TRANSCENDENTAL ENTIRE FUNCTIONS MAPPING EVERY ALGEBRAIC NUMBER FIELD INTO ITSELF

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T. Schneider [1] has shown that a transcendental function with a limited rate of growth cannot assume algebraic values at too many algebraic points. It is not clear however whether a transcendental function may assume algebraic values at all algebraic points in an open set in its domain of analyticity. We show, using a method of B. H. Neumann and R. Rado [2] that this can be the case. Indeed we construct transcendental entire functions which, together with all their derivatives, assume, at every point in every algebraic number field, values in that field.

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Let

$$Q : q_1(z), q_2(z), \cdots$$

be a sequence of all non-constant polynomials with rational integer coefficients and let $q_n(z)$ be of degree d_n. Choose positive integers $k_1, k_2, \cdots$ such that

(1) $$k_n + d_1 + d_2 + \cdots d_n < k_{n+1}$$

and then positive rational numbers $a_1, a_2, \cdots$ such that

(2) $$a_n \max_{|z| \leq n} |z^{k_n} q_1(z) q_2(z) \cdots q_n(z)| < 2^{-n} \quad (n = 1, 2, \cdots).$$

Then the series

$$\sum_{n=1}^{\infty} a_n z^{k_n} q_1(z) q_2(z) \cdots q_n(z)$$

converges uniformly in every circle and its sum $f(z)$ has the required properties.

We note firstly that we may speak of a sequence Q of *all* non-constant polynomials with rational integer coefficients by virtue of the countability of the set of all such polynomials, and secondly that a choice of rational

numbers $a_1, a_2, \cdots$ is always possible since the polynomial

$$p_n(z) = z^{k_n} q_1(z) q_2(z) \cdots q_n(z)$$

is certainly bounded in the circle $|z| \leqq n$. $(n = 1, 2, \cdots)$. By (2) the series

$$\sum_{n=1}^{\infty} a_n z^{k_n} q_1(z) q_2(z) \cdots q_n(z) = \sum_{n=1}^{\infty} a_n p_n(z)$$

converges uniformly in every circle, whence its sum $f(z)$ is an *entire* func-
tion, and its derivatives are given by term-by-term differentiation of the
series.

In the power expansion of $f(z)$ about the origin each term $a_n p_n(z)$
contributes by (1) distinct powers of the variable, and at least the greatest
of these has non-zero coefficient. Thus the expansion has infinitely many
non-zero terms and is not the expansion of a polynomial. Thus $f(z)$ is a
transcendental entire function.

Every algebraic number ζ is a $(k+1)$-tuple zero of at least one poly-
nomial in the sequence Q (indeed of infinitely many polynomials). Thus for
every algebraic number ζ and every non-negative rational integer k there
is a positive rational integer $N = N(k, \zeta)$ such that the polynomial $q_{N+1}^{(k)}(\zeta)$
vanishes.

But then the polynomials

$$p_{N+1}^{(k)}(\zeta), \, p_{N+2}^{(k)}(\zeta), \cdots \text{ all vanish.}$$

Hence

$$f^{(k)}(\zeta) = \sum_{n=1}^{\infty} a_n p_n^{(k)}(\zeta) = \sum_{n=1}^{N} a_n p_n^{(k)}(\zeta).$$

But $\sum_{n=1}^{N} a_n p_n^{(k)}(z)$ is a polynomial with rational coefficients whence $f^{(k)}(\zeta)$
is an element of the field $R(\zeta)$ generated by ζ and the field R of rational
numbers.

Thus $f(z)$ and all its derivatives $f^{(k)}(z)$ $(k = 1, 2, \cdots)$ are transcendental
entire functions mapping every algebraic number field $K = R(\zeta)$ into
itself.

References

[1] T. Schneider, *Mathematische Annalen* 121, (1949), 131—140.
[2] B. H. Neumann and R. Rado ,'Monotone functions mapping the set of Rational Numbers
    onto itself', *J. Aust. Math. Soc.* 3 (1963), 281—287.

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