mathfrak{a}^{P_w}_{Q_{\xi w}}].$$

We put

$$\Lambda^{ij}_{kl}=\lambda_{ik}+\xi w\lambda'_{jl},\quad Q^{ij}_{kl}(X)=q_{ik}((\xi wX)^P_Q)q'_{jl}(X^{P_w}_{Q_{\xi w}}),\quad X\in\mathfrak{a}'_0.$$

Using the Iwasawa decomposition of $M_w(\mathbb{A})$ relative to $Q_M^{\xi w}$, we can express the integral over $Q_M^{\xi w}(F)\backslash M_w(\mathbb{A})'$ as the integral over $m'\in L_{\xi w}(F)\backslash L_{\xi w}(\mathbb{A})'$ and sum over k,l of

$$\Lambda^{T,Q}_{m,\xi w}\phi_{ik}(m'k')\phi'_{jl}(m'k')\int^{\#}_{\mathfrak{a}^{\mathbf{M}_w}_{\mathbf{L}_{\xi w}}+H'_0(m')} Q^{ij}_{kl}(Y)e^{\langle \Lambda^{ij}_{kl}+\varrho^P_{Q,\xi w},\xi wY\rangle}\Gamma_Q^P(\xi wY-T,T') \\ \times\int^{\#}_{\mathfrak{a}_{\mathbf{M}_w}+Y} Q_{ij}(X)e^{\langle \Lambda_{ij}+\varrho_{P,w},wX\rangle}\tau_P(wX-T-T')\,dX\,dY,$$

where we have absorbed the integral over k'' into the integral over $k'\in K'$. We may combine the $\#$-integrals over $\mathfrak{a}^{\mathbf{M}_w}_{\mathbf{L}_{\xi w}}$ and $\mathfrak{a}_{\mathbf{M}_w}$ into a $\#$-integral over $\mathfrak{a}_{\mathbf{L}_{\xi w}}$ by (3.2), and we obtain the triple integral

$$\int_{K'}\int_{L_{\xi w}(F)\backslash L_{\xi w}(\mathbb{A})'}\Lambda^{T,Q}_{m,\xi w}\phi_{ik}(m'k')\phi'_{jl}(m'k')\int^{\#}_{\mathfrak{a}_{\mathbf{L}_{\xi w}}+H'_0(m')} Q^{ij}_{kl}(X)Q_{ij}(X)e^{\langle\lambda_i+\lambda_{ik}+\rho_Q,\xi wX\rangle} \\ \times e^{\langle\lambda'_j+\lambda'_{jl}-\rho_{Q_{\xi w}},X\rangle}\Gamma_Q^P(\xi wX-T,T')\tau_P(wX-T-T')\,dX\,dm'\,dk'.$$

We conclude that

$$\int^{\#}_{P_w(F)\backslash G'(\mathbb{A})}\sum_{\delta\in Q_M^{\xi w}(F)\backslash M_w(F)}\Lambda^{T,Q}_{m,\xi w}\varphi(\delta g)\varphi'_{P_w}(\delta g)\Gamma_Q^P(H_Q(\xi w\delta g)-T,T')\tau_P(H_P(wg)-T-T')\,dg$$

is equal to the #-integral over $Q_{\xi w}(F)\backslash G'(\mathbb{A})$ of

$$\Lambda^{T,Q}_{m,\xi w}\varphi(g)\varphi'_{Q_{\xi w}}(g)\Gamma^P_Q(H_Q(\xi wg) - T, T')\tau_P(H_P(wg) - T - T').$$

Summing this over all $Q \subset P$, $w \in {}_MW^G_{G'}$ and $\xi \in {}_LW^M_{M_w}$, we see that

$$\begin{aligned}\sum_P \sum_{w\in {}_MW^G_{G'}} &\mathbf{P}^{G',T+T'}_{P,w}(\varphi \otimes \varphi') \\ &= \sum_Q \sum_{P\supset Q} \sum_{w\in {}_LW^G_{G'}} \int^{\#}_{Q_w(F)\backslash G'(\mathbb{A})} \Lambda^{T,Q}_{m,w}\varphi(g)\varphi'_{Q_w}(g)\Gamma^P_Q(H_Q(wg) - T, T')\tau_P(H_P(wg) - T - T')\, dg.\end{aligned}$$

The cone defining τ_P is the positive Weyl chamber $\mathcal{C}_P$ in $\mathfrak{a}_P$ and is contained in the positive Weyl chamber $\mathcal{C}_Q$ of $\mathfrak{a}_Q$. We may apply Lemma 3.1 to take the sum over P inside the #-integral. The relation (2.4) applied to $P = G$, $H = H_Q(wg) - T - T'$ and $X = -T'$ shows that the right-hand side equals

$$\sum_Q \sum_{w\in {}_LW^G_{G'}} \mathbf{P}^{G',T}_{Q,w}(\varphi \otimes \varphi'). \qquad \square$$

Let $\mathbf{m} = (m_1, \dots, m_r)$ be a composition of n. For the parabolic subgroup $Q = P_{\mathbf{m}}$ of G' and $w = w_i \in {}_0W^G_{G'}$, we define the parabolic subgroup $Q(w)$ of G by $Q(w) = P_{\mathbf{m}(w)}$, where

$$\mathbf{m}(w) = (m_1, \dots, m_{j-1}, m_j + 1, m_{j+1}, \dots, m_r)$$

if $i \in [m_1 + \dots + m_{j-1} + 2, m_1 + \dots + m_j]$, and

$$\mathbf{m}(w) = (m_1, \dots, m_j, 1, m_{j+1}, \dots, m_r)$$

if $i = m_1 + \dots + m_j + 1$.

PROPOSITION 3.4. *If $(\varphi, \varphi') \in \mathscr{A}(G) \oplus \mathscr{A}(G')$ satisfies*

$$\Re\langle \mathrm{pr}_w(\lambda + w\lambda') + \rho_P - w\rho_Q, \varpi^\vee\rangle < 0$$

for all parabolic subgroups P of G and Q of G', $w \in {}_0W^G_{G'}$, $\lambda \in \mathscr{E}^{\mathrm{cusp}}_P(\varphi)$, $\lambda' \in \mathscr{E}^{\mathrm{cusp}}_Q(\varphi')$ and $\varpi^\vee \in \hat\Delta^\vee_P \cap \hat\Delta^\vee_{Q(w)}$, then $(\varphi, \varphi') \in \mathscr{A}(G \times G')^*$, $\varphi(g)\varphi'(g)$ is integrable over $G'(F)\backslash G'(\mathbb{A})$, and

$$\mathbf{P}^{G'}(\varphi \otimes \varphi') = \int_{G'(F)\backslash G'(\mathbb{A})} \varphi(g)\varphi'(g)\, dg.$$

Proof. By Lemma 2.2(i), if the integrals

$$\int_{P_w(F)\backslash G'(\mathbb{A})} \Lambda^{T,P}_{m,w}\varphi(g)\varphi'(g)\tau_P(H_P(wg) - T)\, dg \tag{3.5}$$

are absolutely convergent for all P and $w \in {}_MW^G_{G'}$, then $\varphi(g)\varphi'(g)$ is integrable over $G'(F)\backslash G'(\mathbb{A})$. Fix P, $w \in {}_MW^G_{G'}$ and $\xi \in {}_0W^M_{M_w}$. Put $\beta = \xi w$ and take $i \in [1, n+1]$ such that $\beta = w_i$. In order for (3.5) to be absolutely convergent, it suffices to find elements $\mu_1 \in (\mathfrak{a}^P_0)^*$ and $\mu_2 \in (\mathfrak{a}'_0)^*$ such that the integrals

$$\int_{A'_0(t'_0)^{(i)}} e^{\langle \Re\lambda_1 + \rho_P + \mu_1, \beta H'_0(a)\rangle} e^{\langle \Re\lambda_2 + \rho_Q + \mu^Q_2 - 2\rho'_0, H'_0(a)\rangle}(1 + \|H'_0(a)\|)^d\, da$$

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are convergent for all parabolic subgroups Q of G', $\lambda_1 \in \mathscr{E}_P(\varphi)$ and $\lambda_2 \in \mathscr{E}_Q^{\mathrm{cusp}}(\varphi')$ in view of Lemma 2.4, (2.5) and [MW95, Lemma I.4.1]. Thus it amounts to the same to prove the convergence of the integral

$$\int_{\mathfrak{a}_0'} e^{\langle \Re(\lambda_1+\beta\lambda_2)+\mu_1+\beta\mu_2^Q+\rho_P+\beta(\rho_Q-2\rho_0'),\beta X\rangle}\tau_B(\beta X-t_0)(1+\|X\|)^d\,dX,$$

which is easily seen to be equivalent to showing that

$$\Re\langle \mathrm{pr}_\beta(\lambda_1+\beta\lambda_2)+\mu_1+\beta\mu_2^Q+\rho_P+\beta(\rho_Q-2\rho_0'),\varpi^\vee\rangle<0$$

for all $\varpi^\vee \in \hat{\Delta}_0^\vee$. If $\varpi^\vee \in \hat{\Delta}_P^\vee \smallsetminus \hat{\Delta}_{Q(\beta)}^\vee$, then $\langle \mu_1,\varpi^\vee\rangle=0$ and we can choose $\mu_2 \in (\mathfrak{a}_0')^*$ so that $\langle \beta\mu_2^Q,\varpi^\vee\rangle$ is very negative and the condition above is satisfied. The condition for $\varpi^\vee \notin \hat{\Delta}_P^\vee$ is fulfilled if $\mu_1 \in (\mathfrak{a}_0^P)^*$ is sufficiently regular (depending on μ_2) in the negative Weyl chamber. As shown in [MW95, p. 50], there exist a parabolic subgroup $R \subset P$ and $\varrho_1 \in \mathscr{E}_R^{\mathrm{cusp}}(\varphi)$ such that λ_1 coincides with the restriction of ϱ_1 to $\mathfrak{a}_P$. Let $\varpi^\vee \in \hat{\Delta}_P^\vee \cap \hat{\Delta}_{Q(\beta)}^\vee$. Then

$$\langle \mu_1,\varpi^\vee\rangle=\langle \beta\mu_2^Q,\varpi^\vee\rangle=\langle \rho_P-\rho_R,\varpi^\vee\rangle=\langle \beta(\rho_Q-\rho_0'),\varpi^\vee\rangle=0.$$

Since $\hat{\Delta}_P^\vee \subset \hat{\Delta}_R^\vee$,

$$\Re\langle \mathrm{pr}_\beta(\lambda_1+\beta\lambda_2)+\rho_P+\beta(\rho_Q-2\rho_0'),\varpi^\vee\rangle=\Re\langle \mathrm{pr}_\beta(\varrho_1+\beta\lambda_2)+\rho_R-\beta\rho_Q,\varpi^\vee\rangle<0$$

by assumption. Thus (3.5) is absolutely convergent, so that it is equal to $\mathbf{P}_{P,w}^{G',T}(\varphi\otimes\varphi')$. Summing this over all P and $w \in {}_MW_{G'}^G$, we obtain the desired equality.

For each $\lambda' \in \mathscr{E}_{P_w}(\varphi')$, there exist a parabolic subgroup $Q \subset P_w$ and $\varrho' \in \mathscr{E}_Q^{\mathrm{cusp}}(\varphi')$ such that λ' is equal to the restriction of ϱ' to $\mathfrak{a}_{P_w}$. If $\varpi^\vee \in \hat{\Delta}_P^\vee$, then since $\varpi^\vee \in \hat{\Delta}_R^\vee \cap \hat{\Delta}_{Q(w)}^\vee$,

$$\Re\langle \mathrm{pr}_w(\lambda_1+w\lambda')+\varrho_{P,w},\varpi^\vee\rangle=\Re\langle \mathrm{pr}_w(\varrho_1+w\varrho')+\rho_R-w\rho_Q,\varpi^\vee\rangle<0.$$

Thus (φ,φ') belongs to $\mathscr{A}(G\times G')^*$. □

Remark 3.5. The proof above confirms that if $\varphi : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is rapidly decreasing and $\varphi' : G'(F)\backslash G'(\mathbb{A}) \to \mathbb{C}$ is slowly increasing, then $\varphi(g)\varphi'(g)$ is integrable over $G'(F)\backslash G'(\mathbb{A})$.

PROPOSITION 3.6. *If $\varphi(\lambda)$ and $\varphi'(\lambda')$ are analytic families of automorphic forms and if $\mathcal{O}$ is the set of all triplets (λ,λ',s) such that $(\varphi(\lambda),\varphi'(\lambda')_s) \in \mathscr{A}(G\times G')^*$, then $\mathcal{O}$ is a nonempty open set and $(\lambda,\lambda',s) \mapsto \mathbf{P}^{G'}(\varphi(\lambda)\otimes\varphi'(\lambda')_s)$ is an analytic function on $\mathcal{O}$.*

Proof. If we put $\mathbf{e}=(1,1,\dots,1) \in \mathfrak{a}_G^*$, then for $s \in \mathbb{C}$,

$$\langle \mathrm{pr}_{w_i}(s\mathbf{e}),\varpi_j^\vee\rangle=\begin{cases} js & \text{if } j<i,\\ (j-n-1)s & \text{if } j\geqslant i.\end{cases}$$

It follows that for any fixed λ and λ', the pair $(\varphi(\lambda),\varphi'(\lambda')_s)$ belongs to $\mathscr{A}(G\times G')^*$ for generic values of the parameter s. Remark 2.5 concludes that the integral $\mathbf{P}_{P,w}^{G',T}(\varphi(\lambda)\otimes\varphi'(\lambda')_s)$ is uniformly convergent for λ, λ' and s in compact subsets, which completes the proof. □

3.3 Periods of truncated automorphic forms

PROPOSITION 3.7. (i) *For $\varphi \in \mathscr{A}(G)$ and $\varphi' \in \mathscr{A}(G')$ the function*

$$T \mapsto \int_{G'(F)\backslash G'(\mathbb{A})} \Lambda_m^T \varphi(g)\varphi'(g)\, dg,$$

defined for $T \in (\mathfrak{a}_0^G)^+$ sufficiently positive, is a polynomial exponential function $\sum_\lambda p_\lambda(T) e^{\langle \lambda, T\rangle}$. The exponents may be taken from the set

$$\bigcup_P \bigcup_{w \in {}_M W_{G'}^G} \{\eta_P^w(\lambda + w\lambda' + \varrho_{P,w}) \mid \lambda \in \mathscr{E}_P(\varphi),\ \lambda' \in \mathscr{E}_{P_w}(\varphi')\}.$$

(ii) *If $(\varphi, \varphi') \in \mathscr{A}(G \times G')^*$, then $\mathbf{P}^{G'}(\varphi \otimes \varphi') = p_0(T)$. In particular, the right-hand side is constant.*

Proof. The argument parallels that in the proof of Proposition 8.4.1 of [LR03]. Because of the importance of this result for us, we reproduce the proof here. Since $\Gamma_P(X - T, T')$ is a compactly supported function of $X \in \mathfrak{a}_P^G$, Lemma 2.4 enables us to integrate the equality in Lemma 2.2(ii) against φ' over $G'(F)\backslash G'(\mathbb{A})$. Then we get

$$\begin{aligned}
&\int_{G'(F)\backslash G'(\mathbb{A})} \Lambda_m^{T+T'} \varphi(g)\varphi'(g)\, dg \\
&\quad = \sum_P \sum_{w \in {}_M W_{G'}^G} \int_{P_w(F)\backslash G'(\mathbb{A})} \Lambda_{m,w}^{T,P} \varphi(g)\varphi'(g)\Gamma_P(H_P(wg) - T, T')\, dg \\
&\quad = \sum_P \sum_{w \in {}_M W_{G'}^G} \int_{U_w(\mathbb{A})M_w(F)\backslash G'(\mathbb{A})} \Lambda_{m,w}^{T,P} \varphi(g)\varphi'_{P_w}(g)\Gamma_P(H_P(wg) - T, T')\, dg.
\end{aligned}$$

Expand φ_P and φ'_{P_w} as in (3.3). The inner integral is equal to the integral over $M_w(F)\backslash M_w(\mathbb{A})' \times K'$ and sum over i, j of

$$\Lambda_{m,w}^{T,P} \phi_i(mk)\phi'_j(mk) \int^{\#}_{\mathfrak{a}_{\mathbf{M}_w} + H_{P_w}(m)} Q_{ij}(X) e^{\langle \Lambda_{ij} + \varrho_{P,w}, wX\rangle} \Gamma_P(wX - T, T')\, dX.$$

Lemma 2.2 of [Art81] implies that the #-integral is a polynomial exponential function in T' whose exponents are $\{\eta_Q^w(\Lambda_{ij} + \varrho_{P,w})\}_{Q \supset P}$, which proves (i).

Observe that $\mathbf{P}_{P,w}^{G',T}(\varphi \otimes \varphi')$ is a polynomial exponential function in T by applying (i) to M. The zero exponent does not appear in all terms $P \neq G$ by assumption. Since the regularized period does not depend on T, it is equal to the coefficient of the zero exponent in the term $P = G$. $\square$

Set $d(T) = \min\{\langle \alpha, T\rangle \mid \alpha \in \Delta_0\}$.

PROPOSITION 3.8 (cf. [Art85, Theorem 3.1]). *Let φ be a smooth function on $G(F)\backslash G(\mathbb{A})$ such that it and its derivatives have uniform moderate growth and let φ' be a function of moderate growth on $G'(F)\backslash G'(\mathbb{A})$. Then for each positive integer m there is a constant C_m independent of T such that*

$$\int_{G'(F)\backslash G'(\mathbb{A})} |\Lambda_m^T \varphi(g)\varphi'(g) - \varphi(g)\varphi'(g)F^G(g, T)|\, dg \leqslant C_m e^{-md(T)},$$

where T varies over suitably regular points such that $d(T) \geqslant \epsilon_0 \|T\|$ for some fixed positive number ϵ_0.

Proof. We will freely use the notation and the discussion of Lemma 2.4. Lemma 6.1 of [Art78] implies that σ_Q^Q is zero unless $Q = G$. Thus $\Lambda_m^T\varphi(g) - \varphi(g)F^G(g,T)$ is equal to the sum over pairs $Q \subsetneq P$, elements $w \in {}_LW_{G'}^G$ and $\gamma \in Q_w(F)\backslash G'(F)$ of the product

$$F^Q(w\gamma g,T)\sigma_Q^P(H_Q(w\gamma g)-T)\varphi_{Q,P}(w\gamma g).$$

By the assumption on φ and φ', we can take m so that

$$\sum_I\sum_i e^{-m\langle\beta_I,H_0(a_1)\rangle}|\varphi'(n^*ac)|\int_{N_I(F)\backslash N_I(\mathbb{A})}|R(X_i)\varphi(uwac)|\,du$$

is bounded independently of T for all $Q \subsetneq P$ and w. It follows that

$$|\Lambda_m^T\varphi(g)\varphi'(g)-\varphi(g)\varphi'(g)F^G(g,T)|$$

is bounded by a constant multiple of the sum over $Q \subsetneq P$, w, I and γ of

$$F^Q(w\gamma g,T)\sigma_Q^P(H_Q(w\gamma g)-T)e^{-2m\langle\beta_I,H_0(w\gamma g)\rangle}.$$

Thus the integral

$$\int_{G'(F)\backslash G'(\mathbb{A})}|\Lambda_m^T\varphi(g)\varphi'(g)-\varphi(g)\varphi'(g)F^G(g,T)|\,dg$$

is bounded by a constant multiple of the sum over $Q \subsetneq P$, w and I of

$$\int_{L_w(F)\backslash L_w(\mathbb{A})'}\int_{\mathfrak{a}_{\mathbf{L}_w}+H_{Q_w}(m)}F^Q(wm,T)\sigma_Q^P(wX-T)e^{-2m\langle\beta_I,wX\rangle}\,dX\,dm.$$

Observe that

$$\begin{aligned}\int_{\mathfrak{a}_{\mathbf{L}_w}+H_{Q_w}(m)}\sigma_Q^P(wX-T)e^{-2m\langle\beta_I,wX\rangle}\,dX &= \int_{\mathfrak{a}_Q^G}\sigma_Q^P(X-T)e^{-2m\langle\beta_I,X\rangle}\,dX\\ &\leqslant e^{-2m\langle\beta_I,T\rangle}\prod_{\alpha\in\Delta_Q^P}\int_0^\infty p_\alpha(t)e^{-2mt\langle\beta_I,\varpi_\alpha^\vee\rangle}\,dt.\end{aligned}$$

The integral

$$\int_{L_w(F)\backslash L_w(\mathbb{A})'}F^Q(wm,T)\,dm$$

is certainly bounded by a constant multiple of a fixed power of $e^{\|T\|}$. This completes our proof, as the factor $e^{-2m\langle\beta_I,T\rangle}$ is bounded by $e^{-2md(T)}$. □

We write $\mathcal{C}_-^* = \{\sum_{\alpha\in\Delta_0}x_\alpha\alpha \mid x_\alpha \leqslant 0\}$ for the closed negative Weyl chamber.

Lemma 3.9. *Let*

$$f(T) = p_0(T) + \sum_{i=1}^k p_i(T)e^{\langle\lambda_i,T\rangle}$$

be a polynomial exponential function on $\mathfrak{a}_0^G$, where $0, \lambda_1, \ldots, \lambda_k \in (\mathfrak{a}_{0,\mathbb{C}}^G)^*$ are distinct and $p_i(T) \neq 0$ for $i \in [1,k]$. Then the following conditions are equivalent:

(a) *$f(T)$ converges as $T \to \infty$ in $(\mathfrak{a}_0^G)^+$;*

(b) *$p_0(T)$ is constant and $\Re\lambda_i \in \mathcal{C}_-^* \smallsetminus \{0\}$ for all i.*

If these conditions are fulfilled, then $f(T)$ converges to $p_0(T)$ as $T \to \infty$ in $(\mathfrak{a}_0^G)^+$.

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Proof. Clearly, (b) implies (a). Now we derive (b) from (a). We may assume that $k \geqslant 1$ and that $f(T)$ converges to 0 as $T \to \infty$ in $(\mathfrak{a}_0^G)^+$. Note that $f(t\delta)$ converges as $t \to \infty$ for all $\delta = \sum_\alpha \delta_\alpha \varpi_\alpha^\vee \in \mathfrak{a}_0^G$ with $\delta_\alpha > 0$. Thus the statement immediately reduces to the case of $\mathfrak{a}_0^G = \mathbb{R}$. We may assume that $\Re\lambda_1 \geqslant \Re\lambda_i$ for all $i \in [2, k]$. If $\Re\lambda_1 > 0$, then $f(T)$ is asymptotic to

$$\sum_{\Re\lambda_i = \Re\lambda_1} p_i(T) e^{\lambda_i T}$$

as $T \to \infty$, which reduces the statement to the case where $\Re\lambda_i = 0$ for all i. Put $\lambda_i' = (2\pi\sqrt{-1})^{-1}\lambda_i$. Considering the terms of maximal degree, we can reduce the statement to the case where $p_0, p_1, \ldots, p_k$ are constant. We may assume that $\lambda_1', \ldots, \lambda_l'$ are rational and that $\lambda_{l+1}', \ldots, \lambda_k'$ are irrational. Since $g(T) = p_0 + \sum_{i=1}^l p_i e^{\lambda_i T}$ is periodic, we have $l < k$ by (a). We fix irrational numbers $\mu_1, \ldots, \mu_r$, which are linearly independent over $\mathbb{Q}$, such that

$$\lambda_i' = \sum_{j=1}^r a_{ij}\mu_j, \quad a_{ij} \in \mathbb{Z}, \quad i \in [l+1, k].$$

We fix real numbers $d_1, \ldots, d_r$ satisfying the following conditions:

- $c_m = g(m) + \sum_{i=l+1}^k p_i e^{2\pi\sqrt{-1}\sum_j a_{ij}d_j} \neq 0$ for all $m \in \mathbb{Z}$;
- $g(m) + \sum_{i=l+1}^k p_i e^{-2\pi\sqrt{-1}\sum_j a_{ij}d_j} \neq 0$ for all $m \in \mathbb{Z}$.

Note that the set $\{g(m) \mid m \in \mathbb{Z}\}$ is finite. For any $\epsilon > 0$, Weyl's equidistribution theorem gives an integer m such that $|m|$ is arbitrarily large and $|e^{2\pi\sqrt{-1}\mu_j m} - e^{2\pi\sqrt{-1}d_j}| < \epsilon$ for all $j \in [1, r]$. At the cost of replacing d_j by $-d_j$, we may assume that m is positive. Then

$$f(m) = g(m) + \sum_{i=l+1}^k p_i e^{2\pi\sqrt{-1}\sum_j a_{ij}\mu_j m}$$

is very close to $c_m \neq 0$, which is a contradiction. □

COROLLARY 3.10 (cf. [Lap11, Lemma 9]). *If $\varphi \otimes \varphi' \in \mathscr{A}(G \times G')$ is integrable over $G'(F)\backslash G'(\mathbb{A})$, then*

$$\int_{G'(F)\backslash G'(\mathbb{A})} \varphi(g)\varphi'(g)\, dg = p_0(T),$$

where $p_0(T)$ is as in Proposition 3.7(i).

Proof. The integral

$$\int_{G'(F)\backslash G'(\mathbb{A})} \Lambda_m^T \varphi(g)\varphi'(g)\, dg$$

converges to the left-hand side as $T \to \infty$ in $(\mathfrak{a}_0^G)^+$ by Proposition 3.8, while it converges to the right-hand side by Lemma 3.9. □

4. Proof of Theorem 1.1

4.1 Eisenstein series on GL_m

Let $P = MU$ and $Q = LV$ be parabolic subgroups of G_m. The longest element in W^M is denoted by w_0^M. When $Q \supset P$, we put $w_M^L = w_0^L w_0^M$. We write $W(M, L)$ for the set of elements $\sigma \in W^m$ of minimal length in σW^M such that $\sigma M \sigma^{-1} = L$. We say that Q is associated to P if $W(M, L)$ is

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not empty. Set $W(M) = \bigcup_L W(M, L)$. Explicitly, an element in $W(M)$ is represented by a unique permutation matrix that shuffles the diagonal blocks of M without causing any internal change within each block. For $\sigma \in W(M)$, put ${}^\sigma M = \sigma M \sigma^{-1}$, denote by ${}^\sigma P$ the standard parabolic subgroup of G_m whose Levi subgroup is ${}^\sigma M$, and denote by ${}^\sigma U$ the unipotent radical of ${}^\sigma P$. Let ${}_L W_M$ be the set of elements $\sigma \in W^m$ such that $\sigma\alpha > 0$ for all $\alpha \in \Delta_0^P$ and $\sigma^{-1}\alpha > 0$ for all $\alpha \in \Delta_0^Q$. Let ${}_L W_M^c$ be the set of elements $\sigma \in {}_L W_M$ such that ${}^\sigma M \subset L$.

For $\phi \in \mathscr{A}_P^c(G_m)$, $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ and $Q \supset P$, the Eisenstein series

$$E^Q(g, \phi, \lambda) = \sum_{\gamma \in P(F)\backslash Q(F)} \phi_\lambda(\gamma g)$$

converges absolutely if $\langle \Re\lambda^Q - \rho_P^Q, \alpha^\vee \rangle > 0$ for all $\alpha^\vee \in (\Delta^\vee)_P^Q$. When $Q = G_m$, we also write w_0, w_M and $E(\phi, \lambda)$ instead of $w_0^{G_m}$, $w_M^{G_m}$ and $E^{G_m}(\phi, \lambda)$, respectively. Alongside, we define the intertwining operator $M(\sigma, \lambda)$ for $\sigma \in W(M)$ by

$$M(\sigma, \lambda)\phi(g) = e^{-\langle \sigma\lambda, H_{\sigma P}(g)\rangle} \int_{({}^\sigma U \cap \sigma U \sigma^{-1})(\mathbb{A})\backslash {}^\sigma U(\mathbb{A})} \phi(\sigma^{-1} u g) e^{\langle \lambda, H_P(\sigma^{-1} u g)\rangle}\, du.$$

These admit the meromorphic continuation to $\mathfrak{a}_{P,\mathbb{C}}^*$. The constant term of $E(\phi, \lambda)$ along Q is given by

$$E_Q(\phi, \lambda) = \sum_{\sigma \in {}_L W_M^c} E^Q(M(\sigma, \lambda)\phi, \sigma\lambda) \tag{4.1}$$

(see [MW95, § II.1.7]).

We fix once and for all a nontrivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ and extend it to a character of $N_m(\mathbb{A})$ trivial on $N_m(F)$ by setting

$$\psi(u) = \psi(u_{1,2} + \cdots + u_{m-1,m})$$

for $u \in N_m(\mathbb{A})$. Its restriction to any subgroup of $N_m(\mathbb{A})$ is also denoted by ψ. For a smooth function f on $N_m(F)\backslash G_m(\mathbb{A})$, we put

$$W^\psi(g, f) = \int_{N_m(F)\backslash N_m(\mathbb{A})} f(ug)\overline{\psi(u)}\, du.$$

For any Levi subgroup M, put $\overleftarrow{M} = w_0 M w_0$. For $\phi \in \mathscr{A}_P^c(G_m)$ and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, the integral

$$\mathbb{W}^\psi(g, \phi, \lambda) = \mathbb{W}^\psi(g, \phi_\lambda) = \int_{(N_m \cap \overleftarrow{M})(F)\backslash N_m(\mathbb{A})} \phi_\lambda(w_M^{-1} u g)\overline{\psi(u)}\, du$$

factors through a nondegenerate Fourier coefficient

$$W_M^\psi(g, \phi) = \int_{(N_m \cap M)(F)\backslash (N_m \cap M)(\mathbb{A})} \phi(ug)\overline{\psi(u)}\, du$$

of the inducing data. We have the following identity of meromorphic functions on $\mathfrak{a}_{P,\mathbb{C}}^*$:

$$W^\psi(E(\phi, \lambda)) = \mathbb{W}^\psi(\phi, \lambda). \tag{4.2}$$

For $\phi \in \mathscr{A}_P^c(G)$, $\varphi' \in \mathscr{A}(G')$ and a point $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ at which $\mathbb{W}^\psi(\phi, \lambda)$ is analytic, the integral

$$I(\phi, \varphi_s', \lambda) = \int_{N'(\mathbb{A})\backslash G'(\mathbb{A})} \mathbb{W}^\psi(g, \phi, \lambda) W^{\bar\psi}(g, \varphi_s')\, dg$$

is absolutely convergent for $\Re s$ sufficiently large (depending on λ) (see [JPS79, § 13]). This is a generalization of the integral studied extensively by Jacquet, Piatetski-Shapiro and Shalika in [JPS79, JPS83, JS90, Jac09]. Since $I(\phi, \varphi', \lambda)$ converges absolutely uniformly for λ in a compact set in the domain of convergence of the integral (see [Jac09, Proposition 3.3]), it is holomorphic for λ in that domain.

The proof of Theorem 1.1 begins with the following special case.

LEMMA 4.1. *If $\phi \in \mathscr{A}_P^c(G)$ and $\varphi' \in \mathscr{A}(G')$, then for $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ in general position,*

$$\mathbf{P}^{G'}(E(\phi, \lambda) \otimes \varphi') = I(\phi, \varphi', \lambda).$$

4.2 Fourier expansions of pseudo-Eisenstein series

For $i \in [0, m]$ we regard G_i and G_{m-i} as subgroups of G_m through the natural embedding

$$(g_1, g_2) \mapsto \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}.$$

When $Q = P_{(i,m-i)}$ and ϕ is a smooth function on $Q(F)\backslash G_m(\mathbb{A})$, put

$$W_Q^\psi(g, \phi) = \int_{N_{m-i}(F)\backslash N_{m-i}(\mathbb{A})} \phi\left[\begin{pmatrix} \mathbf{1}_i & \\ & u \end{pmatrix} g\right] \overline{\psi(u)}\, du, \quad g \in G_m(\mathbb{A}).$$

For $i \in [0, n]$ we define the subgroup $\mathscr{P}_i$ of G by

$$\mathscr{P}_i = \left\{ \begin{pmatrix} g & y \\ & u \end{pmatrix} \;\middle|\; g \in G_i,\ u \in N_{n+1-i},\ y \in \mathrm{M}_{i,n+1-i} \right\}.$$

Let $\mathcal{Q}_i = \mathcal{L}_i \mathcal{V}_i$ be the parabolic subgroup of G attached to the composition $(i, n+1-i)$. We put

$$\mathscr{P}'_i = \mathscr{P}_i \cap G', \quad \mathcal{Q}'_i = \mathcal{L}'_i \mathcal{V}'_i = \mathcal{Q}_i \cap G'.$$

PROPOSITION 4.2. *Let ϕ be a smooth function on $\mathscr{P}_n(F)\backslash G(\mathbb{A})$. If the series*

$$\sum_{i=0}^{n} \sum_{\gamma \in \mathscr{P}'_i(F)\backslash G'(F)} W_{\mathcal{Q}_i}^\psi(\gamma g, \phi_{\mathcal{Q}_i})$$

converges absolutely, uniformly on compact subsets of $G(\mathbb{A})$, then it is equal to $\phi(g)$.

Proof. Following the proof of Theorem 5.3 in [Sha74], we get

$$\phi(g) = \sum_{\gamma \in \mathscr{P}'_{n-1}(F)\backslash G'(F)} \mathbb{P}\phi(\gamma g) + \phi_{\mathcal{Q}_n}(g),$$

where

$$\mathbb{P}\phi(g) = \int_{F^n\backslash \mathbb{A}^n} \phi\left[\begin{pmatrix} \mathbf{1}_n & y \\ & 1 \end{pmatrix} g\right] \overline{\psi(y_n)}\, dy.$$

We fix g and consider the function f_g on $G'(\mathbb{A})$ given by $f_g(g') = \mathbb{P}\phi(g'g)$. Since f_g is a smooth function on $\mathscr{P}'_{n-1}(F)\backslash G'(\mathbb{A})$, we may assume that

$$f_g(g') = \sum_{i=0}^{n-1} \sum_{\delta \in (\mathscr{P}'_i \cap G_{n-1})(F)\backslash G_{n-1}(F)} W_{\mathcal{Q}'_i}^\psi(\delta g', f_{g,\mathcal{Q}'_i})$$

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by induction. Since

$$\begin{aligned}
&W^{\psi}_{\mathcal{Q}'_i}(g', f_{g,\mathcal{Q}'_i}) \\
&\quad = \int_{N_{n-i}(F)\backslash N_{n-i}(\mathbb{A})} f_{g,\mathcal{Q}'_i}\left[\begin{pmatrix} \mathbf{1}_i & 0 \\ 0 & u \end{pmatrix} g'\right] \overline{\psi(u)}\, du \\
&\quad = \int_{N_{n-i}(F)\backslash N_{n-i}(\mathbb{A})} \int_{\mathrm{M}_{i,n-i}(F)\backslash \mathrm{M}_{i,n-i}(\mathbb{A})} f_g\left[\begin{pmatrix} \mathbf{1}_i & v \\ 0 & \mathbf{1}_{n-i} \end{pmatrix}\begin{pmatrix} \mathbf{1}_i & 0 \\ 0 & u \end{pmatrix} g'\right] \overline{\psi(u)}\, dv\, du \\
&\quad = \int_{N_{n-i}(F)\backslash N_{n-i}(\mathbb{A})} \int_{\mathrm{M}_{i,n-i}(F)\backslash \mathrm{M}_{i,n-i}(\mathbb{A})} \int_{F^n\backslash \mathbb{A}^n} \phi\left[\begin{pmatrix} \mathbf{1}_i & vu & \\ & u & y \\ & & 1 \end{pmatrix} g'g\right] \overline{\psi(y_n)\psi(u)}\, dy\, dv\, du \\
&\quad = W^{\psi}_{\mathcal{Q}_i}(g'g, \phi_{\mathcal{Q}_i}),
\end{aligned}$$

the series in the right-hand side converges absolutely by assumption. Substituting the above into the expression for ϕ, we get

$$\begin{aligned}
\phi(g) &= \sum_{\gamma\in\mathscr{P}'_{n-1}(F)\backslash G'(F)} f_g(\gamma) + \phi_{\mathcal{Q}_n}(g) \\
&= \sum_{\gamma\in\mathscr{P}'_{n-1}(F)\backslash G'(F)} \sum_{i=0}^{n-1} \sum_{\delta\in(\mathscr{P}'_i\cap G_{n-1})(F)\backslash G_{n-1}(F)} W^{\psi}_{\mathcal{Q}_i}(\delta\gamma g, \phi_{\mathcal{Q}_i}) + W^{\psi}_{\mathcal{Q}_n}(g, \phi_{\mathcal{Q}_n}) \\
&= \sum_{i=0}^{n} \sum_{\gamma\in\mathscr{P}'_i(F)\backslash G'(F)} W^{\psi}_{\mathcal{Q}_i}(\gamma g, \phi_{\mathcal{Q}_i})
\end{aligned}$$

as claimed. □

For a finite-dimensional real vector space V, let $\mathcal{PW}(V^*_{\mathbb{C}})$ be the Paley–Wiener space of functions on $V^*_{\mathbb{C}}$ obtained as Fourier transforms of compactly supported smooth functions on V. Fix a finite-dimensional subspace $\mathcal{V}$ of $\mathscr{A}^c_P(G)$. Let $\mathcal{PW}_{(P,\mathcal{V})}$ be the space of $\mathcal{V}$-valued entire functions on $(\mathfrak{a}^G_{P,\mathbb{C}})^*$ of Paley–Wiener type. We may identify $\mathcal{PW}_{(P,\mathcal{V})}$ with $\mathcal{PW}((\mathfrak{a}^G_{P,\mathbb{C}})^*)\otimes\mathcal{V}$. For $\phi\in\mathcal{PW}_{(P,\mathcal{V})}$ and any fixed $\kappa\in(\mathfrak{a}^G_P)^*$, we define a function F_ϕ on $A_G U(\mathbb{A})M(F)\backslash G(\mathbb{A})$, compactly supported in $H_P(g)$, by

$$F_\phi(g) = \int_{\lambda\in(\mathfrak{a}^G_{P,\mathbb{C}})^*,\ \Re\lambda=\kappa} \phi(\lambda)(g) e^{\langle\lambda, H_P(g)\rangle}\, d\lambda$$

and define the pseudo-Eisenstein series θ_ϕ by

$$\theta_\phi(g) = \sum_{\gamma\in P(F)\backslash G(F)} F_\phi(\gamma g).$$

The sum is actually over a finite set depending on g (see [Art78, Lemma 5.1] and the argument at the end of § 4.3), θ_ϕ is rapidly decreasing, and

$$\theta_\phi(g) = \int_{\Re\lambda=\kappa} E(g, \phi(\lambda), \lambda)\, d\lambda \tag{4.3}$$

for any κ in the region of convergence of the Eisenstein series (see [MW95, § II.1]).

When $P\subset\mathcal{Q}_i$ for $i\in[0,n]$, put $\mathscr{M}^i = M\cap G_i$. Let $P_i = M_i U_i$ be the parabolic subgroup of G whose Levi subgroup M_i is $\mathscr{M}^i\times G_{n+1-i}$ and let $P'_i = M'_i U'_i$ be the parabolic subgroup of G'

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whose Levi subgroup M_i' is $\mathscr{M}^i \times G_{n-i}$. Define a Levi subgroup $\mathscr{M}_i$ of G_{n+1-i} by

$$\mathscr{M}_i = \left\{ g \in G_{n+1-i} \;\middle|\; \begin{pmatrix} \mathbf{1}_i & \\ & {}_t g \end{pmatrix} \in M \right\},$$

where ${}_t g = w_0\, {}^t g w_0$ is the 'second transpose' of g about the second diagonal. For $\phi \in \mathscr{A}_P(G)$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ we put

$$\mathbb{W}_i^{\psi}(g, \phi, \lambda) = \mathbb{W}_i^{\psi}(g, \phi_\lambda) = \int_{(N \cap \mathscr{M}_i)(F) \backslash N_{n+1-i}(\mathbb{A})} \phi_\lambda \left[(w_M^{M_i})^{-1} \begin{pmatrix} \mathbf{1}_i & \\ & u \end{pmatrix} g \right] \overline{\psi(u)}\, du.$$

COROLLARY 4.3. *Let $\phi \in \mathcal{PW}_{(P,\mathcal{V})}$ and let $f \in C_c^\infty(G_\infty)$ be a decomposable function. Then*

$$(f * \theta_\phi)(g) = \sum_{i=0}^{n} \sum_{\sigma \in {}_{\mathcal{L}_i}W_M^c} \sum_{\gamma \in (\mathscr{P}_i' \cap {}^\sigma P)(F) \backslash G'(F)} \int_{\Re\lambda = \kappa} [f * \mathbb{W}_i^{\psi}(M(\sigma, \lambda)\phi(\lambda), \sigma\lambda)](\gamma g)\, d\lambda,$$

where $*$ denotes convolution (see § 4.3 below for details) and the sum converges absolutely, uniformly on compact subsets of $G(\mathbb{A})$.

Proof. By (4.1) and (4.3),

$$\theta_{\phi, \mathcal{Q}_i}(g) = \int_{\Re\lambda = \kappa} \sum_{\sigma \in {}_{\mathcal{L}_i}W_M^c} E^{\mathcal{Q}_i}(g, M(\sigma, \lambda)\phi(\lambda), \sigma\lambda)\, d\lambda.$$

Formally, Proposition 4.2 gives

$$(f * \theta_\phi)(g) = \sum_{i=0}^{n} \sum_{\gamma \in \mathscr{P}_i'(F) \backslash G'(F)} \sum_{\sigma \in {}_{\mathcal{L}_i}W_M^c} \int_{\Re\lambda = \kappa} [f * W_{\mathcal{Q}_i}^{\psi}(E^{\mathcal{Q}_i}(M(\sigma, \lambda)\phi(\lambda), \sigma\lambda))](\gamma g)\, d\lambda.$$

The corollary can be deduced from this and (4.2). To justify the manipulation, we will prove the absolute convergence at the end of the next subsection. $\square$

4.3 Uniform estimates for Jacquet integrals

In the first half of this subsection, we switch to a local setting. Thus $F = F_v$ is a local field. If F is nonarchimedean, we denote by $\mathfrak{o} = \mathfrak{o}_v$ the integer ring of F and by $q = q_v$ the cardinality of the residue field of F.

Fix a composition $\mathbf{n} = (n_1, \ldots, n_t)$ of $n+1$ and put $P = P_{\mathbf{n}}$. For an irreducible unitary representation π of M and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, let π_λ be the representation of M given by $\pi_\lambda(m) = e^{\langle \lambda, H_P(m) \rangle}\pi(m)$. We denote by $I_P^G(\pi_\lambda)$ the corresponding induced representation of G. Suppose that π is generic and fix an isomorphism W_M^{ψ} from π to the space of Whittaker functions of π. For $\phi \in I_P^G(\pi)$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ we may define a Whittaker function $\mathbb{W}^{\psi}(\phi_\lambda)$ of $I_P^G(\pi_\lambda)$ by the holomorphic continuation of the Jacquet integral

$$\mathbb{W}^{\psi}(g, \phi_\lambda) = \int_{\overleftarrow{U}} W_M^{\psi}(1, \phi_\lambda(w_M^{-1} u g)) \overline{\psi(u)}\, du,$$

where $\overleftarrow{U} = U_{\overleftarrow{\mathbf{n}}}$ with $\overleftarrow{\mathbf{n}} = (n_t, \ldots, n_1)$.

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First assume that F is nonarchimedean. Fix ϕ and choose an open compact subgroup K_0 of G so that ϕ is K_0-invariant. We have $\mathbb{W}^\psi(gk,\phi_\lambda)=\mathbb{W}^\psi(g,\phi_\lambda)$ for all $g\in G$, $k\in K_0$ and $\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}$, and thus there exists a constant c which does not depend on λ such that

$$\mathbb{W}^\psi(g,\phi_\lambda)=0 \tag{4.4}$$

unless $\langle\alpha,H_0(g)\rangle\leqslant c$ for all $\alpha\in\Delta_0$. By a gauge estimate (see [JPS79, Proposition 2.3.6]),

$$|W^\psi_M(m,\phi(k))|\leqslant Ce^{\langle\mu,H_0(m)\rangle}$$

for all $m\in M$ and $k\in K$, for some $C>0$ and $\mu\in\mathfrak{a}^*_0$. It follows that $|W^\psi_M(1,\phi_\lambda(g))|\leqslant Ce^{\langle\Re\lambda+\mu+\rho_P,H_0(g)\rangle}$ and hence

$$|\mathbb{W}^\psi(g,\phi_\lambda)|\leqslant C\int_{\overleftarrow{U}} e^{\langle\Re\lambda+\mu+\rho_P,H_0(w_M^{-1}ug)\rangle}\,du. \tag{4.5}$$

If $\kappa\in(\mathfrak{a}^G_P)^*$ is sufficiently positive and $\Re\lambda=\kappa$, then this integral is convergent and defines an element in $I^G_B(e^{w_M(\kappa+\mu-\rho^P_0)})$. Combining (4.4) and (4.5), we can take $C'>0$ and $\mu'\in\mathfrak{a}^*_0$ such that

$$|\mathbb{W}^\psi(g,\phi_\lambda)|\leqslant C'e^{\langle\mu',H_0(g)\rangle} \tag{4.6}$$

for all $g\in G$ and $\Re\lambda=\kappa$.

Now suppose that ψ is of order zero and π is unramified. We denote by $\mathbb{W}^\psi(\pi_\lambda)$ the K-invariant Whittaker function of $I^G_P(\pi_\lambda)$ such that $\mathbb{W}^\psi(1,\pi_\lambda)=1$. For $i\in[1,n]$ we define $\alpha_i\in\mathrm{Rat}(T)$ by

$$\alpha_i(t)=t_it_{i+1}^{-1},\quad t=\mathrm{diag}(t_1,\dots,t_{n+1})\in T.$$

Assume further that $q\geqslant n+1$. Since π is generic and unitary, it follows from [JPS79, Proposition 2.4.1] that for any $\kappa\in(\mathfrak{a}^G_P)^*$ there exists $r>0$ which depends on κ but not on F or π such that

$$|\mathbb{W}^\psi(t,\pi_\lambda)|\leqslant|\alpha_1(t)\cdots\alpha_n(t)|^{-r}\Phi(\alpha_1(t),\dots,\alpha_n(t)) \tag{4.7}$$

for all $t\in T$ and $\Re\lambda=\kappa$, where Φ is the characteristic function of $\mathfrak{o}^n$.

Next, assume that F is archimedean. For $\phi\in I^G_P(\pi)$, $\lambda\in\mathfrak{a}^*_{P,\mathbb{C}}$ and $f\in C^\infty_c(G)$, we define an element $f*\phi_\lambda$ in $I^G_P(\pi_\lambda)$ by

$$(f*\phi_\lambda)(g)=\int_G\phi_\lambda(gx)f(x)\,dx,$$

which we call a convolution section. By [Jac04, Lemma 1], we may define a Whittaker function $\mathbb{W}^\psi(f*\phi_\lambda)$ by the Jacquet integral. Recall that the space $C^\infty_c(G)$ is endowed with a topology of an LF space. For the convenience of the reader, we recall the definition of the topology (see [Trè06, § 13] for details). Choose a sequence of compact subsets $C_1\subset C_2\subset\cdots\subset C_i\subset\cdots\subset G$ such that $\bigcup_i C_i=G$. We may further assume that for any i, C_i is contained in the interior of C_{i+1}. Let $C^\infty_{C_i}(G)$ be the space of smooth functions on G whose support is contained in C_i. Then $C^\infty_{C_i}(G)\subset C^\infty_{C_{i+1}}(G)$ and $\bigcup_i C^\infty_{C_i}(G)=C^\infty_c(G)$. We equip $C^\infty_{C_i}(G)$ with the topology of uniform convergence of all derivatives. Then $C^\infty_{C_i}(G)$ becomes a Fréchet space and the topology on $C^\infty_{C_i}(G)$ agrees with the induced topology from $C^\infty_{C_{i+1}}(G)$. We endow $C^\infty_c(G)$ with the finest topology such that the inclusion $C^\infty_{C_i}(G) \to C^\infty_c(G)$ is continuous for all i. In particular, $C^\infty_{C_i}(G)$ is open in $C^\infty_c(G)$ for all i. Then $C^\infty_c(G)$ becomes an LF space and this topology on $C^\infty_c(G)$ does not depend on the choice of the sequence $\{C_i\}$. It is easy to see that the action $G \times C^\infty_c(G) \to C^\infty_c(G)$ given by left (or right) translation is continuous. If $\mathcal{K} \subset C^\infty_c(G)$ is a compact subset, then by definition, we have $\mathcal{K} \subset C^\infty_{C_i}(G)$ for some i. In fact, the same property holds for any bounded subset of $C^\infty_c(G)$ (see [Trè06, Proposition 14.6]).

LEMMA 4.4 (cf. [Jac04, Proposition 9]). *Let $\phi \in I_P^G(\pi)$. Fix $\kappa \in (\mathfrak{a}_P^G)^*$ and a compact set $\mathcal{K} \subset C^\infty_c(G)$. Then there exist $r_i > 0$ such that for any $N > 0$ there exists $C_N > 0$ such that*

$$|\mathbb{W}^\psi(tk, f * \phi_\lambda)| \leqslant C_N e^{\langle \rho_0, H_0(t)\rangle} \prod_{i=1}^n \frac{|\alpha_i(t)|^{-r_i}}{(1 + |\alpha_i(t)|)^N}$$

for all $t \in T'$, $k \in K$, $\Re\lambda = \kappa$ and $f \in \mathcal{K}$.

Proof. Proposition 9 of [Jac04], together with Casselman's subrepresentation theorem, tells us that for a given bounded set $\mathcal{B} \subset C^\infty_c(G)$, the desired estimate holds for $f = f_1 * \cdots * f_n$ with $f_i \in \mathcal{B}$. However, it is not clear whether the Dixmier–Malliavin theorem [DM78] implies that $\mathcal{K} \subset \mathcal{B} * \cdots * \mathcal{B}$ for some $\mathcal{B}$.

To deduce the lemma from Jacquet's estimates, we resort to a strong factorization. Let A_G be the image of $\mathbb{R}_+^\times$ in the center of G and put $G^1 = \{g \in G \mid |\det g| = 1\}$. Note that $G = A_G \times G^1$. We define an algebra homomorphism $\mathrm{pr} : C^\infty_c(G) \to C^\infty_c(G^1)$ by

$$\mathrm{pr}(f)(g) = \int_{A_G} f(ag)\, da.$$

We may assume that π is trivial on A_G. If $\Re\lambda = \kappa$, then A_G acts trivially on $I_P^G(\pi_\lambda)$ and

$$(f * \phi_\lambda)(g) = \int_{G^1} \phi_\lambda(gx)\mathrm{pr}(f)(x)\, dx.$$

Since $\mathrm{pr}(\mathcal{K})$ is compact, Remarque 4.10 of [DM78] gives a compact set $\mathcal{K}_1 \subset C^\infty_c(G)$ and functions $f_2, \ldots, f_n \in C^\infty_c(G)$ such that $\mathrm{pr}(\mathcal{K}) = \mathrm{pr}(\mathcal{K}_1) * \mathrm{pr}(f_2) * \cdots * \mathrm{pr}(f_n)$. Thus we can replace $\mathcal{K}$ by $\mathcal{K}_1 * f_2 * \cdots * f_n$. □

We go back to the global setting. We denote by S_∞ the set of archimedean places of F.

LEMMA 4.5. *Let $\phi \in \mathscr{A}_P^c(G)$. Fix $r > 0$, $\kappa \in (\mathfrak{a}_P^G)^*$ sufficiently positive and a compact set $\mathcal{K}_v \subset C^\infty_c(G_v)$ for each $v \in S_\infty$. Put $\mathcal{K} = \prod_{v \in S_\infty} \mathcal{K}_v$. Let $\mathscr{S}$ be a fundamental domain for $N'(F)\backslash G'(\mathbb{A})$. Then there exists $s_0 > 0$ (depending on r, κ and $\mathcal{K}$) such that the integral*

$$\int_{\mathscr{S}} \|g\|^r |\det g|^\sigma |\mathbb{W}^\psi(g, f * \phi_\lambda)|\, dg$$

is bounded uniformly for $\sigma \geqslant s_0$ in a compact set, $\Re\lambda = \kappa$ and $f \in \mathcal{K}$.

Proof. We can assume that for all $k \in K$ the function $m \mapsto e^{-\langle \rho_P, H_P(m)\rangle}\phi(mk)$ belongs to an irreducible summand π of $\mathscr{A}^c(M)$ and that $\phi = \bigotimes_v \phi_v$ is factorizable. We view W_M^ψ as an isomorphism from π to the space of Whittaker functions of π. By uniqueness of the Whittaker

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model, we may factorize W_M^ψ into a product $\prod_v W_M^{\psi_v}$ under a fixed isomorphism $\pi \simeq \bigotimes_v \pi_v$. Let $f = \prod_{v\in S_\infty} f_v$, where $f_v \in \mathcal{K}_v$. Then we can decompose $\mathbb{W}^\psi(f * \phi_\lambda)$ as

$$\mathbb{W}^\psi(f * \phi_\lambda) = b^S(\lambda,\pi)^{-1} \prod_{v\in S_\infty} \mathbb{W}^{\psi_v}(f_v * \phi_{v,\lambda}) \prod_{v\in S\smallsetminus S_\infty} \mathbb{W}^{\psi_v}(\phi_{v,\lambda}) \prod_{v\notin S} \mathbb{W}^{\psi_v}(\pi_{v,\lambda}), \tag{4.8}$$

where S is a sufficiently large finite set of places of F containing S_∞ and

$$b^S(\lambda,\pi) = \prod_{1\leqslant i<j\leqslant t} L^S(\lambda_i - \lambda_j + 1,\ \pi_i \times \pi_j^\vee)$$

for $\pi = \bigotimes_{i\in[1,t]} \pi_i$ and $\lambda = (\lambda_1,\dots,\lambda_t) \in \mathfrak{a}^*_{P,\mathbb{C}} \simeq \mathbb{C}^t$.

We claim that there exists a constant $C > 0$ such that

$$|b^S(\lambda,\pi)| \geqslant C$$

for all $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ such that $\langle \Re\lambda, \alpha^\vee\rangle \geqslant 2$ for all $\alpha^\vee \in \Delta_P^\vee$. To see this, fix $i<j$ and $v \notin S$. Let $\{\beta_1,\dots,\beta_{n_i}\}$ and $\{\beta'_1,\dots,\beta'_{n_j}\}$ be the Satake parameters of $\pi_{i,v}$ and $\pi^\vee_{j,v}$, respectively. Since π_v is generic and unitary, we have $q_v^{-1/2} < |\beta_k|, |\beta'_l| < q_v^{1/2}$ for all k,l. Hence $|1-\beta_k\beta'_l q_v^{-s}| < 1 + q_v^{-\Re s+1}$ and

$$|L(s,\pi_{i,v}\times\pi^\vee_{j,v})| = \prod_{k,l} \frac{1}{|1-\beta_k\beta'_l q_v^{-s}|} > \frac{1}{(1+q_v^{-\Re s+1})^{n_i n_j}}.$$

The product $\prod_{v\notin S}(1+q_v^{-\Re s+1})^{-1}$ is absolutely convergent and nonzero for $\Re s > 2$, bounded from below uniformly for $\Re s$ tending to $+\infty$. This implies the assertion.

By the above, all that is required is that

$$\int_{\mathscr{S}} \|g\|^r |\det g|^\sigma \prod_{v\in S_\infty} |\mathbb{W}^{\psi_v}(g_v, f_v * \phi_{v,\lambda})| \prod_{v\in S\smallsetminus S_\infty} |\mathbb{W}^{\psi_v}(g_v,\phi_{v,\lambda})| \prod_{v\notin S} |\mathbb{W}^{\psi_v}(g_v,\pi_{v,\lambda})|\, dg$$

be bounded uniformly for $\sigma \geqslant s_0$ in a compact set, $\Re\lambda = \kappa$ and $f_v \in \mathcal{K}_v$. This integral is bounded by the product over v of

$$\int_{T'_v\times K'_v} \|t\|^r |\det t|^\sigma \mathcal{W}_v(tk) e^{-\langle 2\rho'_0, H'_0(t)\rangle}\, dt\, dk, \tag{4.9}$$

where

$$\mathcal{W}_v(g) = \begin{cases} |\mathbb{W}^{\psi_v}(g, f_v * \phi_{v,\lambda})| & \text{if } v \in S_\infty,\\ |\mathbb{W}^{\psi_v}(g,\phi_{v,\lambda})| & \text{if } v \in S\smallsetminus S_\infty,\\ |\mathbb{W}^{\psi_v}(g,\pi_{v,\lambda})| & \text{if } v\notin S.\end{cases}$$

For any $\mu \in (\mathfrak{a}'_0)^*$ there exists $d > 0$ such that $e^{\langle\mu, H'_0(t)\rangle} \leqslant \|t\|^d$. Consider the isomorphism $T'_v \to (F_v^\times)^n$ given by $t \mapsto (\alpha_1(t),\dots,\alpha_n(t))$. Since its inverse is given by

$$(\alpha_1,\dots,\alpha_n) \mapsto \operatorname{diag}(\alpha_1\cdots\alpha_n, \alpha_2\cdots\alpha_n,\dots,\alpha_n),$$

we have

$$\|t\| \leqslant \|\alpha_1(t)\|\|\alpha_2(t)\|\cdots\|\alpha_n(t)\|, \quad |\det t| = |\alpha_1(t)||\alpha_2(t)|^2\cdots|\alpha_n(t)|^n.$$

If $v \in S_\infty$, then by Lemma 4.4 there exist $r_i, R > 0$ such that for any $N > 0$ there exists $C_N > 0$ such that (4.9) is bounded by

$$C_N \int_{T'_v} \prod_{i=1}^n \|\alpha_i(t)\|^R |\alpha_i(t)|^{i\sigma - r_i}(1+|\alpha_i(t)|)^{-N}\, dt$$

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for all $\Re\lambda = \kappa$ and $f_v \in \mathcal{K}_v$. If $i\sigma > r_i + R$ for all i, then we may choose $N \gg 0$ so that this integral is convergent, uniformly for σ in a compact set. If $v \in S \smallsetminus S_\infty$, then by (4.4) and (4.6) there exist $R > 0$ and $\Phi \in \mathcal{S}(F_v^n)$ such that (4.9) is bounded by

$$\int_{T_v'} \prod_{i=1}^n \|\alpha_i(t)\|^R |\alpha_i(t)|^{i\sigma} |\Phi(\alpha_1(t), \ldots, \alpha_n(t))| \, dt$$

for all $\Re\lambda = \kappa$. If $i\sigma > R$ for all i, then this integral is convergent, uniformly for σ in a compact set. If $v \notin S$, then by (4.7) there exist $r_0, R > 0$ which do not depend on v such that (4.9) is bounded by

$$\int_{T_v'} \prod_{i=1}^n \|\alpha_i(t)\|^R |\alpha_i(t)|^{i\sigma - r_0} \Phi(\alpha_1(t), \ldots, \alpha_n(t)) \, dt$$

for all $\Re\lambda = \kappa$, where Φ is the characteristic function of $\mathfrak{o}_v^n$. If $i\sigma > r_0 + R + 1$ for all i, then the product of this integral over $v \notin S$ is convergent, uniformly for σ in a compact set. $\square$

LEMMA 4.6. *Let $i \in \mathbf{I}_P \smallsetminus \{n+1\}$, $\phi \in \mathcal{PW}_{(P,\mathcal{V})}$ and $\varphi' \in \mathscr{A}(G')$. Let $f \in C_c^\infty(G_\infty)$ be a decomposable function. If $\Re s \gg 0$, then*

$$\int_{(\mathscr{P}_i' \cap P)(F) \backslash G'(\mathbb{A})} |\mathbb{W}_i^\psi(g, f * F_\phi) \varphi_s'(g)| \, dg$$

is convergent.

Proof. In order to save space, we write $\mathcal{G}$ in place of G_{n+1-i} and denote by $\mathscr{M}$, $\mathscr{N}$ and $\mathscr{S}$ fundamental domains for $\mathscr{M}^i(F) \backslash \mathscr{M}^i(\mathbb{A})$, $(N \cap \mathscr{M}_i)(F) \backslash N_{n+1-i}(\mathbb{A})$ and $N_{n-i}(F) \backslash G_{n-i}(\mathbb{A})$, respectively. Let J be the element in W^{n+1-i} such that $(w_M^{M_i})^{-1} = \mathrm{diag}(\mathbf{1}_i, J)$. We denote by K_∞ the standard maximal compact subgroup of G_∞. For an adele point $g \in G(\mathbb{A})$, we denote its infinite part by g_∞ and its finite part by g_f. Observe first that for $m \in \mathscr{M}^i(\mathbb{A})$, $g \in \mathcal{G}(\mathbb{A})$ and $k \in K$,

$$\begin{aligned}
\mathbb{W}_i^\psi\left[\begin{pmatrix} m & \\ & g \end{pmatrix} k, f * F_\phi\right] &= \int_{\mathscr{N}} (f * F_\phi)\left[\begin{pmatrix} m & \\ & Jug \end{pmatrix} k\right] \overline{\psi(u)} \, du \\
&= \int_{\mathscr{N}} \int_{G_\infty} F_\phi\left[\begin{pmatrix} m & \\ & Jug \end{pmatrix} kx\right] f(x) \, dx \, \overline{\psi(u)} \, du \\
&= \int_{\mathscr{N}} \int_{K_\infty} \int_{\mathcal{G}_\infty} \int_{\mathscr{M}_\infty^i} F_\phi\left[\begin{pmatrix} ma & \\ & Jugb \end{pmatrix} hk_f\right] f_{a,h}^{k_\infty}(b) \, da \, db \, dh \, \overline{\psi(u)} \, du,
\end{aligned}$$

where we define $f_{a,h}^{k_\infty} \in C_c^\infty(\mathcal{G}_\infty)$ by

$$f_{a,h}^{k_\infty}(b) = e^{-2\langle \rho_{P_i}, H_{P_i}(a) \rangle} |\det b|^i \int_{U_{i,\infty}} f\left[k_\infty^{-1} v \begin{pmatrix} a & \\ & b \end{pmatrix} h\right] dv, \quad b \in \mathcal{G}_\infty,$$

for $a \in \mathscr{M}_\infty^i$ and $h, k_\infty \in K_\infty$. Put $\sigma = \Re s$ and

$$\varphi''(g) = \int_{U_i'(F) \backslash U_i'(\mathbb{A})} |\varphi'(ug)| \, du.$$

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The integral equals

$$\int_{K'}\int_{\mathscr{S}}\int_{\mathscr{M}}\varphi''_\sigma\left[\begin{pmatrix} m & \\ & g\end{pmatrix}k\right]\left|\mathbb{W}_i^\psi\left[\begin{pmatrix} m & \\ & g\end{pmatrix}k, f*F_\phi\right]\right|e^{-2\langle\rho_{P'_i},H_{P'_i}(m)\rangle}|\det g|^i\,dm\,dg\,dk$$
$$=\int_{K'}\int_{\mathscr{S}}\int_{\mathscr{M}}\varphi''_\sigma\left[\begin{pmatrix} m & \\ & g\end{pmatrix}k\right]e^{-2\langle\rho_{P'_i},H_{P'_i}(m)\rangle}|\det g|^i$$
$$\times\left|\int_{\mathscr{N}}\int_{K_\infty}\int_{\mathcal{G}_\infty}\int_{\mathscr{M}^i_\infty}F_\phi\left[\begin{pmatrix} ma & \\ & Jugb\end{pmatrix}hk_f\right]f^{k_\infty}_{a,h}(b)\,da\,db\,dh\,\overline{\psi(u)}\,du\right|dm\,dg\,dk,$$

which is bounded by

$$c\int_{K'}\int_{\mathscr{S}}\int_{\mathscr{M}}\int_{K_\infty}\int_{\mathscr{M}^i_\infty}\|m\|^r\|a\|^r\|g\|^r|\det m|^\sigma|\det a|^{-\sigma}|\det g|^\sigma$$
$$\times\left|\int_{\mathscr{N}}\int_{\mathcal{G}_\infty}F_\phi\left[\begin{pmatrix} m & \\ & Jugb\end{pmatrix}hk_f\right]f^{k_\infty}_{a,h}(b)\,db\,\overline{\psi(u)}\,du\right|da\,dh\,dm\,dg\,dk$$

for some constants $c,r>0$ which do not depend on σ. Since $f^{k_\infty}_{a,h}$ is identically zero for a outside some compact subset of $\mathscr{M}^i_\infty$, it suffices to show that the integral

$$\int_{\mathscr{S}}\int_{\mathscr{M}}\|m\|^r\|g\|^r|\det m|^\sigma|\det g|^\sigma\left|\int_{\mathscr{N}}\int_{\mathcal{G}_\infty}F_\phi\left[\begin{pmatrix} m & \\ & Jugb\end{pmatrix}\right]f(b)\,db\,\overline{\psi(u)}\,du\right|dm\,dg$$

is bounded uniformly for decomposable functions $f\in C_c^\infty(\mathcal{G}_\infty)$ in a compact set.

Put $\mathcal{P}=\mathcal{M}\mathcal{U}=P\cap\mathcal{G}$. We may assume that

$$\phi(\lambda)\left[\begin{pmatrix} m & \\ & g\end{pmatrix}\right]=\beta(\lambda)\Phi(m)\Psi(g)e^{\langle\rho_{P_i},H_P(m)+H_P(g)\rangle}$$

for some $\beta\in\mathcal{PW}((\mathfrak{a}^G_{P,\mathbb{C}})^*)$, $\Phi\in\mathscr{A}^c(\mathscr{M}^i)$ and $\Psi\in\mathscr{A}^c_{\mathcal{P}}(\mathcal{G})$. We may further assume that for all $k\in K_{n+1-i}$ the function $m\mapsto e^{-\langle\rho_{\mathcal{P}},H_{\mathcal{P}}(m)\rangle}\Psi(mk)$ belongs to an irreducible summand of $\mathscr{A}^c(\mathcal{M})$. For any $\lambda\in(\mathfrak{a}^G_{P,\mathbb{C}})^*$ we write $\lambda=\lambda_1+\lambda_2$, where $\lambda_1\in(\mathfrak{a}^G_{P_i,\mathbb{C}})^*$ and $\lambda_2\in(\mathfrak{a}^{P_i}_{P,\mathbb{C}})^*$. Fix $\kappa\in(\mathfrak{a}^G_P)^*$ sufficiently positive. Then

$$F_\phi\left[\begin{pmatrix} m & \\ & g\end{pmatrix}\right]=\Phi(m)\Psi(g)\int_{\Re\lambda=\kappa}\beta(\lambda)e^{\langle\lambda+\rho_{P_i},H_P(m)+H_P(g)\rangle}\,d\lambda$$
$$=\Phi(m)\Psi(g)\int_{\Re\lambda_2=\kappa_2}\hat\beta((H_{P_i}(m)+H_{P_i}(g))^G,\lambda_2)e^{\langle\lambda_2,H_P(g)\rangle}\,d\lambda_2$$
$$=\Phi(m)\int_{\Re\lambda_2=\kappa_2}\hat\beta((H_{P_i}(m)+H_{P_i}(g))^G,\lambda_2)\Psi_{\lambda_2}(g)\,d\lambda_2,$$

where we put

$$\hat\beta(X_1,\lambda_2)=\int_{\Re\lambda_1=\kappa_1}\beta(\lambda_1+\lambda_2)e^{\langle\lambda_1+\rho_{P_i},X_1\rangle}\,d\lambda_1,\quad X_1\in\mathfrak{a}^G_{P_i},\ \lambda_2\in(\mathfrak{a}^{P_i}_{P,\mathbb{C}})^*.$$

For each $v\in S_\infty$, let $A_{\mathcal{G}_v}$ be the image of $\mathbb{R}_+^\times$ in the center of $\mathcal{G}_v$, and put $\mathcal{G}^1_v=\{g\in\mathcal{G}_v\mid|\det g|=1\}$. Put $A_{\mathcal{G}_\infty}=\prod_{v\in S_\infty}A_{\mathcal{G}_v}$ and $\mathcal{G}^1_\infty=\prod_{v\in S_\infty}\mathcal{G}^1_v$. Then $\mathcal{G}_\infty=A_{\mathcal{G}_\infty}\times\mathcal{G}^1_\infty$ and

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$$\int_{\mathscr{N}} \int_{\mathcal{G}_\infty} F_\phi \left[\begin{pmatrix} m & \\ & Jugb \end{pmatrix} \right] f(b)\, db\, \overline{\psi(u)}\, du$$
$$= \Phi(m) \int_{\mathscr{N}} \int_{\mathcal{G}_\infty} \int_{\Re\lambda_2 = \kappa_2} \hat{\beta}((H_{P_i}(m) + H_{P_i}(gb))^G, \lambda_2) \Psi_{\lambda_2}(Jugb) f(b)\, d\lambda_2\, db\, \overline{\psi(u)}\, du$$
$$= \Phi(m) \int_{\mathscr{N}} \int_{\mathcal{G}^1_\infty} \int_{A_{\mathcal{G}_\infty}} \int_{\Re\lambda_2 = \kappa_2} \hat{\beta}((H_{P_i}(m) + H_{P_i}(ga))^G, \lambda_2) \Psi_{\lambda_2}(Jugb) f_a(b)\, d\lambda_2\, da\, db\, \overline{\psi(u)}\, du,$$

where we define $f_a \in C_c^\infty(\mathcal{G}^1_\infty)$ by $f_a(b) = f(ab)$. Thus the integral is bounded by

$$\int_{\mathscr{S}} \int_{\mathscr{M}} \int_{A_{\mathcal{G}_\infty}} \int_{\Re\lambda_2 = \kappa_2} \|m\|^r \|g\|^r |\det m|^\sigma |\det g|^\sigma |\Phi(m) \hat{\beta}((H_{P_i}(m) + H_{P_i}(ga))^G, \lambda_2)|$$
$$\times \left| \int_{\mathscr{N}} \int_{\mathcal{G}^1_\infty} \Psi_{\lambda_2}(Jugb) f_a(b)\, db\, \overline{\psi(u)}\, du \right| d\lambda_2\, da\, dm\, dg.$$

There is a compact subset C of $A_{\mathcal{G}_\infty}$ (depending on a given compact set $\Xi \subset C_c^\infty(\mathcal{G}_\infty)$ of decomposable functions) such that f_a is identically zero for all $f \in \Xi$ unless $a \in C$. We can take $\hat{\beta}_1 \in \mathcal{S}(\mathfrak{a}^G_{P_i})$ and $\beta_2 \in \mathcal{S}((\mathfrak{a}^{P_i}_{P,\mathbb{C}})^*)$ such that $|\hat{\beta}(X_1 + H_{P_i}(a)^G, \lambda_2)| \leqslant |\hat{\beta}_1(X_1) \beta_2(\lambda_2)|$ for all $a \in C$. Therefore, it suffices to show that the integral

$$\int_{\mathscr{S}} \int_{\mathscr{M}} \int_{\Re\lambda_2 = \kappa_2} \|m\|^r \|g\|^r |\det m|^\sigma |\det g|^\sigma |\Phi(m) \hat{\beta}_1((H_{P_i}(m) + H_{P_i}(g))^G) \beta_2(\lambda_2)|$$
$$\times \left| \int_{\mathscr{N}} \int_{\mathcal{G}^1_\infty} \Psi_{\lambda_2}(Jugb) f(b)\, db\, \overline{\psi(u)}\, du \right| d\lambda_2\, dm\, dg$$

is bounded uniformly for decomposable functions $f \in C_c^\infty(\mathcal{G}^1_\infty)$ in a compact set.

For $g \in \mathcal{G}(\mathbb{A})$, let a_g be the unique element in A_{G_i} satisfying $|\det a_g| = |\det g|^{i/(n+1-i)}$. Then $H_{P_i}(g)^G = H_{P_i}(a_g^{-1})^G$ and

$$\int_{\mathscr{M}} \|m\|^r |\det m|^\sigma |\Phi(m) \hat{\beta}_1((H_{P_i}(m) + H_{P_i}(g))^G)|\, dm$$
$$\leqslant c' \|a_g\|^r |\det a_g|^\sigma \int_{\mathscr{M}} \|m\|^r |\det m|^\sigma |\Phi(m) \hat{\beta}_1(H_{P_i}(m)^G)|\, dm$$
$$\leqslant c'' \|g\|^{r'} |\det g|^{i\sigma/(n+1-i)} \int_{A_{\mathscr{M}^i}} \|a\|^r |\det a|^\sigma |\hat{\beta}_1(H_{P_i}(a)^G)|\, da$$

for some constants $c', c'', r' > 0$ which do not depend on σ. Since the last integral is convergent, it remains to show that the integral

$$\int_{\Re\lambda_2 = \kappa_2} |\beta_2(\lambda_2)| \int_{\mathscr{S}} \|g\|^{r+r'} |\det g|^{(n+1)\sigma/(n+1-i)} \left| \int_{\mathscr{N}} \int_{\mathcal{G}^1_\infty} \Psi_{\lambda_2}(Jugb) f(b)\, db\, \overline{\psi(u)}\, du \right| dg\, d\lambda_2$$

is bounded uniformly for decomposable functions $f \in C_c^\infty(\mathcal{G}^1_\infty)$ in a compact set, which reduces to Lemma 4.5. □

We conclude this subsection by completing our proof of Corollary 4.3. It remains to prove that for each i and σ, the sum

$$\sum_{\gamma \in (\mathscr{P}'_i \cap {}^\sigma P)(F) \backslash G'(F)} \int_{\Re\lambda = \kappa} [f * \mathbb{W}^\psi_i(M(\sigma, \lambda)\phi(\lambda), \sigma\lambda)](\gamma g)\, d\lambda$$

converges absolutely, uniformly on compact subsets of $G(\mathbb{A})$. We will write ${}^\sigma P_i = ({}^\sigma P)_i$ and ${}^\sigma P_i' = ({}^\sigma P)_i'$ for brevity. Since the function

$$X \mapsto \int_{\Re\lambda=\kappa} f * [M(\sigma,\lambda)\phi(\lambda)]_{\sigma\lambda}(e^X g)\, d\lambda, \quad X \in \mathfrak{a}_{{}^\sigma P}$$

is compactly supported modulo $\mathfrak{a}_G$, uniformly for $g \in G(\mathbb{A})$ with $H_{{}^\sigma P}(g) = 0$, we can take a subset $\mathcal{X} \subset \mathfrak{a}_{{}^\sigma P_i}$ whose projection to $\mathfrak{a}^G_{{}^\sigma P_i}$ is compact and such that the support of the function

$$X \mapsto \int_{\Re\lambda=\kappa} [f * \mathbb{W}_i^\psi(M(\sigma,\lambda)\phi(\lambda),\sigma\lambda)](e^X g)\, d\lambda, \quad X \in \mathfrak{a}_{{}^\sigma P_i}$$

is contained in $\mathcal{X}$ for all $g \in G(\mathbb{A})$ with $H_{{}^\sigma P_i}(g) = 0$. Fix a compact subset $C \subset G(\mathbb{A})$ and choose $T \in \mathfrak{a}_{{}^\sigma P_i}$ so that

$$\{X - H_{{}^\sigma P_i}(kh) - T \mid X \in \mathcal{X}, k \in K, h \in C\} \subset \hat{\mathcal{C}}_{{}^\sigma P_i},$$

where $\hat{\mathcal{C}}_{{}^\sigma P_i}$ is the cone defining $\hat{\tau}_{{}^\sigma P_i}$. By [Art78, Lemma 5.1], the set of $\gamma \in {}^\sigma P_i(F)\backslash G(F)$ such that $\hat{\tau}_{{}^\sigma P_i}(H_{{}^\sigma P_i}(\gamma) - T) = 1$ is finite. Choose a finite subset $\Gamma \subset G'(F)$ so that

$${}^\sigma P_i'(F)\Gamma = \{\gamma \in G'(F) \mid \hat{\tau}_{{}^\sigma P_i}(H_{{}^\sigma P_i}(\gamma) - T) = 1\}.$$

If

$$\int_{\Re\lambda=\kappa} [f * \mathbb{W}_i^\psi(M(\sigma,\lambda)\phi(\lambda),\sigma\lambda)](\gamma h)\, d\lambda \neq 0$$

for $\gamma \in G'(F)$ and $h \in C$, then $H_{{}^\sigma P_i}(\gamma h) \in \mathcal{X}$. Since

$$H_{{}^\sigma P_i}(\gamma h) = H_{{}^\sigma P_i}(\gamma) + H_{{}^\sigma P_i}(k(\gamma)h),$$

where $k(\gamma) \in K$ is a K-part of the Iwasawa decomposition of γ, we have $H_{{}^\sigma P_i}(\gamma) - T \in \hat{\mathcal{C}}_{{}^\sigma P_i}$ and hence $\gamma \in {}^\sigma P_i'(F)\Gamma$. It therefore suffices to show that the sum

$$\sum_{\gamma \in N_{n-i}(F)\backslash G_{n-i}(F)} \int_{\Re\lambda=\kappa} [f * \mathbb{W}_i^\psi(M(\sigma,\lambda)\phi(\lambda),\sigma\lambda)] \left[\begin{pmatrix} \mathbf{1}_i & \\ & \gamma \end{pmatrix} g \right] d\lambda$$

converges absolutely, uniformly on compact subsets of $G(\mathbb{A})$.

For each $h \in G_\infty$, we have

$$[f * \mathbb{W}_i^\psi(M(\sigma,\lambda)\phi(\lambda),\sigma\lambda)](gh) = [f_h * \mathbb{W}_i^\psi(M(\sigma,\lambda)\phi(\lambda),\sigma\lambda)](g),$$

where $f_h(x) = f(h^{-1}x)$. Since the map $(f,h) \mapsto f_h$ is continuous, the subset $\{f_h \mid h \in C_\infty\} \subset C_c^\infty(G_\infty)$ is compact for any compact subset $C_\infty \subset G_\infty$. We have thus reduced to showing the absolute convergence of the sum above with $g = \mathbf{1}_{n+1}$, uniformly for decomposable functions $f \in C_c^\infty(G_\infty)$ in a compact set. We may assume that $\phi(\lambda) = \beta(\lambda)\phi$ for some $\beta \in \mathcal{PW}((\mathfrak{a}^G_{P,\mathbb{C}})^*)$ and $\phi \in \mathscr{A}_P^c(G)$. We may further assume that for all $k \in K$ the function $m \mapsto e^{-\langle \rho_P, H_P(m)\rangle}\phi(mk)$ belongs to an irreducible summand $\pi \simeq \bigotimes_v \pi_v$ of $\mathscr{A}^c(M)$ and that $\phi = \bigotimes_v \phi_v$ is factorizable. For each v and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ we define the representation $\pi_{v,\lambda}$ of M_v on the same space by $\pi_{v,\lambda}(m) = e^{\langle \lambda, H_P(m)\rangle}\pi_v(m)$. We realize the local induced representation $I_{P_v}^{G_v}(\pi_{v,\lambda})$ on a space $\mathcal{H}_P(\pi_v)$ (independent of λ) of π_v-valued functions on K_v, and decompose $M(\sigma,\lambda)$ into a product of local intertwining operators $M_v(\sigma,\lambda) : \mathcal{H}_P(\pi_v) \to \mathcal{H}_{{}^\sigma P}(\sigma\pi_v)$. We denote by ${}^L\mathfrak{u}_\sigma$ the subspace of the Lie algebra of the L-group of U consisting of all those root spaces whose roots are sent to the negative roots under σ and by $r_\sigma^\vee$ the contragredient of the adjoint action r_σ of the L-group of M on ${}^L\mathfrak{u}_\sigma$. Then

$$M(\sigma,\lambda)\phi(\lambda) = \beta(\lambda)\frac{L^S(0,\pi_\lambda,r_\sigma^\vee)}{L^S(1,\pi_\lambda,r_\sigma^\vee)}\left(\bigotimes_{v\in S} M_v(\sigma,\lambda)\phi_v\right)\otimes\left(\bigotimes_{v\notin S}\phi'_{v,0}\right)$$

for a sufficiently large finite set S of places of F containing S_∞, where $\phi'_{v,0}$ is a K_v-fixed element in $\mathcal{H}_{{}^\sigma P}(\sigma\pi_v)$ used to define the restricted tensor product $I_{{}^\sigma P}^G(\sigma\pi_\lambda) \simeq \bigotimes_v \mathcal{H}_{{}^\sigma P}(\sigma\pi_v)$.

As in the proof of Lemma 4.5, we can show that the function $\lambda \mapsto L^S(0,\pi_\lambda,r_\sigma^\vee)/L^S(1,\pi_\lambda,r_\sigma^\vee)$ is holomorphic and bounded on a vertical strip containing $\Re\lambda = \kappa$. By Cauchy's integral formula, all its derivatives are also bounded on $\Re\lambda = \kappa$. If $v \in S$, then since ϕ_v is K_v-finite, there exists a finite set of K_v-types $\mathfrak{F}_v$ independent of λ such that any K_v-type occurring in the K_v-span of $M_v(\sigma,\lambda)\phi_v$ belongs to $\mathfrak{F}_v$ for all λ. Hence, noting that $\mathcal{H}_{{}^\sigma P}(\sigma\pi_v)$ is admissible, we can write $M_v(\sigma,\lambda)\phi_v$ as a finite sum $\sum_j c_{v,j}(\lambda)\phi'_{v,j}$ for some functions $c_{v,j}(\lambda)$ in λ and K_v-finite elements $\phi'_{v,j}$ in $\mathcal{H}_{{}^\sigma P}(\sigma\pi_v)$. We may assume that the $\phi'_{v,j}$ are linearly independent. Then $c_{v,j}(\lambda)$ is holomorphic in the region of absolute convergence of $M_v(\sigma,\lambda)$. We shall show that $c_{v,j}(\lambda)$ and all its derivatives are bounded on $\Re\lambda = \kappa$. If $v \in S \smallsetminus S_\infty$, then this is clear since there exists a lattice $\Lambda_v \subset (\mathfrak{a}_P^G)^*$ such that $M_v(\sigma,\lambda+\sqrt{-1}\lambda_0) = M_v(\sigma,\lambda)$ for all $\lambda_0 \in \Lambda_v$. Suppose that $v \in S_\infty$. Fixing an inner product on $\sigma\pi_v$, we equip $\mathcal{H}_{{}^\sigma P}(\sigma\pi_v)$ with the inner product $(\cdot,\cdot)$ given by integration over K_v, the norm $\|\cdot\|$ associated to $(\cdot,\cdot)$, and the supremum norm $\|\cdot\|_\infty$. Note that $\|\cdot\| \leqslant \|\cdot\|_\infty$. For each j, we can find a K_v-finite element $\phi''_{v,j}$ in $\mathcal{H}_{{}^\sigma P}(\sigma\pi_v)$ such that

$$c_{v,j}(\lambda) = (M_v(\sigma,\lambda)\phi_v, \phi''_{v,j}).$$

Hence

$$|c_{v,j}(\lambda)| \leqslant \|M_v(\sigma,\lambda)\phi_v\|\|\phi''_{v,j}\| \leqslant \|M_v(\sigma,\lambda)\phi_v\|_\infty\|\phi''_{v,j}\|_\infty$$

by the Cauchy–Schwarz inequality. On the other hand, it follows from [Wal92, Lemma 10.1.11] that $\|M_v(\sigma,\lambda)\phi_v\|_\infty$ is bounded on a vertical strip containing $\Re\lambda = \kappa$. By Cauchy's integral formula again, $c_{v,j}(\lambda)$ and all its derivatives are also bounded on $\Re\lambda = \kappa$.

Thus, after replacing $\beta(\lambda)$ if necessary and rewriting ${}^\sigma P$ as P, etc., we have reduced to showing the absolute convergence of the sum

$$\sum_{\gamma\in N_{n-i}(F)\backslash G_{n-i}(F)}\int_{\Re\lambda=\kappa}[f*\mathbb{W}_i^\psi(\phi(\lambda),\lambda)]\left[\begin{pmatrix}\mathbf{1}_i & \\ & \gamma\end{pmatrix}\right]d\lambda,$$

uniformly for $f \in \mathcal{K}$, where $i \in \mathbf{I}_P \smallsetminus \{n+1\}$, $\kappa \in (\mathfrak{a}_P^G)^*$ with $\kappa^{\mathcal{Q}_i} \in (\mathfrak{a}_P^{\mathcal{Q}_i})^*$ sufficiently positive, $\phi(\lambda) = \beta(\lambda)\phi$ with $\beta(\lambda) \in \mathcal{S}((\mathfrak{a}_{P,\mathbb{C}}^G)^*)$ and $\phi \in \mathscr{A}_P^c(G)$, and $\mathcal{K} \subset C_c^\infty(G_\infty)$ is a compact subset of decomposable functions. We retain the notation in the proof of Lemma 4.6. There we observe that

$$[f*\mathbb{W}_i^\psi(\phi(\lambda),\lambda)]\left[\begin{pmatrix}\mathbf{1}_i & \\ & g\end{pmatrix}\right] = \beta(\lambda)\int_{\mathscr{N}}\int_{K_\infty}\int_{\mathcal{G}_\infty}\int_{\mathscr{M}_\infty^i}\phi_\lambda\left[\begin{pmatrix}a & \\ & Jugb\end{pmatrix}h\right]f_{a,h}^1(b)\,da\,db\,dh\,\overline{\psi(u)}\,du$$

for $g \in \mathcal{G}(\mathbb{A})$. Since $\mathcal{K}$ is compact, there exists a compact subset of G_∞ which contains the support of any $f \in \mathcal{K}$, so that $f_{a,h}^1$ is identically zero for all $f \in \mathcal{K}$ and $h \in K_\infty$ if a is outside some compact subset of $\mathscr{M}_\infty^i$. Since ϕ is K_∞-finite and the subset $\{f_{a,h}^1 \mid f \in \mathcal{K}, a \in \mathscr{M}_\infty^i, h \in K_\infty\} \subset C_c^\infty(\mathcal{G}_\infty)$

is compact, it suffices to show that for any compact subset $\Omega \subset \mathscr{M}^i_\infty$ and any compact subset $\Xi \subset C_c^\infty(\mathcal{G}_\infty)$ of decomposable functions, the sum

$$\sum_{\gamma \in N_{n-i}(F)\backslash G_{n-i}(F)} \left| \int_{\Re\lambda=\kappa} \int_{\mathscr{N}} \int_{\mathcal{G}_\infty} \beta(\lambda)\phi_\lambda \left[\begin{pmatrix} a & \\ & Ju\gamma b \end{pmatrix} \right] f(b)\, db\, \overline{\psi(u)}\, du\, d\lambda \right|$$

is bounded uniformly for $a \in \Omega$ and $f \in \Xi$. We may assume that

$$\phi\left[\begin{pmatrix} m & \\ & g \end{pmatrix} \right] = \Phi(m)\Psi(g) e^{\langle \rho_{P_i}, H_P(m)+H_P(g)\rangle}$$

for some $\Phi \in \mathscr{A}^c(\mathscr{M}^i)$ and $\Psi \in \mathscr{A}^c_{\mathcal{P}}(\mathcal{G})$. We may further assume that for all $k \in K_{n+1-i}$ the function $m \mapsto e^{-\langle \rho_{\mathcal{P}}, H_{\mathcal{P}}(m)\rangle}\Psi(mk)$ belongs to an irreducible summand of $\mathscr{A}^c(\mathcal{M})$. Then the computation in the proof of Lemma 4.6 yields

$$\begin{aligned}
&\int_{\Re\lambda=\kappa} \int_{\mathscr{N}} \int_{\mathcal{G}_\infty} \beta(\lambda)\phi_\lambda \left[\begin{pmatrix} a & \\ & Ju\gamma b \end{pmatrix} \right] f(b)\, db\, \overline{\psi(u)}\, du\, d\lambda \\
&\quad = \int_{\Re\lambda_2=\kappa_2} \int_{\mathscr{N}} \int_{A_{\mathcal{G}_\infty}} \int_{\mathcal{G}^1_\infty} \Phi(a)\hat{\beta}((H_{P_i}(a)+H_{P_i}(z))^G, \lambda_2)\Psi_{\lambda_2}(Ju\gamma b) f_z(b)\, db\, dz\, \overline{\psi(u)}\, du\, d\lambda_2
\end{aligned}$$

for $\gamma \in G_{n-i}(F)$. There is a compact subset $\Omega' \subset A_{\mathcal{G}_\infty}$ such that f_z is identically zero for all $f \in \Xi$ unless $z \in \Omega'$. We can take $\beta' \in \mathcal{S}((\mathfrak{a}^{P_i}_{P,\mathbb{C}})^*)$ such that $|\Phi(a)\hat{\beta}((H_{P_i}(a)+H_{P_i}(z))^G, \lambda_2)| \leqslant \beta'(\lambda_2)$ for all $a \in \Omega$ and $z \in \Omega'$. Then the absolute value of the integral above is bounded by

$$\int_{\Re\lambda_2=\kappa_2} \int_{A_{\mathcal{G}_\infty}} \beta'(\lambda_2) \left| \int_{\mathscr{N}} \int_{\mathcal{G}^1_\infty} \Psi_{\lambda_2}(Ju\gamma b) f_z(b)\, db\, \overline{\psi(u)}\, du \right| dz\, d\lambda_2$$

for all $a \in \Omega$ and $f \in \Xi$. Since the subset $\{f_z \mid f \in \Xi, z \in A_{\mathcal{G}_\infty}\} \subset C_c^\infty(\mathcal{G}^1_\infty)$ is compact, it suffices to show that for any compact subset $\Xi^1 \subset C_c^\infty(\mathcal{G}^1_\infty)$ of decomposable functions, the sum

$$\sum_{\gamma \in N_{n-i}(F)\backslash G_{n-i}(F)} \left| \int_{\mathscr{N}} \int_{\mathcal{G}^1_\infty} \Psi_{\lambda_2}(Ju\gamma b) f(b)\, db\, \overline{\psi(u)}\, du \right|$$

is bounded uniformly for $\Re\lambda_2 = \kappa_2$ and $f \in \Xi^1$. Choose a compact neighborhood $\mathscr{V}$ of $\mathbf{1}_{n-i}$ in $G_{n-i}(\mathbb{A})$ whose translates by $G_{n-i}(F)$ do not meet it. Since κ_2 is sufficiently positive in $(\mathfrak{a}^{P_i}_P)^*$, the proof of Lemma 4.5 gives a function ξ on $N_{n-i}(\mathbb{A})\backslash G_{n-i}(\mathbb{A})$ such that

$$\left| \int_{\mathscr{N}} \int_{\mathcal{G}^1_\infty} \Psi_{\lambda_2}(Jugb) f(b)\, db\, \overline{\psi(u)}\, du \right| \leqslant \xi(gx)$$

for $g \in G_{n-i}(\mathbb{A})$, $x \in \mathscr{V}$, $\Re\lambda_2 = \kappa_2$, and $f \in \Xi^1$, and such that the integral

$$\int_{N_{n-i}(F)\backslash G_{n-i}(\mathbb{A})} \xi(g)|\det g|^s\, dg$$

converges for sufficiently large $s \in \mathbb{R}$. More precisely, we apply the argument to $C_c^\infty(\mathcal{G}^1_\infty)$ rather than to $C_c^\infty(\mathcal{G}_\infty)$, noting that $A_{\mathcal{G}_\infty}$ acts trivially on Ψ_{λ_2}. The uniform convergence of the sum above now follows from [JPS79, proof of Proposition 12.2]. This completes the proof of Corollary 4.3.

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4.4 Periods of pseudo-Eisenstein series

Let $f \in C_c^\infty(G_\infty)$. When $\phi(\lambda) \in \mathscr{A}_P^c(G)$ depends holomorphically on $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, we define $f \star \phi(\lambda) \in \mathscr{A}_P^c(G)$ by

$$(f \star \phi(\lambda))_\lambda = f * \phi(\lambda)_\lambda$$

for $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, where we recall that $\phi(\lambda)_\lambda(g) = \phi(\lambda)(g)e^{\langle \lambda, H_P(g)\rangle}$ for $g \in G(\mathbb{A})$.

Lemma 4.7. *Let $\phi \in \mathcal{PW}_{(P,\mathcal{V})}$ and $\varphi' \in \mathscr{A}(G')$. Let $f \in C_c^\infty(G_\infty)$ be a decomposable function. Fix $s \in \mathbb{C}$ with $\Re s$ large enough. Put*

$$s_i = \Bigg(\underbrace{-s, \ldots, -s}_{i}, \underbrace{\frac{is}{n+1-i}, \ldots, \frac{is}{n+1-i}}_{n+1-i}\Bigg) \in (\mathfrak{a}_{Q_i,\mathbb{C}}^G)^*.$$

For $i \in [0, n]$ and $\sigma \in {}_{\mathcal{L}_i}W_M^c$ we write ${}^\sigma P_i = ({}^\sigma P)_i$, ${}^\sigma P_i' = ({}^\sigma P)_i'$ and ${}^\sigma\mathscr{M}^i = {}^\sigma M \cap G_i$ for brevity. Assume that $\phi(\sigma^{-1}\lambda)$ vanishes to a higher order on the affine subspaces

$$\beta_{{}^\sigma P_i}^G + s_i + (\mathfrak{a}_{{}^\sigma P,\mathbb{C}}^{{}^\sigma P_i})^*$$

for $i \in [1, n]$, $\sigma \in {}_{\mathcal{L}_i}W_M^c$ and $\lambda' \in \mathscr{E}_{{}^\sigma P_i'}(\varphi')$, where β is the restriction of $\rho_{{}^\sigma P_i'} - \rho_{{}^\sigma P_i} - \lambda'$ to $\mathfrak{a}_{{}^\sigma\mathscr{M}^i}$. Then

$$\int_{G'(F)\backslash G'(\mathbb{A})} (f * \theta_\phi)(g)\varphi_s'(g)\, dg = \int_{\Re\lambda=\kappa} I(f \star \phi(\lambda), \varphi_s', \lambda)\, d\lambda.$$

Proof. To simplify notation, we put $F_{i,\sigma}(g, \sigma\lambda) = [f * \mathbb{W}_i^\psi(M(\sigma,\lambda)\phi(\lambda), \sigma\lambda)](g)$. Corollary 4.3, together with Lemmas 4.5 and 4.6, tells us that

$$\begin{aligned}
&\int_{G'(F)\backslash G'(\mathbb{A})} (f * \theta_\phi)(g)\varphi_s'(g)\, dg \\
&\quad = \sum_{i=0}^n \sum_{\sigma \in {}_{\mathcal{L}_i}W_M^c} \int_{(\mathscr{P}_i' \cap {}^\sigma P)(F)\backslash G'(\mathbb{A})} \int_{\Re\lambda=\kappa} F_{i,\sigma}(g, \sigma\lambda)\varphi_s'(g)\, d\lambda\, dg \\
&\quad = \sum_{i=0}^n \sum_{\sigma \in {}_{\mathcal{L}_i}W_M^c} \int_{(\mathscr{P}_i' \cap {}^\sigma P)(F)\backslash G'(\mathbb{A})} \int_{\nu \in (\mathfrak{a}_{{}^\sigma P,\mathbb{C}}^G)^*, \Re\nu=\sigma\kappa} F_{i,\sigma}(g, \nu) W_{Q_i'}^{\bar\psi}(g, \varphi_{{}^\sigma P_i', s}')\, d\nu\, dg.
\end{aligned}$$

Note that we can apply Lemma 4.6 to $M(\sigma,\lambda)\phi(\lambda)$ by the argument about intertwining operators at the end of § 4.3. The function $\varphi_{{}^\sigma P_i'}'$ has a decomposition of the form

$$\varphi_{{}^\sigma P_i'}'\left[u\begin{pmatrix} m & \\ & h\end{pmatrix}k\right] = \sum_\mu Q_\mu(H_{{}^\sigma P_i'}(m)^G)\phi_\mu'\left[\begin{pmatrix} m & \\ & h\end{pmatrix}k\right],$$

where

$$u \in {}^\sigma U_i'(\mathbb{A}), \quad m \in {}^\sigma\mathscr{M}^i(\mathbb{A}), \quad h \in G_{n-i}(\mathbb{A}), \quad k \in K', \quad \mu \in (\mathfrak{a}_{{}^\sigma P_i,\mathbb{C}}^G)^*, \quad Q_\mu \in \mathbb{C}[\mathfrak{a}_{{}^\sigma P_i}^G]$$

and $\phi_\mu' \in \mathscr{A}_{{}^\sigma P_i'}(G')$ is an eigenfunction under $\mathfrak{a}_{{}^\sigma\mathscr{M}^i}$ satisfying

$$\phi_\mu'(e^X g) = e^{\langle 2\rho_{{}^\sigma P_i'} - \rho_{{}^\sigma P_i} - \mu, X\rangle}\phi_\mu'(g)$$

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for $X \in \mathfrak{a}_{\sigma\mathscr{M}^i}$. Fix $i \in [1, n]$, $\sigma \in {}_{\mathcal{L}_i}W_M^c$ and μ. We write the corresponding summand as

$$\int_{K'} \int_{N_{n-i}(\mathbb{A})\backslash G_{n-i}(\mathbb{A})} \int_{{}^\sigma\mathscr{M}^i(F)\backslash{}^\sigma\mathscr{M}^i(\mathbb{A})^1} \int_{\mathfrak{a}_{\sigma\mathscr{M}^i}} \int_{\Re\nu=\sigma\kappa} F_{i,\sigma}\left[e^X \begin{pmatrix} m & \\ & h \end{pmatrix} k, \nu\right] \\ \times Q_\mu(X^G) W^{\bar\psi}_{\mathcal{Q}'_i}\left[e^X \begin{pmatrix} m & \\ & h\end{pmatrix} k, \phi'_{\mu,s}\right] e^{-2\langle\rho_{\sigma P'_i}, X\rangle} |\det h|^i \, d\nu \, dX \, dm \, dh \, dk.$$

Put $g = \left(\begin{smallmatrix} m & \\ & h \end{smallmatrix}\right)k$. We can use Fourier inversion for the inner integration to obtain

$$\begin{aligned} &\int_{\mathfrak{a}_{\sigma\mathscr{M}^i}} \int_{\Re\nu=\sigma\kappa} F_{i,\sigma}(e^X g, \nu) Q_\mu(X^G) e^{\langle 2\rho_{\sigma P'_i} - \rho_{\sigma P_i} - \mu - s_i, X\rangle} W^{\bar\psi}_{\mathcal{Q}'_i}(g, \phi'_{\mu,s}) e^{-2\langle\rho_{\sigma P'_i}, X\rangle} \, d\nu \, dX \\ &\quad = W^{\bar\psi}_{\mathcal{Q}'_i}(g, \phi'_{\mu,s}) \int_{\mathfrak{a}^G_{\sigma P_i}} \int_{\Re\nu=\sigma\kappa} F_{i,\sigma}(e^X g, \nu) Q_\mu(X) e^{-\langle\rho_{\sigma P_i} + \mu + s_i, X\rangle} \, d\nu \, dX \\ &\quad = W^{\bar\psi}_{\mathcal{Q}'_i}(g, \phi'_{\mu,s}) \int_{\mathfrak{a}^G_{\sigma P_i}} \int_{\Re\nu=\sigma\kappa} (D_\mu \bullet F_{i,\sigma})(e^X g, \nu) e^{-\langle\rho_{\sigma P_i} + \mu + s_i, X\rangle} \, d\nu \, dX \\ &\quad = W^{\bar\psi}_{\mathcal{Q}'_i}(g, \phi'_{\mu,s}) \int_{\nu \in (\mathfrak{a}^{\sigma P_i}_{\sigma P,\mathbb{C}})^*, \ \Re\nu = (\sigma\kappa)^{\sigma P_i}_{\sigma P}} (D_\mu \bullet F_{i,\sigma})(g, \mu + s_i + \nu) \, d\nu, \end{aligned}$$

where D_μ is a differential operator with constant coefficients on $(\mathfrak{a}^G_{\sigma P_i,\mathbb{C}})^*$ and $D_\mu \bullet F_{i,\sigma}$ is defined by

$$(D_\mu \bullet F_{i,\sigma})(g, \nu) e^{-\langle\nu + \rho_{\sigma P}, H_{\sigma P}(g)\rangle} = D_\mu[F_{i,\sigma}(g,\nu) e^{-\langle\nu+\rho_{\sigma P}, H_{\sigma P}(g)\rangle}].$$

Thus the inner integration vanishes by assumption, and only the zeroth term contributes. Since Lemma 4.5 allows us to interchange the inner integral with the outer integral, our proof is complete. □

For $\varphi' \in \mathscr{A}(G')$, let $\mathfrak{D} = \mathfrak{D}_{\varphi'}$ be the set of elements $\Lambda \in (\mathfrak{a}^G_P)^* \cap (\rho_P + (\mathfrak{a}^*_P)^+)$ satisfying

$$\langle \sigma\Lambda + \mathrm{pr}_w(w\lambda') + \varrho_{Q,w}, \varpi^\vee\rangle \neq 0$$

for all proper parabolic subgroups Q of G, $w \in {}_L W^G_{G'}$, $\sigma \in {}_L W^c_M$, $\lambda' \in \mathscr{E}_{Q_w}(\varphi')$ and $\varpi^\vee \in \hat\Delta^\vee_Q$. Whenever $\phi \in \mathscr{A}^c_P(G)$ and $\Re\lambda \in \mathfrak{D}$, the Eisenstein series $E(\phi, \lambda)$ converges absolutely and the regularized period $\mathbf{P}^{G'}(E(\phi,\lambda) \otimes \varphi')$ is well-defined.

Lemma 4.8. *If $\phi \in \mathscr{A}^c_P(G)$ and $\varphi' \in \mathscr{A}(G')$, then $\mathbf{P}^{G'}(E(\phi,\lambda)\otimes\varphi')$ is bounded on $\{\lambda \mid \Re\lambda \in \mathcal{D}\}$ for any compact set $\mathcal{D} \subset \mathfrak{D}$.*

Proof. Since

$$\Lambda^{T,Q}_{m,w} E(\phi,\lambda) = \sum_{\sigma \in {}_L W^c_M} \Lambda^{T,Q}_{m,w} E^Q(M(\sigma,\lambda)\phi, \sigma\lambda)$$

by (4.1), we get

$$\mathbf{P}^{G',T}_{Q,w}(E(\phi,\lambda)\otimes\varphi') = \sum_{\sigma\in {}_L W^c_M} \int^{\#}_{Q_w(F)\backslash G'(\mathbb{A})} \Lambda^{T,Q}_{m,w} E^Q(g, M(\sigma,\lambda)\phi, \sigma\lambda) \varphi'_{Q_w}(g) \tau_Q(H_Q(wg) - T) \, dg.$$

Expand φ'_{Q_w} as in (3.3). Then $\mathbf{P}^{G',T}_{Q,w}(E(\phi,\lambda)\otimes\varphi')$ is the sum over σ and j of

$$\int_{K'} \int_{L_w(F)\backslash L_w(\mathbb{A})'} \Lambda^{T,Q}_{m,w} E^Q(mk, M(\sigma,\lambda)\phi, (\sigma\lambda)^Q) \phi'_j(mk) f_j(H_{Q_w}(m), T) \, dm \, dk,$$

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where

$$f_j(H_{Q_w}(m), T) = \int^{\#}_{\mathfrak{a}_{\mathbf{L}_w} + H_{Q_w}(m)} Q'_j(X) e^{\langle (\sigma\lambda)_Q + w\lambda'_j + \varrho_{Q,w}, wX \rangle} \tau_Q(wX - T)\, dX.$$

Recall that $t_Q = \dim \mathfrak{a}_Q^G$. By the calculations in [JLR99, § II] and § 3.1, there exist a positive integer N and a polynomial $p_j(H_{Q_w}(m), T)$ on $(\mathfrak{a}^G_{Q,\mathbb{C}})^*$ of degree at most $t_Q(N-1)$ such that $f_j(H_{Q_w}(m), T)$ is of the form

$$\frac{e^{\langle \eta_Q^w(\sigma\lambda + w\lambda'_j + \varrho_{Q,w}), T\rangle} e^{\langle \kappa_{Q,w}(\sigma\lambda, \lambda'_j), H_{Q_w}(m)\rangle}}{\prod_{\varpi^\vee \in \hat{\Delta}_Q^\vee} \langle \sigma\lambda + \mathrm{pr}_w(w\lambda'_j) + \varrho_{Q,w}, \varpi^\vee \rangle^N} p_j(H_{Q_w}(m), T)(\lambda).$$

This expression is bounded for $\Re\lambda \in \mathcal{D}$. We have seen how to estimate truncated Eisenstein series in Remark 2.5, where we can take $c(\lambda)$ independently of $\Im\lambda$ for λ in the domain of absolute convergence. □

LEMMA 4.9. *Let $\phi \in \mathcal{PW}_{(P,\mathcal{V})}$ and $\varphi' \in \mathscr{A}(G')$. If $\phi(\lambda)$ vanishes to a higher order on the hyperplanes*

$$\langle \sigma\lambda + \mathrm{pr}_w(w\lambda') + \varrho_{Q,w},\ \varpi^\vee \rangle = 0$$

for all proper parabolic subgroups Q of G, $w \in {}_LW^G_{G'}$, $\sigma \in {}_LW^c_M$, $\lambda' \in \mathscr{E}_{Q_w}(\varphi')$ and $\varpi^\vee \in \hat{\Delta}^\vee_Q$, then for $\kappa \in (\mathfrak{a}^G_P)^*$ in the realm of absolute convergence of the Eisenstein series,

$$\int_{G'(F)\backslash G'(\mathbb{A})} \theta_\phi(g)\varphi'(g)\, dg = \int_{\Re\lambda = \kappa} \mathbf{P}^{G'}(E(\phi(\lambda), \lambda) \otimes \varphi')\, d\lambda.$$

Proof. The proof is nearly identical, word for word, to that of Lemma 9.1.1 in [LR03]. By definition,

$$\int_{G'(F)\backslash G'(\mathbb{A})} \theta_\phi(g)\varphi'(g)\, dg = \int_{G'(F)\backslash G'(\mathbb{A})} \int_{\Re\lambda = \kappa} E(g, \phi(\lambda), \lambda)\varphi'(g)\, d\lambda\, dg.$$

From Lemma 2.2(i), we can write this as the sum over Q and $w \in {}_LW^G_{G'}$ of

$$\int_{G'(F)\backslash G'(\mathbb{A})} \int_{\Re\lambda = \kappa} \sum_{\gamma \in Q_w(F)\backslash G'(F)} \Lambda^{T,Q}_{m,w} E(\gamma g, \phi(\lambda), \lambda)\varphi'(g)\tau_Q(H_Q(w\gamma g) - T)\, d\lambda\, dg,$$

provided that this expression converges for all Q and w. Since only finitely many γ contribute for a given g in view of Remark 2.1, we may bring the sum over $Q_w(F)\backslash G'(F)$ outside the inner integral and combine it with the outer integration to obtain

$$\int_{Q_w(F)\backslash G'(\mathbb{A})} \int_{\Re\lambda = \kappa} \Lambda^{T,Q}_{m,w} E(g, \phi(\lambda), \lambda)\varphi'(g)\tau_Q(H_Q(wg) - T)\, d\lambda\, dg.$$

Again, the convergence of the latter as an iterated integral will justify all the manipulations.

From the proof of Lemma 4.8, we conclude that the integral

$$\int_{\Re\lambda = \kappa} \int^{\#}_{Q_w(F)\backslash G'(\mathbb{A})} \Lambda^{T,Q}_{m,w} E^Q(g, M(\sigma, \lambda)\phi(\lambda), \sigma\lambda)\varphi'_{Q_w}(g)\tau_Q(H_Q(wg) - T)\, dg\, d\lambda \tag{4.10}$$

converges. Our task is to show that for each $\sigma \in {}_LW^c_M$, the integral

$$\int_{Q_w(F)\backslash G'(\mathbb{A})} \int_{\Re\lambda = \kappa} \Lambda^{T,Q}_{m,w} E^Q(g, M(\sigma, \lambda)\phi(\lambda), \sigma\lambda)\varphi'(g)\tau_Q(H_Q(wg) - T)\, d\lambda\, dg \tag{4.11}$$

converges and equals (4.10). We denote by $R^+(Z_M, G)$ the set of elements in $\mathrm{Rat}(Z_M)$ obtained by decomposing the Lie algebra of U under the adjoint action of Z_M. Let $D_\sigma \subset (\mathfrak{a}_{P,\mathbb{C}}^G)^*$ be the set of λ satisfying the following two properties:

– $\langle \Re\lambda, \alpha^\vee\rangle \gg 0$ for all $\alpha \in R^+(Z_M, G)$ such that $\sigma\alpha < 0$;
– $\Re(\sigma\lambda)^Q$ is sufficiently positive in $(\mathfrak{a}_{\sigma P}^Q)^*$.

These properties guarantee that both $M(\sigma,\lambda)\phi(\lambda)$ and $E^Q(M(\sigma,\lambda)\phi(\lambda),\sigma\lambda)$ converge absolutely for $\lambda \in D_\sigma$. Since $\sigma^{-1}\alpha > 0$ for all $\alpha \in \Delta_0^Q$, if $\Re\lambda = \kappa$, then $\lambda \in D_\sigma$.

We show that there exists $\lambda_1 \in (\mathfrak{a}_P^G)^*$ of the form $\lambda_1 = \sigma^{-1}(\lambda_2 + \lambda_3)$ with $\lambda_2 \in (\mathfrak{a}_{\sigma P}^Q)^*$ and $\lambda_3 \in (\mathfrak{a}_Q^G)^*$ such that if $\Re\lambda = \lambda_1$, then $\lambda \in D_\sigma$ and the #-integral in (4.10) and the integral

$$\int_{Q_w(F)\backslash G'(\mathbb{A})} \Lambda_{m,w}^{T,Q} E^Q(g, M(\sigma,\lambda)\phi(\lambda), \sigma\lambda)\varphi'(g)\tau_Q(H_Q(wg) - T)\, dg$$

converge absolutely. To prove this, we fix $\lambda_2 \in (\mathfrak{a}_{\sigma P}^Q)^*$ regular enough in the positive Weyl chamber. If $\alpha \in R^+(Z_M, G)$ is such that $\sigma\alpha < 0$, then $\sigma\alpha \notin (\mathfrak{a}_{\sigma P}^Q)^*$ since $\sigma \in {}_L W_M^c$. If $\lambda_3 \in (\mathfrak{a}_Q^G)^*$ is sufficiently regular in the negative Weyl chamber (depending on λ_2), then

$$\langle \Re\lambda, \alpha^\vee\rangle = \langle \sigma\Re\lambda, \sigma\alpha^\vee\rangle = \langle \lambda_2, (\sigma\alpha^\vee)^Q\rangle + \langle \lambda_3, (\sigma\alpha^\vee)_Q\rangle \gg 0,$$

and since all coweights in $\hat{\Delta}_Q^\vee$ are nonnegative linear combinations of coroots in $\Delta_Q^\vee$,

$$\Re\langle \sigma\lambda + \mathrm{pr}_\beta(\beta\lambda') + \rho_Q - \beta\rho_R, \varpi^\vee\rangle = \langle \lambda_3 + \mathrm{pr}_\beta(\beta\Re\lambda') + \rho_Q - \beta\rho_R, \varpi^\vee\rangle < 0$$

for all parabolic subgroups R of G', $\beta \in {}_0W_{G'}^G$, $\lambda' \in \mathscr{E}_R^{\mathrm{cusp}}(\varphi')$ and $\varpi^\vee \in \hat{\Delta}_Q^\vee$. The proof of Proposition 3.4 confirms that the integrals are absolutely convergent.

Since $\phi(\lambda)$ vanishes on the hyperplane singularities of the #-integral, we may shift the contour of integration in (4.10) to $\Re\lambda = \lambda_1$. The shift of contour takes place inside the domain D_σ. Thus (4.10) is equal to the absolutely convergent integral

$$\int_{\Re\lambda=\lambda_1}\int_{Q_w(F)\backslash G'(\mathbb{A})} \Lambda_{m,w}^{T,Q} E^Q(g, M(\sigma,\lambda)\phi(\lambda), \sigma\lambda)\varphi'(g)\tau_Q(H_Q(wg) - T)\, dg\, d\lambda.$$

We may therefore interchange the order of integration. We are now free to shift the contour of the inner integration back to $\Re\lambda = \kappa$ to obtain (4.11), as required, which also shows that (4.11) converges as an iterated integral. □

4.5 Regularized periods of cuspidal Eisenstein series

Lemma 4.10. *Let D be a tempered distribution on a Euclidean space V whose Fourier transform $\hat{D}$ is given by integration against a bounded function A. Suppose that $(D, f) = 0$ whenever $\hat{f}$ has a zero of order higher than m_i on each of finitely many prescribed affine hyperplanes $\lambda_i + V_{i,\mathbb{C}}$ of $V_{\mathbb{C}}^*$. Then $D = 0$.*

Proof. Take a nonzero polynomial function h which has a zero of order m_i on $\lambda_i + V_{i,\mathbb{C}}$. Since $(\hat{D}, \phi h) = 0$ by assumption, we get $(Ah, \phi) = (A, \phi h) = 0$ for all $\phi \in \mathcal{PW}(V_{\mathbb{C}}^*)$. This implies that Ah is identically zero, and hence so is A. □

We are ready to prove Lemma 4.1. Recall that $\mathbf{e} = (1, 1, \dots, 1) \in \mathfrak{a}_G^*$. Fix a point $\kappa \in \mathfrak{D}_{\varphi'}$ and choose $\sigma \in \mathbb{R}$ so that if $\lambda_0 \in \mathfrak{a}_{P,\mathbb{C}}^*$ satisfies $\Re\lambda_0 = \kappa + \sigma\mathbf{e}$, then $\mathbf{P}^{G'}(E(\phi, \lambda_0) \otimes \varphi')$ is well-defined

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and $I(\phi, \varphi', \lambda_0)$ converges absolutely. Fix such an element λ_0. We write $\lambda_0 = \lambda_1 + s\mathbf{e}$, where $\lambda_1 \in (\mathfrak{a}^G_{P,\mathbb{C}})^*$ and $s \in \mathbb{C}$. Note that

$$\mathbf{P}^{G'}(E(\phi, \lambda_0) \otimes \varphi') = \mathbf{P}^{G'}(E(\phi, \lambda_1) \otimes \varphi'_s), \quad I(\phi, \varphi', \lambda_0) = I(\phi, \varphi'_s, \lambda_1).$$

We are taking for granted the extension of Lemma 4.9 to the convolution sections $f * \phi_\lambda$ for right and left K_∞-finite functions $f \in C_c^\infty(G_\infty)$. If $\beta \in \mathcal{PW}((\mathfrak{a}^G_{P,\mathbb{C}})^*)$ satisfies the conditions of Lemmas 4.7 and 4.9, then these lemmas yield

$$\int_{\Re\lambda=\kappa} \beta(\lambda)\mathbf{P}^{G'}(E(f\star\phi, \lambda) \otimes \varphi'_s)\, d\lambda = \int_{\Re\lambda=\kappa} \beta(\lambda) I(f\star\phi, \varphi'_s, \lambda)\, d\lambda$$

for all $\phi \in \mathscr{A}^c_P(G)$ and right and left K_∞-finite decomposable functions $f \in C_c^\infty(G_\infty)$. Lemmas 4.5, 4.8 and 4.10 give rise to the equality

$$\mathbf{P}^{G'}(E(f\star\phi, \lambda_1) \otimes \varphi'_s) = I(f\star\phi, \varphi'_s, \lambda_1).$$

Theorem 1 of [Har66] gives a right and left K_∞-finite decomposable function f belonging to $C_c^\infty(G_\infty)$ such that $f * \phi_{\lambda_1} = \phi_{\lambda_1}$. It follows that

$$\mathbf{P}^{G'}(E(\phi, \lambda_0) \otimes \varphi') = I(\phi, \varphi', \lambda_0).$$

Since $\mathbf{P}^{G'}(E(\phi, \lambda) \otimes \varphi')$ possesses a meromorphic continuation to $\mathfrak{a}^*_{P,\mathbb{C}}$, so does $I(\phi, \varphi', \lambda)$. Hence the equality holds for generic values of the parameter λ, which proves Lemma 4.1.

4.6 Regularized periods of general automorphic forms

Fix P, $\phi \in \mathscr{A}^c_P(G)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_t) \in \mathfrak{a}^*_{P,\mathbb{C}}$. We can find a holomorphic function $d(\lambda)$, not identically zero, such that for every $g \in G(\mathbb{A})$ the function $d(\lambda)E(g, \phi, \lambda)$ is holomorphic at λ'. Put $F(\lambda) = d(\lambda)E(\phi, \lambda)$. Consider its Taylor expansion at $\lambda = \lambda'$,

$$F(g, \lambda) = \sum_{k=(k_1,\ldots,k_t)} F_k(g, \lambda') \prod_{i=1}^{t} (\lambda_i - \lambda'_i)^{k_i}.$$

The coefficients $F_k(\lambda')$ are automorphic forms on $G(\mathbb{A})$. Let $\mathscr{E}(G)$ be the space of automorphic forms generated by these functions as we let P, ϕ, λ', $d(\lambda)$ and k vary. Franke has demonstrated the following result for all reductive groups.

Theorem 4.11 (Franke [Fra98]). *$\mathscr{A}(G) = \mathscr{E}(G)$.*

The reader can consult [Wal97] for a survey of his work. Theorem 1.1 therefore follows from the lemma below.

Lemma 4.12. *With notation as above, we have the identity*

$$\mathbf{P}^{G'}(F_k(\lambda') \otimes \varphi'_s) = I(s, F_k(\lambda'), \varphi')$$

as a meromorphic function in s.

Proof. Fix a point $s_0 \in \mathbb{C}$ and a neighborhood Ω of λ' satisfying the following conditions:

- the integral $I(s_0, F(\lambda), \varphi')$ converges absolutely and uniformly for $\lambda \in \Omega$;
- $(F(\lambda), \varphi'_{s_0}) \in \mathscr{A}(G \times G')^*$ for $\lambda \in \Omega$.

The period $\mathbf{P}^{G'}(F_k(\lambda') \otimes \varphi'_{s_0})$ equals the zero coefficient of the polynomial exponential function

$$\int_{G'(F)\backslash G'(\mathbb{A})} \Lambda_m^T F_k(g, \lambda') \varphi'_{s_0}(g)\, dg$$

in T by Proposition 3.7(ii). We can write the coefficient $F_k(\lambda')$ as a Cauchy integral

$$F_k(\lambda') = \frac{1}{(2\pi\sqrt{-1})^t} \int_{\Gamma_1} \cdots \int_{\Gamma_t} \frac{F(\lambda)}{\prod_{i=1}^t (\lambda_i - \lambda'_i)^{k_i+1}}\, d\lambda_1 \cdots d\lambda_t,$$

where Γ_i is a sufficiently small positively oriented circle about λ'_i in the complex plane such that $\Gamma_1 \times \cdots \times \Gamma_t \subset \Omega$. This integral can be interchanged with the mixed truncation operator in view of Remark 2.1. Furthermore, we can justify the interchange of the Cauchy integral with the period integral by invoking Fubini's theorem. The arguments are the same as those introduced by Arthur in [Art82, pp. 47–48]. Therefore the integral is equal to

$$\frac{1}{(2\pi\sqrt{-1})^t} \int_{\Gamma_1} \cdots \int_{\Gamma_t} \int_{G'(F)\backslash G'(\mathbb{A})} \frac{\Lambda_m^T F(g, \lambda)}{\prod_{i=1}^t (\lambda_i - \lambda'_i)^{k_i+1}}\, \varphi'_{s_0}(g)\, dg\, d\lambda_1 \cdots d\lambda_t.$$

Proposition 3.7(ii) and Lemma 4.1 tell us that for $\lambda \in \Omega$, the zero coefficient of

$$\int_{G'(F)\backslash G'(\mathbb{A})} \Lambda_m^T F(g, \lambda)\, \varphi'_{s_0}(g)\, dg$$

is equal to

$$\mathbf{P}^{G'}(F(\lambda) \otimes \varphi'_{s_0}) = I(s_0, F(\lambda), \varphi').$$

It follows that $\mathbf{P}^{G'}(F_k(\lambda') \otimes \varphi'_{s_0})$ is equal to

$$\frac{1}{(2\pi\sqrt{-1})^t} \int_{\Gamma_1} \cdots \int_{\Gamma_t} \frac{1}{\prod_{i=1}^t (\lambda_i - \lambda'_i)^{k_i+1}} \times \int_{N'(\mathbb{A})\backslash G'(\mathbb{A})} W^{\psi}(g, F(\lambda)) W^{\bar{\psi}}(g, \varphi') |\det g|^{s_0}\, dg\, d\lambda_1 \cdots d\lambda_t.$$

The absolute convergence ensures that we can interchange the Cauchy integral with the integral over $N'(\mathbb{A})\backslash G'(\mathbb{A})$ and with the integral defining the Whittaker function. This gives the result. □

5. Odds and ends

The following corollary can be derived as a direct consequence of Theorem 1.1.

COROLLARY 5.1. *The regularized period does not depend on the choices of B, T, K and K'. Moreover, it defines a $G'(\mathbb{A})$-invariant linear functional on $\mathscr{A}(G \times G')^*$.*

Remark 5.2. One can prove Corollary 5.1 without recourse to Theorem 1.1 by using exactly the same argument as in the proof of Theorem 9(i) of [JLR99].

COROLLARY 5.3. *Let $\varphi \in \mathscr{A}(G)$ and $\varphi' \in \mathscr{A}(G')$. Define $\tilde{\varphi} \in \mathscr{A}(G)$ by $\tilde{\varphi}(g) = \varphi({}^t g^{-1})$ for $g \in G(\mathbb{A})$ and define $\tilde{\varphi}' \in \mathscr{A}(G')$ by $\tilde{\varphi}'(g') = \varphi'({}^t g'^{-1})$ for $g' \in G'(\mathbb{A})$. Then $I(s, \varphi, \varphi')$ possesses a meromorphic continuation to the whole complex plane and satisfies the functional equation*

$$I(-s, \tilde{\varphi}, \tilde{\varphi}') = I(s, \varphi, \varphi').$$

Proof. We write $\Lambda^T_{m,B}$ for the mixed truncation to indicate the dependence on the choice of B. For any parabolic subgroup P of G, put ${}^tP = \{ {}^tg \mid g \in P\}$. Observe that

$$\tilde{\varphi}_P(g) = \varphi_{{}^tP}({}^tg^{-1}), \quad H_P(x) = -H_{{}^tP}({}^tx^{-1}), \quad \hat{\tau}_P(X) = \hat{\tau}_{{}^tP}(-X).$$

It follows that

$$\Lambda^T_{m,B}\tilde{\varphi}(g) = \Lambda^{-T}_{m,{}^tB}\varphi({}^tg^{-1}).$$

We can deduce the stated identity from Proposition 3.7 and Corollary 5.1. □

For $\phi \in \mathscr{A}^c_P(G)$, $\phi' \in \mathscr{A}^c_{P'}(G')$, $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ and $\lambda' \in \mathfrak{a}^*_{P',\mathbb{C}}$, we set

$$I(\phi, \phi', \lambda, \lambda') = \int_{N'(\mathbb{A})\backslash G'(\mathbb{A})} \mathbb{W}^{\psi}(g, \phi, \lambda)\mathbb{W}^{\bar{\psi}}(g, \phi', \lambda')\, dg.$$

COROLLARY 5.4. *Let $\phi \in \mathscr{A}^c_P(G)$ and $\phi' \in \mathscr{A}^c_{P'}(G')$.*

(i) *We have*

$$\mathbf{P}^{G'}(E(\phi, \lambda) \otimes E(\phi', \lambda')) = I(\phi, \phi', \lambda, \lambda').$$

(ii) *$I(\phi, \phi', \lambda, \lambda')$ extends to a meromorphic function on $\mathfrak{a}^*_{P,\mathbb{C}} \times \mathfrak{a}^*_{P',\mathbb{C}}$.*

(iii) *For $w \in W(M)$ and $w' \in W(M')$,*

$$I(M(w, \lambda)\phi, M(w', \lambda')\phi', w\lambda, w'\lambda') = I(\phi, \phi', \lambda, \lambda').$$

Proof. Assertion (i) follows from Theorem 1.1 and (4.2). By Proposition 3.6 the stated properties of $I(\phi, \phi', \lambda, \lambda')$ are inherited from the relevant properties of the Eisenstein series. □

Let P be a parabolic subgroup of G_m. We denote by $\Pi^1_c(M)$ the set of irreducible summands of $\mathscr{A}^c(M)$. For a representation π of $M(\mathbb{A})$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, let π_λ be the representation of $M(\mathbb{A})$ on the space of π given by $\pi_\lambda(m)v = e^{\langle \lambda, H_P(m)\rangle}\pi(m)v$. Put

$$\Pi_c(M) = \{\pi_\lambda \mid \pi \in \Pi^1_c(M),\ \lambda \in \sqrt{-1}\mathfrak{a}^*_P\}.$$

For $\pi \in \Pi^1_c(M)$ we write $\mathscr{A}^\pi_P(G_m)$ for the subspace of functions $\phi \in \mathscr{A}^c_P(G_m)$ such that for all $k \in K_m$ the function $m \mapsto e^{-\langle \rho_P, H_P(m)\rangle}\phi(mk)$ belongs to the space of π. For $\pi \in \Pi^1_c(M)$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, we denote by $I^{G_m}_P(\pi_\lambda)$ the representation of $G_m(\mathbb{A})$ given by right translations on the space

$$\mathscr{A}^{\pi_\lambda}_P(G_m) = \{\phi_\lambda \mid \phi \in \mathscr{A}^\pi_P(G_m)\}.$$

The modulus function of $P(\mathbb{A})$ is built into the definition in order for the representation $I^{G_m}_P(\pi_\lambda)$ to be unitary whenever the inducing representation is unitary, which is to say, whenever λ belongs to $\sqrt{-1}\mathfrak{a}^*_P$.

When $\pi = \bigotimes_{i\in[1,t]} \pi_i \in \Pi_c(M)$ and $\phi \in \mathscr{A}^\pi_P(G_m)$, we define a normalized Eisenstein series by

$$E^*(\phi, \lambda) = b(\lambda, \pi)E(\phi, \lambda), \quad b(\lambda, \pi) = \prod_{1\leqslant i<j\leqslant t} L(\lambda_i - \lambda_j + 1, \pi_i \times \pi_j^\vee).$$

Put

$$\mathcal{C}^*_P = \{\lambda \in \mathfrak{a}^*_{P,\mathbb{C}} \mid \langle \Re\lambda, \alpha^\vee\rangle \geqslant 0 \text{ for all } \alpha^\vee \in \Delta^\vee_P\}.$$

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Proposition 5.5. *Let $\pi \in \Pi_c(M)$ and $\phi \in \mathscr{A}_P^{\pi}(G_m)$.*

(i) *$W^{\psi}(E^*(\phi,\lambda))$ is holomorphic on $\mathcal{C}_P^*$.*
(ii) *If $\Lambda \in \mathcal{C}_P^*$, then $W^{\psi}(E^*(\phi,\lambda))$ can be made nonzero at $\lambda = \Lambda$ for a suitable choice of $\phi \in \mathscr{A}_P^{\pi}(G_m)$.*
(iii) *$E^*(\phi,\lambda)$ is holomorphic on $\sqrt{-1}\mathfrak{a}_P^*$.*

Proof. We may assume that $\phi = \bigotimes_v \phi_v$ is decomposable. As in (4.8) we can decompose $\mathbb{W}^{\psi}(\phi,\lambda)$ into an Euler product

$$\mathbb{W}^{\psi}(g,\phi,\lambda) = \prod_v \mathbb{W}^{\psi_v}(g_v,\phi_v,\lambda), \quad g \in G_m(\mathbb{A}).$$

For given ϕ and g, let S be a finite set of places of F containing all the archimedean places such that for $v \notin S$, π_v is unramified, ψ_v has conductor $\mathfrak{o}_v$, ϕ_v is $K_{m,v}$-invariant, $W_M^{\psi_v}(1,\phi_v) = 1$ and $g_v \in K_{m,v}$. Then

$$W^{\psi}(E^*(\phi,\lambda)) = \prod_{v \in S} b(\lambda,\pi_v)\mathbb{W}^{\psi_v}(g_v,\phi_v,\lambda),$$

where we define $b(\lambda,\pi_v)$ by taking the local L-factors in place of the global L-functions in the definition of $b(\lambda,\pi)$. Since π_v is unitary and generic, $b(\lambda,\pi_v)$ is holomorphic on $\mathcal{C}_P^*$. The local Whittaker function $\mathbb{W}^{\psi_v}(\phi_v,\lambda)$ is known to extend to an entire function on $\mathfrak{a}_{P,\mathbb{C}}^*$ which can made nonzero at $\lambda = \Lambda$ by choosing ϕ_v to be supported in a small neighborhood modulo P_v inside $P_v w_M^{-1} N_{m,v}$.

To prove the last statement, we may suppose that $\pi \in \Pi_c^1(M)$. Put $\mathfrak{T} = \{1 \leqslant i < j \leqslant t \mid \pi_i \simeq \pi_j\}$. Since the poles of $E^*(\phi,\lambda)$ on $\sqrt{-1}\mathfrak{a}_P^*$ are among those of $b(\lambda,\pi)$ and since $E(\phi,\lambda)$ is concentrated on parabolic subgroups associated to P, it suffices to show that for all parabolic subgroups Q associated to P,

$$E_Q(\phi,\lambda) \prod_{(i,j)\in\mathfrak{T}} (\lambda_i - \lambda_j)^{-1}$$

is holomorphic on $\sqrt{-1}\mathfrak{a}_P^*$. Since $\lambda_i - \lambda_j$ $(i < j)$ are distinct prime elements in the ring of power series $\mathbb{C}[[\lambda_1,\dots,\lambda_t]]$, we need only check the holomorphy of $(\lambda_i - \lambda_j)^{-1}E_Q(\phi,\lambda)$ near the imaginary axis for all $(i,j) \in \mathfrak{T}$. Fix $(i_0,j_0) \in \mathfrak{T}$. Identifying $W(M)$ with $\mathfrak{S}_t$, we put

$$W(M,L)_0 = \{\sigma \in W(M,L) \mid \sigma(i_0) < \sigma(j_0)\}.$$

Let σ_0 be the transposition interchanging i_0 and j_0. Since $W(M,L)$ is a disjoint union of $W(M,L)_0$ and $W(M,L)_0\sigma_0$, the formula (4.1) for the constant term yields

$$E_Q(\phi,\lambda) = \sum_{\sigma \in W(M,L)_0} (M(\sigma,\lambda)\phi)_{\sigma\lambda} + \sum_{\sigma \in W(M,L)_0} (M(\sigma,\sigma_0\lambda)M(\sigma_0,\lambda)\phi)_{\sigma\sigma_0\lambda}.$$

Taking $\lim_{\lambda_{i_0}\to\lambda_{j_0}} M(\sigma_0,\lambda) = -1$ into account (see [MW89, II.1(4)]), we get the required result. □

Lemma 5.6. *If $\phi \in \mathscr{A}_P^c(G)$, $\phi' \in \mathscr{A}_{P'}^c(G')$, $\lambda \in \sqrt{-1}\mathfrak{a}_P^*$ and $\lambda' \in \sqrt{-1}\mathfrak{a}_{P'}^*$, then the pair of normalized cuspidal Eisenstein series $(E^*(\phi,\lambda), E^*(\phi',\lambda'))$ belongs to $\mathscr{A}(G \times G')^*$.*

Proof. Since the real parts of the exponents of $E(\phi,\lambda)$ and $E(\phi',\lambda')$ are zero and since $\langle \varrho_{Q,w}, \varpi^\vee\rangle \neq 0$ for all Q, $w \in {}_LW_{G'}^G$ and $\varpi^\vee \in \hat{\Delta}_Q^\vee$, the lemma follows. □

For irreducible automorphic representations $\pi \simeq \bigotimes_v \pi_v$ and $\pi' \simeq \bigotimes_v \pi'_v$ of $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively, the basic analytic properties of the tensor product L-function

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v)$$

have been established through the works [JPS83, MW89, CP04]. The infinite product converges absolutely in some right half-plane, continues to a meromorphic function on the whole complex plane and satisfies a functional equation. Let $\pi = \bigotimes_{i \in [1,t]} \pi_i \in \Pi_c(M)$ and $\pi' = \bigotimes_{j \in [1,t']} \pi'_j \in \Pi_c(M')$. Since π and π' are unitary, the induced representations $I_P^G(\pi)$ and $I_{P'}^{G'}(\pi')$ are irreducible (see [Ber84, Vog86]). Since

$$L(s, I_P^G(\pi) \times I_{P'}^{G'}(\pi')) = \prod_i \prod_j L(s, \pi_i \times \pi'_j),$$

the L-function $L(s, I_P^G(\pi) \times I_{P'}^{G'}(\pi'))$ is holomorphic away from the lines $\Re s = 0, 1$.

COROLLARY 5.7. *If $\pi \in \Pi_c(M)$ and $\pi' \in \Pi_c(M')$, then the following conditions are equivalent:*

- *there are functions $\phi \in \mathscr{A}_P^\pi(G)$ and $\phi' \in \mathscr{A}_{P'}^{\pi'}(G')$ such that $\mathbf{P}^{G'}(E^*(\phi, 0) \otimes E^*(\phi', 0)) \neq 0$;*
- *$L(1/2, I_P^G(\pi) \times I_{P'}^{G'}(\pi')) \neq 0$.*

Proof. Put $\varphi = E^*(\phi, 0)$ and $\varphi' = E^*(\phi', 0)$. Provided that ϕ and ϕ' are factorizable, we have an Euler factorization

$$I(s, \varphi, \varphi') = L\left(s + \frac{1}{2},\ I_P^G(\pi) \times I_{P'}^{G'}(\pi')\right) \prod_v \frac{I(s, W_{\varphi_v}^{\psi_v}, W_{\varphi'_v}^{\bar{\psi}_v})}{L\left(s + \frac{1}{2}, I_P^G(\pi_v) \times I_{P'}^{G'}(\pi'_v)\right)}.$$

The right-hand side is a finite product. The local L-factor coincides with the 'g.c.d.' of the local zeta integrals (see [JPS83, CP04]). That is, the ratio

$$\frac{I(s, W_{\varphi_v}^{\psi_v}, W_{\varphi'_v}^{\bar{\psi}_v})}{L\left(s + \frac{1}{2}, I_P^G(\pi_v) \times I_{P'}^{G'}(\pi'_v)\right)}$$

is not only entire for all φ_v and φ'_v but also nonzero at each fixed point $s \in \mathbb{C}$ for a suitable choice of φ_v and φ'_v, from which we can infer the corollary. $\square$

We write $\Pi_d(G_m)$ for the set of irreducible summands of the discrete spectrum of G_m.

COROLLARY 5.8. *Let $\pi \in \Pi_d(G)$, $\pi' \in \Pi_d(G')$, $\varphi \in \pi$ and $\varphi' \in \pi'$. Exclude the case where π is one-dimensional. Then the integral*

$$\int_{G'(F)\backslash G'(\mathbb{A})} \varphi(g)\varphi'(g)\, dg$$

is absolutely convergent. It is zero unless $\pi \in \Pi_c(G)$ and $\pi' \in \Pi_c(G')$.

Remark 5.9. If π is one-dimensional, then the period integral obviously diverges.

Proof. The classification of the discrete spectrum of G_m was established by Mœglin and Waldspurger in [MW89]. The representations in $\Pi_d(G_m)$ are parametrized by pairs (t, σ) where t

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divides m and $\sigma \in \Pi_c(G_{m/t})$. Let Q be the parabolic subgroup of G_m attached to the composition $(m/t, \dots, m/t)$. Put

$$\Lambda_t = \left(\frac{t-1}{2}, \frac{t-3}{2}, \dots, \frac{1-t}{2}\right).$$

The representation $I_Q^{G_m}(\sigma^{\otimes t}_{\Lambda_t})$ has a unique irreducible quotient, which belongs to $\Pi_d(G_m)$. For $\phi \in I_Q^{G_m}(\sigma^{\otimes t})$, the square-integrable automorphic form $E_{-1}(\phi)$ is defined to be the limit

$$E_{-1}(\phi) = \lim_{\lambda \to \Lambda_t} \left[E(\phi, \lambda) \prod_{i=1}^{t-1} (\lambda_i - \lambda_{i+1} - 1) \right].$$

Let $n+1 = dm$ and $n = d'm'$. Assume that $m > 1$. Let P (respectively P') be the parabolic subgroup of G (respectively of G') attached to the composition $(m, \dots, m)$ (respectively $(m', \dots, m')$). Note that $\rho_P = m\Lambda_d$ and $\rho_{P'} = m'\Lambda_{d'}$. Let $\rho \in \Pi_c(G_m)$, $\rho' \in \Pi_c(G_{m'})$, $\phi \in I_P^G(\rho^{\otimes d})$ and $\phi' \in I_{P'}^{G'}(\rho'^{\otimes d'})$. We can assume that $\varphi = E_{-1}(\phi)$ and $\varphi' = E_{-1}(\phi')$. By [Jac84], $E_{-1}(\phi)$ (respectively $E_{-1}(\phi')$) is concentrated on P (respectively on P'), and its only cuspidal exponent has real part $-\Lambda_d$ (respectively $-\Lambda_{d'}$). Put

$$e_i = (\underbrace{1, \dots, 1}_{i}, \underbrace{0, \dots, 0}_{n+1-i}), \quad i \in [1, n].$$

If $w \in {}_0W_{G'}^G$ and $\varpi_i^\vee \in \hat{\Delta}_P^\vee \cap \hat{\Delta}_{P'(w)}^\vee$, then i is divisible by m and

$$\begin{aligned} \langle -\Lambda_d - w\Lambda_{d'} + \rho_P - w\rho_{P'}, \varpi_i^\vee \rangle &= \langle (1 - m^{-1})\rho_P - (1 + m'^{-1}) w\rho_{P'}, e_i \rangle \\ &= \langle (1 - m^{-1})\rho_0 - (1 + m'^{-1}) w\rho_0', e_i \rangle \\ &\leqslant \langle (1 - m^{-1})\rho_0 - (1 + n^{-1}) w\rho_0', e_i \rangle. \end{aligned}$$

Since $m > 1$ and $m \leqslant i \leqslant n + 1 - m$,

$$(1 - m^{-1}) \sum_{j=1}^{i} (n + 2 - 2j) - (1 + n^{-1}) \sum_{j=1}^{i} (n + 1 - 2j) = -\left(\frac{1}{m} + \frac{1}{n}\right)(n + 1 - i)i + \left(1 + \frac{1}{n}\right)i < 0$$

and

$$(1 - m^{-1}) \sum_{j=1}^{i} (n + 2 - 2j) - (1 + n^{-1}) \sum_{j=1}^{i-1} (n + 1 - 2j) = \left(1 - \frac{i}{m} + \frac{1 - i}{n}\right)(n + 1 - i) < 0.$$

Thus Proposition 3.4 can be applied to prove the convergence.

At this stage, we can derive the last statement as a direct corollary of Theorem 1.1, noting that the representations occurring in the residual spectrum are not generic. From the representation-theoretic point of view we can argue as follows. The period integral

$$\varphi \otimes \varphi' \mapsto \int_{G'(F)\backslash G'(\mathbb{A})} \varphi(g)\varphi'(g) |\det g|^s \, dy$$

varies analytically in s and defines an element of the space $\mathrm{Hom}_{G'(\mathbb{A})}(\pi \otimes \pi' \otimes |\det|^s, \mathbb{C})$ for s near the imaginary axis. Note that if $d > 1$, then none of the local components of π is generic. Therefore, if $dd' > 1$, then there does not exist any such invariant functional for generic values s by [JPS83, Lemma 2.11], so that the function $s \mapsto \mathbf{P}^{G'}(\varphi \otimes \varphi'_s)$ must vanish identically. □

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