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First Countable Continua and Proper Forcing

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Abstract. Assuming the Continuum Hypothesis, there is a compact, first countable, connected space of weight $\aleph_1$ with no totally disconnected perfect subsets. Each such space, however, may be destroyed by some proper forcing order which does not add reals.

1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff. The following terms are defined as in [12].

Definition 1.1 A space X is *weird* if and only if X is compact and not scattered, and no perfect subset of X is totally disconnected. A subset P of X is *perfect* if and only if P is closed and has no isolated points. As usual, $\mathfrak{c}$ denotes the (von Neumann) cardinal $2^{\aleph_0}$.

Big weird spaces (of size $2^{\mathfrak{c}}$) were produced from CH in Fedorchuk, Ivanov, and van Mill [10]. Small weird spaces (of size $\aleph_1$) were constructed from $\diamondsuit$ in [12], which proved the following.

Theorem 1.2 *Assuming $\diamondsuit$, there is a connected weird space which is hereditarily separable and hereditarily Lindelöf.*

The weird spaces of [10, 12], and the earlier Fedorchuk [9] are all separable spaces of weight $\aleph_1$. Our $\diamondsuit$ example is also first countable because it is compact and hereditarily Lindelöf. In contrast, the CH weird spaces of [9, 10] have no convergent ω-sequences. We do not know whether CH can replace $\diamondsuit$ in Theorem 1.2, but weakening hereditarily Lindelöf to first countable, we do get the following.

Theorem 1.3 *Assuming CH, there is a separable first countable connected weird space of weight $\aleph_1$.*

This theorem cannot be proved by a classical CH construction. Classical CH arguments build the item of interest directly from an enumeration in type ω_1 of some natural set of size $\mathfrak{c}$ (*e.g.*, $\mathbb{R}$, $\mathbb{R}^{<\omega_1}$, etc.). The result, then, is preserved by any forcing

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that does not add reals. These arguments include any CH proof found in Sierpiński's text [15] as well as most CH proofs in the current literature, including the constructions of the big weird spaces of [9, 10]. In contrast, every space satisfying Theorem 1.3 is destroyed by some proper forcing order which does not add reals.

Our proof of Theorem 1.3 uses classical CH arguments to make X weird, but then, to make X first countable, we adapt the method of Gregory [11] and Devlin and Shelah [2]. The methods of [11] and [2] are, as Hellsten, Hyttinen, and Shelah [13] pointed out, essentially the same. We review the method in Section 2, and use it to prove Theorem 1.3 in Section 4. Although [11] and [2] derive results from $2^{\aleph_0} < 2^{\aleph_1}$, for Theorem 1.3, we need CH. Section 5 explains why.

In Section 3, we show that each space satisfying Theorem 1.3 can be destroyed by a proper forcing which does not add reals. In $V[G]$, we add a point of uncountable character. More precisely, if X is a compactum in V, then in each generic extension $V[G]$, we still have the same set X with the natural topology obtained by using the open sets from V as a base. If X is first countable in V, then it must remain first countable in $V[G]$, but X need not be compact in $V[G]$. We get the point of uncountable character in the natural corresponding *compact* space $\widetilde{X}$ in $V[G]$. This compact space determined by X was described by Bandlow [1] (and later in [3, 4, 6]), and can be defined as follows.

Definition 1.4 If X is a compactum in V and $V[G]$ is a forcing extension of V, then in $V[G]$ the corresponding compactum $\widetilde{X}$ is characterized by:

(1) X is dense in $\widetilde{X}$.
(2) Every $f \in C(X, [0, 1]) \cap V$ extends to an $\widetilde{f} \in C(\widetilde{X}, [0, 1])$ in $V[G]$.
(3) The functions $\widetilde{f}$ (for $f \in V$) separate the points of $\widetilde{X}$.

In forcing, $\overset{\diamond}{X}$ denotes the $\widetilde{X}$ of $V[G]$, while $\check{X}$ denotes the X of $V[G]$.

For example, if X is the $[0, 1]$ of V, then $\widetilde{X}$ will be the unit interval of $V[G]$; note that in statement (2), asserted in $V[G]$, the "$[0, 1]$" really refers to the unit interval of $V[G]$. If in V we have $X \subseteq [0, 1]^\kappa$, then $\widetilde{X}$ is simply the closure of X in the $[0, 1]^\kappa$ of $V[G]$. If in V X is the Stone space of a boolean algebra $\mathcal{B}$, then $\widetilde{X}$ will be the Stone space, computed in $V[G]$, of the same $\mathcal{B}$. In general, the weights of X and $\widetilde{X}$ will be the same (assuming that cardinals are not collapsed), but their characters need not be.

Following Eisworth and Roitman [7, 8], we call a partial order $\mathbb{P}$ *totally proper* if and only if $\mathbb{P}$ is proper and forcing with it does not add reals.

Theorem 1.5 *If X is compact, connected, and infinite, and X does not have a Cantor subset, then for some totally proper $\mathbb{P}$: $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\overset{\diamond}{X}$ is not first countable".*

The proof is in Section 3. Observe the importance of connectivity here. Suppose in V that X is the double arrow space, obtained from $[0, 1]$ by doubling the points of $(0, 1)$. Then in any $V[G]$, $\widetilde{X}$ is the compactum obtained from $[0, 1]$ by doubling the points of $(0, 1) \cap V$, and is hence first countable.

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2 Predictors

In the following, λ^{ω_α} denotes the set of functions from ω_α into λ. Something like the next definition and theorem are implicit in [11] and [2].

Definition 2.1 Let κ, λ be any cardinals and $\Psi\colon \kappa^{<\omega_1} \to \lambda$. If $f \in \kappa^{\omega_1}$, $g \in \lambda^{\omega_1}$, and $C \subseteq \omega_1$, then Ψ, f *predict* g *on* C if and only if $g(\xi) = \Psi(f\restriction\xi)$ for all $\xi \in C$. Ψ is a (κ, λ)*–predictor* if and only if for all $g \in \lambda^{\omega_1}$ there is an $f \in \kappa^{\omega_1}$ and a club C such that Ψ, f predict g on C.

Theorem 2.2 *The following are equivalent whenever* $2 \le \kappa \le \mathfrak{c}$ *and* $2 \le \lambda \le \mathfrak{c}$*:*

(1) *There is a* (κ, λ)*-predictor.*
(2) *There is a* $(\mathfrak{c}, \mathfrak{c})$*-predictor.*
(3) $2^{\aleph_0} = 2^{\aleph_1}$.

Proof (3) $\Rightarrow$ (1): Let $C = \omega_1 \setminus \omega$. List λ^{ω_1} as $\{g_\alpha : \alpha < \mathfrak{c}\}$ and choose $f_\alpha \in \kappa^{\omega_1}$ so that the $f_\alpha\restriction\omega$, for $\alpha < \mathfrak{c}$, are all distinct. Then we can define $\Psi\colon \kappa^{<\omega_1} \to \lambda$ so that $\Psi(f_\alpha\restriction\xi) = g_\alpha(\xi)$ for all $\xi \in C$.

(1) $\Rightarrow$ (2): Fix a (κ, λ)–predictor $\Psi\colon \kappa^{<\omega_1} \to \lambda$. We shall define $\Phi\colon (\kappa^\omega)^{<\omega_1} \to (\lambda^\omega)$ so that it is a $(\kappa^\omega, \lambda^\omega)$–predictor in the sense of Definition 2.1. For $p \in (\kappa^\omega)^\xi$ and $n \in \omega$, define $p_{(n)} \in \kappa^\xi$ by: $p_{(n)}(\mu) = (p(\mu))(n) \in \kappa$. Then for $p \in (\kappa^\omega)^{<\omega_1}$, define $\Phi(p) = \langle \Psi(p_{(n)}) : n \in \omega\rangle \in \lambda^\omega$.

(2) $\Rightarrow$ (3): Fix a $(\mathfrak{c}, \mathfrak{c})$–predictor $\Psi\colon \mathfrak{c}^{<\omega_1} \to \mathfrak{c}$. Let $\Gamma\colon \mathfrak{c}^{<\omega_1} \times \mathfrak{c}^{<\omega_1} \to \mathfrak{c}$ be any 1-1 function. If $K \subseteq \omega_1$ is unbounded and $\xi < \omega_1$, let $\mathrm{next}(\xi, K)$ be the least element of K which is greater than ξ.

For each $B \in \mathfrak{c}^{\omega_1}$, choose $G(n, B), F(n, B) \in \mathfrak{c}^{\omega_1}$ and clubs $C(n, B) \subseteq \omega_1$ for $n \in \omega$ as follows: Let $G(0, B) = B$. Given $G(n, B)$, let $C(n, B)$ be a club of limit ordinals and let $F(n, B) \in \mathfrak{c}^{\omega_1}$ be such that $(G(n, B))(\xi) = \Psi((F(n, B)) \restriction \xi)$ for all $\xi \in C(n, B)$. Then define $G(n + 1, B)$ so that

$$(G(n+1, B))(\xi) = \Gamma\big(F(n, B) \restriction \mathrm{next}(\xi, C(n, B)),\ G(n, B) \restriction \mathrm{next}(\xi, C(n, B))\big)$$

for each ξ.

Now, fix $B, B' \in \mathfrak{c}^{\omega_1}$, and consider the statement:

$$(\text{☆}(\xi)) \qquad \forall n \in \omega\big[G(n, B)\restriction\xi = G(n, B')\restriction\xi\big]$$

So, ☆(0) is true trivially, and ☆(ξ) implies ☆(ζ) whenever $\zeta < \xi$. We shall prove inductively that ☆(1) implies ☆(η) for all $\eta < \omega_1$. If we do this, then ☆(1) will imply $B = B'$, so we shall have $2^{\aleph_0} = 2^{\aleph_1}$, since there are $2^{\aleph_1}$ possible values for B but only $2^{\aleph_0}$ possible values for the sequence $\langle (G(n, B))(0) : n \in \omega\rangle$.

The induction is trivial at limits, so it is sufficient to fix η with $1 \le \eta < \omega_1$, assume ☆(η), and prove ☆$(\eta + 1)$, that is, prove $(G(n, B))(\eta) = (G(n, B'))(\eta)$ for all n. Fix n. For $\xi < \eta$, we have $(G(n + 1, B))(\xi) = (G(n + 1, B'))(\xi)$, which implies:

(a) $\mathrm{next}(\xi, C(n, B)) = \mathrm{next}(\xi, C(n, B'))$; call this γ_ξ.
(b) $F(n, B)\restriction\gamma_\xi = F(n, B')\restriction\gamma_\xi$.
(c) $G(n, B)\restriction\gamma_\xi = G(n, B')\restriction\gamma_\xi$.

Applying (a) for all $\xi < \eta$: $\eta \in C(n,B)$ if and only if $\eta \in C(n,B')$. If $\eta \notin C(n,B), C(n,B')$, then fix ξ with with $\xi < \eta < \gamma_\xi$; now (c) implies $(G(n,B))(\eta) = (G(n,B'))(\eta)$. If $\eta \in C(n,B), C(n,B')$, then η is a limit ordinal and (b) implies $F(n,B){\restriction}\eta = F(n,B'){\restriction}\eta$; now $(G(n,B))(\eta) = (G(n,B'))(\eta) = \Psi((F(n,B)) \restriction \eta)$. ■

The non-existence of a $(2,2)$-predictor is the weak version of $\diamondsuit$ discussed by Devlin and Shelah in [2], where they use it to prove that, assuming $2^{\aleph_0} < 2^{\aleph_1}$, every ladder system on ω_1 has a non-uniformizable coloring. By Shelah [14, p. 196], each such coloring may be uniformized in some totally proper forcing extension.

A direct proof of (3) $\Rightarrow$ (2), resembling the above proof of (3) $\Rightarrow$ (1), would obtain C fixed at $\omega_1 \setminus \{0\}$, since one may choose the f_α so that the $f_\alpha(0)$, for $\alpha < \mathfrak{c}$, are all distinct. Gregory [11] used the failure of (2), with this specific C, to derive a result about trees under $2^{\aleph_0} < 2^{\aleph_1}$; see Theorem 3.14 below.

3 Some Totally Proper Orders

We consider *forcing posets*, $(\mathbb{P}; \leq, \mathbb{1})$, where $\leq$ is a transitive and reflexive relation on $\mathbb{P}$ and $\mathbb{1}$ is a largest element of $\mathbb{P}$. As usual, if $p, q \in \mathbb{P}$, then $p \not\perp q$ means that p, q are compatible (that is, have a common extension), and $p \perp q$ means that p, q are incompatible.

Definition 3.1 Assume that X is compact, connected, and infinite. Let $\mathbb{K} = \mathbb{K}_X$ be the forcing poset consisting of all closed, connected, infinite subsets of X, with $p \leq q$ if and only if $p \subseteq q$ and $\mathbb{1}_\mathbb{K} = X$. In $\mathbb{K}$, define $p \perp\!\!\!\perp q$ if and only if $p \cap q = \varnothing$.

Note that $p \perp q$ if and only if $p \cap q$ is totally disconnected. The stronger relation $p \perp\!\!\!\perp q$ will be useful in the proof that $\mathbb{K}$ is totally proper whenever X does not have a Cantor subset. First, we verify that $\mathbb{K}$ is separative; this follows easily from the following lemma, which is probably well-known; a proof is in [12].

Lemma 3.2 *If P is compact, connected, and infinite, and $U \subseteq P$ is a nonempty open set, then there is a closed $R \subseteq U$ such that R is connected and infinite.*

In particular, in $\mathbb{K}$, if $p \not\leq q$, then we may apply this lemma with $U = p \setminus q$ to get $r \leq p$ with $r \perp q$, proving the following.

Corollary 3.3 *If X is compact, connected, and infinite, then $\mathbb{K}_X$ is separative and atomless.*

We collect some useful properties of the relation $\perp\!\!\!\perp$ on $\mathbb{K}$ in the following:

Definition 3.4 A binary relation $⫳$ on a forcing poset is a *strong incompatibility relation* if and only if

(1) $p ⫳ q$ implies $p \perp q$.
(2) Whenever $p \perp q$, there are p_1, q_1 with $p_1 \leq p, q_1 \leq q$, and $p_1 ⫳ q_1$.
(3) $p ⫳ q \;\&\; p_1 \leq p \;\&\; q_1 \leq q \;\to\; p_1 ⫳ q_1$.

This definition does not require $⫳$ to be symmetric, but note that the relation $p ⫳ q \;\&\; q ⫳ p$ is symmetric and is also a strong incompatibility relation.

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Lemma 3.5 *The relation $\perp\!\!\!\perp$ is a strong incompatibility relation on $\mathbb{K}_X$.*

Proof Conditions (1) and (3) are obvious. For (2), suppose that $p \perp q$. Let $F = p \cap q$, which is totally disconnected. Then by Lemma 3.2 there is an infinite connected $p_1 \subseteq p \backslash F$. Likewise, we get $q_1 \subseteq q \backslash F$. ■

Definition 3.6 If $\mathbb{P}$ is a forcing poset with a strong incompatibility relation $\not\wr$, then a *strong Cantor tree* in $\mathbb{P}$ (with respect to $\not\wr$) is a subset $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P}$ such that each $p_{s^\frown\mu} < p_s$ for $\mu = 0, 1$, and each $p_{s^\frown 0} \not\wr p_{s^\frown 1}$. Then $\mathbb{P}$ has the *weak Cantor tree property (WCTP)* (with respect to $\not\wr$) if and only if whenever $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P}$ is a strong Cantor tree, there is at least one $f \in 2^\omega$ such that $\mathbb{P}$ contains some $q = q_f$ with $q \leq p_{f\upharpoonright n}$ for each $n \in \omega$.

Note that if $\mathbb{P}$ has the WCTP, then the set of f for which q_f is defined must meet every perfect subset of the Cantor set 2^ω, since otherwise we could find a subtree of the given Cantor tree which contradicts the WCTP.

Lemma 3.7 *If X is compact, connected, and infinite, and X does not have a Cantor subset, then $\mathbb{K}_X$ has the WCTP.*

Definition 3.8 $\mathbb{P}$ has the *Cantor tree property (CTP)* if and only if $\mathbb{P}$ has the WCTP with respect to the usual $\perp$ relation.

$\mathbb{K}_X$ need not have the CTP (see Theorem 5.4). A countably closed $\mathbb{P}$ clearly has the CTP. In the case of trees, the CTP was also discussed in [13] (where it was called "$\aleph_0$ fan closed") and in [12]. The following modifies [13, Lemma 3] and [12, Lemma 5.5]:

Lemma 3.9 *If $\mathbb{P}$ has the WCTP, then $\mathbb{P}$ is totally proper.*

Proof Define $q \leq' p$ if and only if there is no r such that $r \leq q$ and $r \perp p$. When $\mathbb{P}$ is separative, this is equivalent to $q \leq p$.

Fix a suitably large regular cardinal θ, and let $M \prec H(\theta)$ be countable with $(\mathbb{P}; \leq, \mathbb{1}, \not\wr) \in M$, and fix $p \in \mathbb{P} \cap M$. It suffices (see [8]) to find a $q \leq p$ such that whenever $A \subseteq \mathbb{P}$ is a maximal antichain and $A \in M$, there is an $r \in A \cap M$ with $q \leq' r$. If $\mathbb{P}$ has an atom $q \leq p$ such that $q \in M$, then we are done. Otherwise, since $M \prec H(\theta)$, $\mathbb{P}$ must be atomless below p. Let $\{A_n : n \in \omega\}$ list all the maximal antichains which are in M. Build a strong Cantor tree $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P} \cap M$ such that, $p_{()} \leq p$, and such that, when $n \in \omega$ and $s \in 2^n$, p_s extends some element of $A_n \cap M$. Then choose $f \in 2^\omega$ such that there is some $q \in \mathbb{P}$ with $q \leq p_{f\upharpoonright n}$ for each $n \in \omega$. ■

Proof of Theorem 1.5 Let $\mathbb{P} = \mathbb{K}_X$. Working in $V[G]$, let $G' = \{\widetilde{p} : p \in G\}$; then $\bigcap G' = \{y\}$ for some $y \in \widetilde{X} \setminus X$. Since $\mathbb{P}$ does not add ω–sequences, $\bigcap E \supsetneqq \{y\}$ whenever E is a countable subset of G'. Thus, $\chi(y, \widetilde{X})$ is uncountable. ■

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These totally proper partial orders yield natural weakenings of PFA:

Definition 3.10 If $\mathfrak{P}$ is a class of forcing posets, then $\mathrm{MA}_{\mathfrak{P}}(\aleph_1)$ is the statement that whenever $\mathbb{P} \in \mathfrak{P}$ and $\mathcal{D}$ is a family of $\leq \aleph_1$ dense subsets of $\mathbb{P}$, then there is a filter on $\mathbb{P}$ meeting each $D \in \mathcal{D}$.

Trivially, PFA $\Rightarrow$ $\mathrm{MA}_{\mathrm{WCTP}}(\aleph_1)$ $\Rightarrow$ $\mathrm{MA}_{\mathrm{CTP}}(\aleph_1)$, but in fact $\mathrm{MA}_{\mathrm{WCTP}}(\aleph_1) \Leftrightarrow \mathrm{MA}_{\mathrm{CTP}}(\aleph_1)$ (see Lemma 3.13). Also, $\mathrm{MA}_{\mathrm{CTP}}(\aleph_1) \Rightarrow 2^{\aleph_0} = 2^{\aleph_1}$ (see Corollary 3.15), so, the natural iteration of (totally proper) CTP orders with countable supports must introduce reals at limit stages. By the proof of Theorem 5.9 in [12], PFA does not follow from $\mathrm{MA}_{\mathrm{CTP}}(\aleph_1) + \mathrm{MA}(\aleph_1) + 2^{\aleph_0} = \aleph_2$, which in fact can be obtained by ccc forcing over L.

We now consider some CTP trees.

Definition 3.11 Order $\lambda^{<\omega_1}$ by: $p \leq q$ if and only if $p \supseteq q$. Let $\mathbb{1} = \varnothing$, the empty sequence.

So, $\lambda^{<\omega_1}$ is a tree, with the root $\mathbb{1}$ at the top. Viewed as a forcing order, it is equivalent to countable partial functions from ω_1 to λ. We often view $p \in \lambda^{<\omega_1}$ as a countable sequence and let $\mathrm{lh}(p) = \mathrm{dom}(p)$. Then $\mathrm{lh}(\mathbb{1}) = 0$.

Kurepa showed that SH is equivalent to the non-existence of Suslin trees. A similar proof shows that $\mathrm{MA}_{\mathrm{CTP}}(\aleph_1)$ is equivalent to the non-existence of Gregory trees:

Definition 3.12 A *Gregory tree* is a forcing poset $\mathbb{P}$ which is a subtree of $\mathfrak{c}^{<\omega_1}$ and satisfies:

(1) $\mathbb{P}$ has the CTP.
(2) $\mathbb{P}$ is atomless.
(3) $\mathbb{P}$ has no uncountable chains.

It is easily seen that if any of conditions (1), (2), or (3) are dropped, such trees may be constructed in ZFC. However, we have the following.

Lemma 3.13 *The following are equivalent:*

(1) $\mathrm{MA}_{\mathrm{CTP}}(\aleph_1)$.
(2) $\mathrm{MA}_{\mathrm{WCTP}}(\aleph_1)$.
(3) *There are no Gregory trees.*

Proof (1) $\Rightarrow$ (3): Let $\mathbb{P}$ be a Gregory tree. As with Suslin trees under $\mathrm{MA}(\aleph_1)$, a filter G meeting the sets $D_\xi := \{p \in \mathbb{P} : \mathrm{lh}(p) \geq \xi\}$ yields an uncountable chain, and hence a contradiction, but to apply $\mathrm{MA}_{\mathrm{CTP}}(\aleph_1)$, we must prove that each D_ξ is dense in $\mathbb{P}$. To do this, induct on ξ. The case $\xi = 0$ is trivial. For the successor stages, use the fact that $\mathbb{P}$ is atomless. For the limit stages, use the CTP.

(3) $\Rightarrow$ (2): Fix $\mathbb{P}$ with the WCTP and dense sets $D_\xi \subseteq \mathbb{P}$ for $\xi < \omega_1$. We need to produce a filter $G \subseteq \mathbb{P}$ meeting each D_ξ. This is trivial if $\mathbb{P}$ has an atom, so assume that $\mathbb{P}$ is atomless.

Inductively define a subtree T of $2^{<\omega_1}$ together with a function $F\colon T \to \mathbb{P}$ as follows: $F(\mathbb{1}) = \mathbb{1}_{\mathbb{P}}$. If $t \in T$ and $\mathrm{lh}(t) = \xi$, then $t^\frown 0 \in T$ and $t^\frown 1 \in T$, and $F(t^\frown 0)$,

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$F(t^\frown 1)$ are extensions of $F(t)$ such that each $F(t^\frown i) \in D_\xi$ and $F(t^\frown 0) \not\parallel F(t^\frown 1)$; to accomplish this, given t and $F(t)$: first choose two $\perp$ extensions of $F(t)$, then extend these to be $\not\parallel$, and then extend these to be in D_ξ. If $\eta < \omega_1$ is a limit ordinal and $\mathrm{lh}(t) = \eta$, then $t \in T$ if and only if $\forall \xi < \eta\, [t{\restriction}\xi \in T]$ and $\exists q \in \mathbb{P}\, \forall \xi < \eta\, [q \leq F(t{\restriction}\xi)]$; then choose $F(t)$ to be some such q.

T is clearly atomless, and T has the CTP because $\mathbb{P}$ has the WCTP. If there are no Gregory trees, then T has an uncountable chain, so fix $g \in 2^{\omega_1}$ such that $g{\restriction}\xi \in T$ for all $\xi < \omega_1$, and let $G = \{y \in \mathbb{P} : \exists \xi < \omega_1\, [F(g{\restriction}\xi) \leq y]\}$. ■

Theorem 3.14 **(Gregory [11])** *If* $2^{\aleph_0} < 2^{\aleph_1}$, *then there is a Gregory tree.*

Corollary 3.15 $\mathrm{MA}_{\mathrm{CTP}}(\aleph_1)$ *implies that* $2^{\aleph_0} = 2^{\aleph_1}$.

4 A Weird Space

We now prove Theorem 1.3. The basic construction is an inverse limit in ω_1 steps, and we follow approximately the terminology in [5, 12]. We build a compact space $X_{\omega_1} \subseteq [0,1]^{\omega_1}$ by constructing inductively $X_\alpha \subseteq [0,1]^{1+\alpha} \cong [0,1] \times [0,1]^\alpha$. Usually, one has $X_\alpha \subseteq [0,1]^\alpha$ in these constructions, but for finite α, the notation will be slightly simpler if we start at stage 0 with $X_0 = [0,1] = [0,1]^1$; of course, $1 + \alpha = \alpha$ for infinite α.

Definition 4.1 $\pi^\beta_\alpha : [0,1]^{1+\beta} \twoheadrightarrow [0,1]^{1+\alpha}$ is the natural projection.

As usual, $\pi \colon X \twoheadrightarrow Y$ means that π is a *continuous* map from X *onto* Y. These constructions always have $\pi^\beta_\alpha(X_\beta) = X_\alpha$ whenever $0 \leq \alpha \leq \beta \leq \omega_1$. This determines X_γ for limit γ, so the meat of the construction involves describing how to build $X_{\alpha+1}$ given X_α.

A classical CH argument can ensure that X_{ω_1} is weird, but by Theorem 1.5, such an argument cannot make X_{ω_1} first countable. However, the same classical argument will let us construct a binary tree of spaces, resulting in a weird space $X_g \subseteq [0,1]^{\omega_1}$ for each $g \in 2^{\omega_1}$. We shall show that if no X_g were first countable, then there would be a $(\mathfrak{c}, 2)$ – predictor $\Psi \colon [0,1]^{<\omega_1} \to 2$; so CH ensures that some X_g is first countable.

Our tree will give us an X_p for each $p \in 2^{\leq\omega_1}$. We now list requirements (R1), (R2), (R3), …, (R17) on the construction; a proof that all the requirements can be satisfied, and that they yield a weird space, concludes this section. We begin with the requirements involving the inverse limit:

(R1) $X_{\mathbb{1}} = [0,1]$, where $\mathbb{1}$ is the empty sequence.

(R2) X_p is an infinite closed connected subspace of $[0,1]^{1+\mathrm{lh}(p)}$.

(R3) $\pi^\beta_\alpha{\restriction}X_p \colon X_p \twoheadrightarrow X_{p{\restriction}\alpha}$, and is irreducible, whenever $\beta = \mathrm{lh}(p) \geq \alpha$.

When $\gamma = \mathrm{lh}(p) \leq \omega_1$ is a limit, (R2) and (R3) force

$$(\circledast) \qquad X_p = \{x \in [0,1]^\gamma : \forall \alpha < \gamma\, [\pi^\gamma_\alpha(x) \in X_{p{\restriction}\alpha}]\}.$$

To simplify notation for the restricted projection maps, we shall use the following definition.

Definition 4.2 If $\beta = \mathrm{lh}(p) \geq \alpha$ and $r = p{\restriction}\alpha$, define $\pi^p_r = \pi^\beta_\alpha{\restriction}X_p \colon X_p \twoheadrightarrow X_r$.

As in [12], each of $X_{p^\frown 0}$ and $X_{p^\frown 1}$ is obtained from X_p as the graph of a "$\sin(1/x)$" curve. We choose h_q, u_q, and v^n_q for $n < \omega$ and $q \in 2^{<\omega_1}$ of successor length, satisfying, for $i = 0, 1$:

(R4) $u_{p^\frown i} \in X_p$ and $h_{p^\frown i} \in C(X_p \backslash \{u_{p^\frown i}\},\ [0,1])$ and $X_{p^\frown i} = \overline{h_{p^\frown i}}$.
(R5) $v^n_{p^\frown i} \in X_p \backslash \{u_{p^\frown i}\}$, and $\langle v^n_{p^\frown i} : n \in \omega\rangle \to u_{p^\frown i}$, and all points of $[0,1]$ are limit points of $\langle h_{p^\frown i}(v^n_{p^\frown i}) : n \in \omega\rangle$.

As usual, we identify $h_{p^\frown i}$ with its graph. So, if $\alpha = \mathrm{lh}(p)$, then $X_{p^\frown i}$ is a subset of $[0,1]^{1+\alpha} \times [0,1]$, which we identify with $[0,1]^{1+\alpha+1}$. We shall say that the point $u_{p^\frown i}$ gets *expanded* in the passage from X_p to $X_{p^\frown i}$; the other points get *fixed*. (R3) follows from (R4) plus (❁). Also, if $\delta < \alpha$, then $\pi^p_{p\restriction\delta} \colon X_p \twoheadrightarrow X_{p\restriction\delta}$, and $(\pi^p_{p\restriction\delta})^{-1}\{x\}$ is a singleton unless x is in the countable set

$$\{\pi^{p\restriction\xi}_{p\restriction\delta}(u_{p\restriction(\xi+1)}) \colon \delta \leq \xi < \alpha\}.$$

We now explain how points in $X_g \subset [0,1]^{\omega_1}$ can predict g, in the sense of Definition 2.1. We shall get A_q and B_q for $q \in 2^{<\omega_1}$ of successor length, satisfying the following:

(R6) For $i = 0, 1$: $A_{p^\frown i}, B_{p^\frown i} \subseteq X_p$ and $A_{p^\frown i} = X_p \setminus B_{p^\frown i}$.
(R7) For $i = 0, 1$ and $\xi < \mathrm{lh}(p)$: $A_{p^\frown i} \supseteq (\pi^p_{p\restriction\xi})^{-1}(A_{p\restriction(\xi+1)})$.
(R8) $B_{p^\frown 0} \cap B_{p^\frown 1} = \varnothing$.
(R9) For $i = 0, 1$: $u_{p^\frown i} \in B_{p^\frown i}$.

Observe that some care must be exercised here in the inductive construction; otherwise, at some stage (R7) might imply that $A_{p^\frown i} = X_p$, so that $B_{p^\frown i} = \varnothing$, making (R9) impossible.

Requirements (R6),(R7), and (R9) imply that points in $A_{p^\frown i}$ are forever fixed in the passage from X_p to any future X_q with $q \leq p^\frown i$; only points in $B_{p^\frown i}$ can get expanded. Points which are forever fixed must wind up having countable character, and (R8) lets us use a point of uncountable character in X_g to predict g.

Lemma 4.3 *Assume that we have* (R1)–(R9)*, and assume that* $2^{\aleph_0} < 2^{\aleph_1}$. *Then* X_g *is first countable for some* $g \in 2^{\omega_1}$.

Proof We shall define $\Psi \colon [0,1]^{<\omega_1} \to 2$, and prove that Ψ is a $(\mathfrak{c}, 2)$-predictor if every X_g contains a point of uncountable character.

Say $\mathrm{lh}(p) = \alpha < \omega_1$ and $\delta < \alpha$. If $x \in B_{p^\frown i} \subseteq X_p$, then by (R6) and (R7), $\pi^\alpha_\delta(x) \in B_{p\restriction(\delta+1)} \subseteq X_{p\restriction\delta}$. Applying (R8), if $x \in [0,1]^{1+\alpha}$ and $x \in B_{p^\frown i} \cap B_{r^\frown j}$, then $p = r$ and $i = j$; to prove this, consider the least $\delta < \alpha$ such that $p(\delta) \neq r(\delta)$.

Set $\Psi(x) = 0$ if $\mathrm{lh}(x) < \omega$. Now, say $x \in [0,1]^\alpha$, where $\omega \leq \alpha < \omega_1$ (so $1 + \alpha = \alpha$). If there exist $p \in 2^\alpha$ and $i \in 2$ such that $x \in B_{p^\frown i}$, then these p, i are unique, and set $\Psi(x) = i$. If there are no such p, i, then set $\Psi(x) = 0$.

Now, assume that for each g, we can find $z = z_g \in X_g$ with $\chi(z, X_g) = \aleph_1$. Let $C = \omega_1 \setminus \omega$. We shall show that Ψ, z predict g on C. For $\xi \in C$, let $p^\frown i = g{\restriction}(\xi + 1)$. Then $z{\restriction}\xi = \pi^g_p(z) \in X_p$, and $z{\restriction}\xi$ must be in $B_{p^\frown i}$, since if it were in $A_{p^\frown i}$, then $(\pi^g_p)^{-1}(\pi^g_p(z)) = \{z\}$, so that $\chi(z, X_g) = \aleph_0$. Thus, $\Psi(z{\restriction}\xi) = i = g(\xi)$. ∎

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Since every X_g clearly has weight $\aleph_1$, we are done if we can make every X_g weird. Since points in $A_{p^\frown i}$ are forever fixed, we must make sure that $A_{p^\frown i}$ has no Cantor subsets. Conditions (R6) and (R8) say that $A_{p^\frown 0} \cup A_{p^\frown 1} = X_p$, so $A_{p^\frown 0}$ and $A_{p^\frown 1}$ must be Bernstein sets. Note that Condition (R7) may present a problem at limit stages. When $\mathrm{lh}(p) = \alpha$, we have $A_{p^\frown i} \supseteq \bigcup_{\xi<\alpha} (\pi^p_{p\restriction\xi})^{-1}(A_{p\restriction(\xi+1)})$. Points in $A_{p\restriction(\xi+1)}$ are forever fixed, so each $(\pi^p_{p\restriction\xi})^{-1}(A_{p\restriction(\xi+1)})$ will have no Cantor subsets. Without further requirements, though, $\bigcup_{\xi<\alpha} (\pi^p_{p\restriction\xi})^{-1}(A_{p\restriction(\xi+1)})$ may contain a Cantor subset. So, we make sure each such union is disjoint from some set in a tree of Bernstein sets.

Definition 4.4 For any topological space Y and $p \in 2^{<\omega_1}$, a *Bernstein tree in Y rooted in p* is a family of subsets of Y, $\{D^q : q \le p\}$, satisfying the following.

(1) For each q, neither D^q nor $Y \backslash D^q$ contains a Cantor subset.
(2) Each $D^{q^\frown 0} \cap D^{q^\frown 1} = \varnothing$.
(3) If $r \le q$, then $D^r \subseteq D^q$.

Note that if Y itself does not contain a Cantor subset, then (1) is trivial, and we may take all $D^q = \varnothing$ to satisfy (2) and (3).

Now, in our construction, we also build D^q_p for $q \le p \in 2^{<\omega_1}$ satisfying the following.

(R10) For each $p \in 2^{<\omega_1}$: $\{D^q_p : q \le p\}$ is a Bernstein tree in X_p rooted in p.
(R11) If $q \le p \le r$ and $\pi = \pi^p_r : X_p \twoheadrightarrow X_r$ and $x \in X_p$ with $\pi^{-1}(\pi(x)) = \{x\}$, then $x \in D^q_p$ if and only if $\pi(x) \in D^q_r$.
(R12) For each $p \in 2^{<\omega_1}$ and $i \in 2$: $B_{p^\frown i} = D^{p^\frown i}_p$ and $A_{p^\frown i} = X_p \setminus D^{p^\frown i}_p$.

Of course, (R12) simply defines $A_{p^\frown i}$ in terms of the D^q_p, and then (R10) guarantees that no $A_{p^\frown i}$ has a Cantor subset, but we need to verify that the conditions (R1 – R12) can indeed be satisfied. We begin with three easy lemmas about Bernstein trees. A standard inductive construction in $\mathfrak{c}$ steps shows the following.

Lemma 4.5 *If Y is a separable metric space, then there is a Bernstein tree in Y rooted in $\mathbb{1}$.*

Using the fact that every uncountable Borel subset of the Cantor set contains a perfect subset, we get the following.

Lemma 4.6 *Assume that Y is any topological space, Z is a Borel subset of Y, and $\{D^q : q \le p\}$ is a family of subsets of Y satisfying* (2) *and* (3) *of Definition* 4.4*. Then $\{D^q : q \le p\}$ is a Bernstein tree in Y if and only if both $\{D^q \cap Z : q \le p\}$ is a Bernstein tree in Z and $\{D^q \backslash Z : q \le p\}$ is a Bernstein tree in $Y \backslash Z$.*

Combining these two lemmas gives the following.

Lemma 4.7 *If Y is a separable metric space, Z is a Borel subset of Y, and $\{E^q : q \le p\}$ is a Bernstein tree in Z rooted in p, then there is a Bernstein tree $\{D^q : q \le p\}$ in Y rooted in p such that each $D^q \cap Z = E^q$.*

Returning to the construction, we have the following.

Lemma 4.8 *There exist X_p for $p \in 2^{\le\omega_1}$ satisfying Conditions* (R1)–(R12).

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Proof We start with $X_{\mathbb{1}} = [0,1]$, and we obtain the $D^q_{\mathbb{1}}$ by applying Lemma 4.5.

If $\alpha = \mathrm{lh}(p) > 0$ and we have done the construction for $p\restriction\xi$ for all $\xi < \mathrm{lh}(p)$, then X_p is determined either by (R4) when $\mathrm{lh}(p)$ is a successor or by (❀) when $\mathrm{lh}(p)$ is a limit. If $\alpha < \omega_1$, we construct the D^q_p to satisfy (R10) and (R11) as follows. For $\xi < \alpha$, use π_ξ for $\pi^p_{p\restriction\xi}$. Let $Z_\xi = \{x \in X_p : \pi_\xi^{-1}(\pi_\xi(x)) = \{x\}\}$, and let $Z = \bigcup_{\xi<\alpha} Z_\xi$. Observe that Z and all the Z_ξ are Borel sets. Let $\{E^q_p : q \le p\}$ be the Bernstein tree in Z rooted in p defined by saying that for $x \in Z_\xi$: $x \in E^q_p$ if and only if $\pi_\xi(x) \in D^q_{p\restriction\xi}$. Note that, by (R11) applied inductively, this is independent of which ξ is used. To obtain the D^q_p from the E^q_p, apply Lemma 4.7. Note that, by (R11) applied inductively once again, these D^q_p work for X_p.

The $A_{p^\frown i}$ and $B_{p^\frown i}$ (for $i = 0,1$) are now defined by (R12), and we must verify that this definition satisfies (R7). Assume that $\xi < \mathrm{lh}(p) = \alpha$, $x \in X_p$, and $\pi_\xi(x) \in A_{p\restriction(\xi+1)}$. We must show that $x \in A_{p^\frown i}$, equivalently, by (R12), that $x \notin D^{p^\frown i}_p$. Now $\pi_\xi(x) \in A_{p\restriction(\xi+1)}$ implies that $\pi_\xi^{-1}(\pi_\xi(x)) = \{x\}$ (using (R4)–(R9) inductively), so that $x \notin D^{p^\frown i}_p$ if and only if $\pi_\xi(x) \notin D^{p^\frown i}_{p\restriction\xi}$. By (R10) for $p\restriction\xi$ and Definition 4.4(3), $D^{p^\frown i}_{p\restriction\xi} \subseteq D^{p\restriction(\xi+1)}_{p\restriction\xi}$. So $A_{p\restriction(\xi+1)} = X_{p\restriction\xi} \setminus D^{p\restriction(\xi+1)}_{p\restriction\xi}$ gives us (R7).

Since the $B_{p^\frown i}$ are nonempty, there is no problem choosing the $u_{p^\frown i}$, $v^n_{p^\frown i}$, and $h_{p^\frown i}$ to satisfy (R4), (R5), and (R9), and then the $X_{p^\frown i}$ are defined by (R4). ■

Finally, we must make each X_g weird. Observe the following.

Lemma 4.9 *Conditions* (R1)–(R5) *imply that if $F \subseteq X_p$ is closed and connected, then $(\pi^q_p)^{-1}(F)$ is connected for all $q \le p$.*

Now, we shall make sure that whenever F is a perfect subset of X_g, there is some $\alpha < \omega_1$ such that $(\pi^g_{g\restriction(\alpha+1)})^{-1}(\{u_{g\restriction(\alpha+1)}\} \times [0,1]) \subseteq F$ (recall that our construction gave us $\{u_{g\restriction(\alpha+1)}\} \times [0,1] \subset X_{g\restriction(\alpha+1)} \subset X_{g\restriction\alpha} \times [0,1]$) . By Lemma 4.9, this implies that F is not totally disconnected. The argument in [12] obtained this α by using $\diamondsuit$ to capture F. Here, we replace this use of $\diamondsuit$ by a classical CH argument. First, as in [12], construct $\mathcal{F}_p$ for $p \in 2^{<\omega_1}$ so that the following hold.

(R13) $\mathcal{F}_p$ is a countable family of uncountable closed subsets of X_p.

(R14) If $F \in \mathcal{F}_p$ and $q \le p$ then $(\pi^q_p)^{-1}(F) \in \mathcal{F}_q$.

(R15) For each $F \in \mathcal{F}_p$, either $u_{p^\frown i} \notin F$, or $u_{p^\frown i} \in F$ and $v^n_{p^\frown i} \in F$ for all but finitely many n.

(R16) $\{u_{p^\frown i}\} \times [0,1] \in \mathcal{F}_{p^\frown i}$.

We may satisfy (R13), (R14), and (R16) simply by defining

$$\mathcal{F}_p = \left\{ (\pi^p_{p\restriction\xi})^{-1}\{u_{p\restriction(\xi+1)}\} : \xi < \mathrm{lh}(p) \right\}.$$

Requirements (R4), (R14), and (R15) imply the following.

Lemma 4.10 *$\pi^q_p : (\pi^q_p)^{-1}(F) \twoheadrightarrow F$ is irreducible whenever $F \in \mathcal{F}_p$ and $q \le p$.*

Then we use CH rather than $\diamondsuit$ to get the following.

(R17) Whenever $p \in 2^{<\omega_1}$ and F is an uncountable closed subset of X_p, there is a β with $\mathrm{lh}(p) < \beta < \omega_1$ such that for all $q < p$ with $\mathrm{lh}(q) = \beta$ and for each $x \in \{u_{q^\frown 0}, u_{q^\frown 1}\} \cup \{v^n_{q^\frown i} : n \in \omega \;\&\; i \in 2\}$, the projections $\pi = \pi^q_p$ satisfy $\pi(x) \in F$ and $|\pi^{-1}(\pi(x))| = 1$.

Proof of Theorem 1.3 Assuming that we can obtain (R1)–(R17), note that each X_g is separable, because each $\pi^g_{\mathbb{1}}\colon X_g \twoheadrightarrow X_{\mathbb{1}}$ is irreducible. Then to finish, by Lemma 4.3 it suffices to show that each X_g is weird. Fix a perfect $H \subseteq X_g$; we shall show that it is not totally disconnected. First, fix $\alpha < \omega_1$ such that, if we set $p = g{\restriction}\alpha$ and $F = \pi^g_p(H)$, then F is perfect (the set of all such α form a club). Then fix $\beta > \alpha$ as in (R17), let $q = g{\restriction}\beta$, and let $i = g(\beta)$, so that $q^\frown i = g \restriction (\beta + 1)$. Let $K = (\pi^q_p)^{-1}(F)$. Then $\pi^g_q(H) \subseteq K$, and this inclusion may well be proper. However, $u_{q^\frown i} \in \pi^g_q(H)$ and $v^n_{q^\frown i} \in \pi^g_q(H)$ for each $n \in \omega$ because $\pi(u_{q^\frown i}) \in F$, $\pi(v^n_{q^\frown i}) \in F$, and $|\pi^{-1}(\pi(u_{q^\frown i}))| = |\pi^{-1}(\pi(v^n_{q^\frown i}))| = 1$. It follows (using (R5)) that $E := \{u_{q^\frown i}\} \times [0,1] \subseteq \pi^g_{q^\frown i}(H)$. Since $E \in \mathcal{F}_{q^\frown i}$ by (R16) and H maps onto E, Lemma 4.10 implies that $(\pi^g_{q^\frown i})^{-1}(E) \subseteq H$. Since $(\pi^g_{q^\frown i})^{-1}(E)$ is connected by Lemma 4.9, H cannot be totally disconnected.

Next, to obtain conditions (R1)–(R17), we must augment the proof of Lemma 4.8. Fix in advance a map ψ from $\omega_1 \backslash \{0\}$ onto $\omega_1 \times \omega_1$, such that $\alpha < \beta$ whenever $\psi(\beta) = (\alpha, \xi)$. Now, given X_p, use CH and let $\{F^p_\xi : \xi < \omega_1\}$ be a listing of all uncountable closed subsets of X_p. Whenever $0 < \beta < \omega_1$, $\psi(\beta) = (\alpha, \xi)$, and $\mathrm{lh}(q) = \beta$, we may set $p = q{\restriction}\alpha$ and $F = F^p_\xi \subseteq X_p$. It is sufficient to show how to accomplish (R17) with these specific α, β, p, q, F.

Choose a perfect $K \subset F$ which is disjoint from $\{\pi^{(q\restriction\zeta)}_p(u_{q\restriction(\zeta+1)}) : \alpha \le \zeta < \beta\}$. Then π^q_p is 1-1 on $(\pi^q_p)^{-1}(K)$, so choosing all $u_{q^\frown i}$ and $v^n_{q^\frown i}$ in $(\pi^q_p)^{-1}(K)$ will ensure (R17). Now *fix* $i \in 2$, and write u and v^n for $u_{q^\frown i}$ and $v^n_{q^\frown i}$. To ensure (R15) and (R9), we modify the argument of [12]. Let $\{Q^n : n \in \omega\}$ list $\mathcal{F}_q$. Let d be a metric on $(\pi^q_p)^{-1}(K)$. For each $s \in 2^{<\omega}$, choose a perfect $L_s \subseteq (\pi^q_p)^{-1}(K)$. Make these into a tree, in the sense that each $L_{s^\frown 0} \cap L_{s^\frown 1} = \varnothing$, each $\mathrm{diam}(L_s) \le 2^{-\mathrm{lh}(s)}$, and $L_{s^\frown 0} \subseteq L_s$ and $L_{s^\frown 1} \subseteq L_s$. Also make sure that whenever $\mathrm{lh}(s) = n+1$, we have either $L_s \subseteq Q^n$ or $L_s \cap Q^n = \varnothing$. Let $v[s^\frown \ell]$ be any point in $L_{s^\frown \ell} \backslash L_s$. For $f \in 2^\omega$, let $\{u[f]\} = \bigcap_n L_{f\restriction n}$. For any $f \in 2^\omega$, if we set $u = u[f]$ and $v^n = v[f \restriction (n+1)]$, then (R15) will hold. Now, (R9) requires $u \in B_{q^\frown i}$. Since $B_{q^\frown i}$ is a Bernstein set and $\{u[f] : f \in 2^\omega\}$ is a Cantor set, we may choose f so that $u[f] \in B_{q^\frown i}$. ■

If $H \subseteq X_g$ is closed and for some initial segment $p = g{\restriction}\alpha$ the projection $\pi^g_p(H) \in \mathcal{F}_p$, then by irreducibility, $H = (\pi^g_p)^{-1}(\pi^g_p(H))$, so that H is a G_δ. To make X_g hereditarily Lindelöf, it suffices to capture projections for each closed $H \subseteq X_g$ this way, but it is not clear whether this can be done without using $\diamondsuit$.

5 Remarks and Examples

One cannot replace "CH" by "$2^{\aleph_0} < 2^{\aleph_1}$" in the statement of Theorem 1.3, since by Proposition 5.3, it is consistent with any cardinal arithmetic that every non-scattered compactum of weight less than $\mathfrak{c}$ contains a copy of the Cantor set.

Definition 5.1 As usual, $\mathrm{cov}(\mathcal{M})$ is the least κ such that $\mathbb{R}$ is the union of κ meager sets.

Note that $\mathrm{cov}(\mathcal{M})$ is the least κ such that $\mathrm{MA}(\kappa)$ for countable partial orders fails. Using this, we easily see the following.

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Lemma 5.2 *If* $\kappa < \mathrm{cov}(\mathcal{M})$ *and* $E_\alpha \subset [0,1]$ *is meager for each* $\alpha < \kappa$, *then* $[0,1] \setminus \bigcup_{\alpha<\kappa} E_\alpha$ *contains a copy of the Cantor set.*

Proposition 5.3 *If* X *is compact and not scattered, and* $w(X) < \mathrm{cov}(\mathcal{M})$, *then* X *contains a copy of the Cantor set.*

Proof Replacing X by a subspace, we may assume that we have an irreducible map $\pi\colon X \twoheadrightarrow [0,1]$. Let $\mathcal{B}$ be an open base for X with $|\mathcal{B}| < \mathrm{cov}(\mathcal{M})$ and $\varnothing \neq \mathcal{B}$.

Whenever $U, V \in \mathcal{B}$ with $\overline{U} \cap \overline{V} = \varnothing$, let $E_{U,V} = \pi(\overline{U}) \cap \pi(\overline{V})$. Then $E_{U,V} \subset [0,1]$ is nowhere dense because π is irreducible. Applying Lemma 5.2, let $K \subset [0,1]$ be a copy of the Cantor set disjoint from all the $E_{U,V}$. Note that $|\pi^{-1}\{y\}| = 1$ for all $y \in K$. Thus, $\pi^{-1}(K)$ is homeomorphic to K. ∎

Note that one can force "$\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$" by adding $\mathfrak{c}$ Cohen reals, which does not change cardinal arithmetic, but in the statement of Proposition 5.3, "$\mathrm{cov}(\mathcal{M})$" cannot be replaced by "$\mathfrak{c}$". If CH holds in V, then one may force $\mathfrak{c}$ to be arbitrarily large by adding random reals, and *any* random real extension $V[G]$ will have a compact non-scattered space of weight $\aleph_1$ which does not contain a Cantor subset. In fact, Dow and Fremlin [4] show that if X is a compact F-space in V, then in a random real extension $V[G]$, the corresponding compact space $\widetilde{X}$ has no convergent ω–sequences and hence no Cantor subsets.

The weird space constructed in [12] also failed to satisfy the CSWP (the complex version of the Stone–Weierstrass Theorem). Using the method there, we can modify the proof of Theorem 1.3 to get the following.

Theorem 5.4 *Assuming CH, there is a separable, first countable, connected, weird space* X *of weight* $\aleph_1$ *such that* X *fails the CSWP and* $\mathbb{K}_X$ *fails the CTP.*

Proof First, in the proof of Theorem 1.3, replace $[0,1]$ by $\overline{D}$, the closed unit disc in the complex plane, so that we may view X as a subspace of the $\aleph_1$–dimensional polydisc. Then, as in [12], by carefully choosing the functions $h_{p^\frown i}$, one can ensure that the restriction to X of the natural analog of the disc algebra refutes the CSWP of X. To refute the CTP, construct in $\overline{D}$ a Cantor tree $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{K}_{\overline{D}}$ such that each p_s is a wedge of the disc with center 0 and radius $2^{-\mathrm{lh}(s)}$; then each $\bigcap_{n\in\omega} p_{f\restriction n} = \{0\}$. Then, since we may assume the point 0 is not expanded in the construction of X, the inverse images of the p_s yield a counterexample to the CTP of $\mathbb{K}_X$. ∎

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