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PRODUCTS OF SHIFTED PRIMES SIMULTANEOUSLY TAKING PERFECT POWER VALUES

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Abstract

Let r be an integer greater than 1, and let A be a finite, nonempty set of nonzero integers. We obtain a lower bound for the number of positive squarefree integers n, up to x, for which the products $\prod_{p|n}(p+a)$ (over primes p) are perfect rth powers for all the integers a in A. Also, in the cases where $A = \{-1\}$ and $A = \{+1\}$, we will obtain a lower bound for the number of such n with exactly r distinct prime factors.

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1. Introduction

If we pick a large integer close to x at random, the probability that it is a perfect rth power is around $x^{1/r}/x$. We might expect the shifted primes $p+a$ to behave more or less like random integers in terms of their multiplicative properties. Thus, if we take a large squarefree integer n close to x, we might naively expect that $\sigma(n) = \prod_{p|n}(p+1) \approx n$ is an rth power with probability close to $x^{1/r}/x$. However, as we will see, the probability is much higher than this, indeed more than $x^{0.7038}/x$, for *any* given r. We will even show that the likelihood of $\phi(n)$ and $\sigma(n)$ *simultaneously* being (different) rth powers is more than $x^{0.2499}/x$. (As usual, ϕ denotes Euler's totient function and σ denotes the sum-of-divisors function.) It would seem that rth powers are 'popular' values for products of shifted primes in general.

Counting those n with exactly r prime factors, we will show that the number of such n up to x for which $\phi(n)$ is a perfect rth power is at least of the order of $x^{1/r}/(\log x)^{r+2}$, and likewise for $\sigma(n)$. Thus the number of positive integers n such that $n \le x$ and $n = pq$, where p and q are distinct primes, and $(p-1)(q-1)$ is a square, is at least of

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the order of $x^{1/2}/(\log x)^4$. This may be seen as an 'approximation' to the well-known conjecture that there are infinitely many primes p for which $p-1$ is a square. It is easily seen that, for any given $r \geq 2$, there is at most one prime p such that $p+1$ is a perfect rth power, namely $3+1=2^2$, $7+1=2^3$, and so on.

Notation The expressions $F = O(G)$, $F \ll G$, and $G \gg F$ all mean that $|F| \leq cG$, where c is a positive constant. Further, $F \asymp G$ means that $F \ll G \ll F$. Where the constant c is not absolute but depends on one or more parameters, this dependence may be indicated, as in, for example, $F \asymp_\epsilon G$, where the implied constants depend on ϵ. If $f(x)$ and $g(x)$ are functions and $g(x)$ is nonzero for all sufficiently large x, we write $f(x) \sim g(x)$ to mean that $\lim_{x\to\infty} f(x)/g(x) = 1$, and $f(x) = o(g(x))$ to mean that $\lim_{x\to\infty} f(x)/g(x) = 0$. Other notation will be introduced as needed.

Given an integer $r \geq 2$ and a finite, nonempty set A of nonzero integers, let

$$\mathcal{B}(x; A, r) = \Big\{n \in S \cap [1, x] : \prod_{p|n}(p+a) \in \mathbb{Z}^r \ \forall a \in A\Big\},$$

where S denotes the set of squarefree integers and $\mathbb{Z}^r$ denotes the set of perfect rth powers. Banks *et al.* [4] proved, among other results, that

$$|\mathcal{B}(x; \{-1\}, 2)| \geq x^{0.7039-o(1)} \quad \text{and} \quad |\mathcal{B}(x; \{+1\}, 2)| \geq x^{0.7039-o(1)},$$

and that

$$|\mathcal{B}(x; \{-1, +1\}, 2)| \geq x^{1/4-o(1)}.$$

The first theorem generalizes both of these results.

Theorem 1.1. *Fix an integer $r \geq 2$, and a finite, nonempty set A of nonzero integers. As x tends to infinity,*

$$|\mathcal{B}(x; A, r)| \geq x^{1/2|A|-o(1)}.$$

Moreover, if $|A| = 1$, then as x tends to infinity,

$$|\mathcal{B}(x; A, r)| \geq x^{0.7039-o(1)}.$$

In the cases where $A = \{-1\}$ or $A = \{+1\}$, $\mathcal{B}(x; A, r)$ is the set of positive squarefree integers n up to x for which $\phi(n)$ or $\sigma(n)$ respectively is an rth power. There is no condition on the number of prime factors of n, but the next theorem concerns the sets

$$\mathcal{B}^*(x; -1, r) = \mathcal{B}(x; \{-1\}, r) \cap \{n : \omega(n) = r\},$$
$$\mathcal{B}^*(x; +1, r) = \mathcal{B}(x; \{+1\}, r) \cap \{n : \omega(n) = r\},$$

where $\omega(n)$ is the number of distinct prime factors of n.

Theorem 1.2. *Fix an integer $r \geq 2$. For all sufficiently large x,*

$$|\mathcal{B}^*(x; -1, r)| \gg \frac{rx^{1/r}}{(\log x)^{r+2}} \quad \text{and} \quad |\mathcal{B}^*(x; +1, r)| \gg \frac{rx^{1/r}}{(\log x)^{r+2}}. \tag{1.1}$$

The implied constants are absolute.

The proof of Theorem 1.1, in Section 3, is an extension of the proof by Banks *et al.* [4] of the aforementioned special cases of Theorem 1.1. It employs some of the ideas of Erdős [9, 10] upon which Alford *et al.* [1] based their proof that there are infinitely many Carmichael numbers. (A Carmichael number is a composite number n for which $a^n \equiv a \bmod n$ for all integers a.) Theorem 1.2 can be proved along the same lines. Indeed, in [5] it is shown that for all sufficiently large x, the lower bound $|\mathcal{B}^*(x; -1, r)| \geq 4x^{1/r}/(9e(\log x)^{2r})$ holds for $2 \leq r \leq (\log x/(12 \log\log x))^{1/2}$. However, our proof of Theorem 1.2, in Section 4, introduces a new method, which, as we will explain, is an application of the ideas of Goldston *et al.* [11].

2. Preliminaries

Theorem 1.1 is a consequence of the first four results of this section, and we use the fifth in the proof of Theorem 1.2.

An integer n is called y-smooth if $p \leq y$ for every prime p dividing n. Given a polynomial $F(X) \in \mathbb{Z}[X]$ and numbers $x \geq y \geq 2$, let

$$\pi_F(x, y) = |\{p \leq x : F(p) \text{ is } y\text{-smooth}\}|.$$

In the case where $F = X - 1$, Erdős [9] proved that there exists a number $\eta \in (0, 1)$ such that $\pi_F(x, x^\eta) \gg_\eta \pi(x)$ (where $\pi(x)$ is the number of primes up to x), for all large x depending on the choice of η. Several authors have improved upon this, the next two results being the best so far obtained.

Theorem 2.1. *Fix a nonzero integer a and let $F(X) = X + a$. There exists an absolute constant c such that*

$$\pi_F(x, y) > \frac{x}{(\log x)^c}$$

for all sufficiently large x, provided that $y \geq x^{0.2961}$.

Proof. See [2, Theorem 1]. □

Theorem 2.2. *Let $F \in \mathbb{Z}[X]$. Let g be the largest of the degrees of the irreducible factors of F in $\mathbb{Z}[X]$, and let k be the number of distinct irreducible factors of F in $\mathbb{Z}[X]$ of degree g. Suppose that $F(0) \neq 0$ if $g = k = 1$, and let ϵ be any positive real number. Then*

$$\pi_F(x, y) \asymp \frac{x}{\log x}$$

for all sufficiently large x, provided that $y \geq x^{g+\epsilon-1/2k}$.

Proof. See [6, Theorem 1.2]. □

For a finite additive abelian group G, denote by $n(G)$ the length of the longest sequence of (not necessarily distinct) elements of G, no nonempty subsequence of which sums to 0, the additive identity of G. For instance, if $G = (\mathbb{Z}/2\mathbb{Z})^m$, then $n(G) \leq m$, for any sequence of $m + 1$ elements of G contains a nonempty subsequence whose elements sum to $(0, \ldots, 0)$ mod 2, as can be seen by considering that such a sequence contains $2^{m+1} - 1 > 2^m = |G|$ nonempty subsequences. For any group G of order m, then any sequence of m elements contains a nonempty subsequence whose sum is 0, hence $n(G) \leq m - 1$. The next theorem, due to van Emde Boas and Kruyswijk [8], gives a nontrivial upper bound for $n(G)$.

THEOREM 2.3. *If G is a finite abelian group and m is the maximal order of an element in G, then $n(G) < m(1 + \log(|G|/m))$.*

PROOF. See [8]. A proof is also given in [1, Theorem 1.1]. □

The following proposition shows that under certain conditions there are many sequences in G whose elements sum to 0.

PROPOSITION 2.4. *Let G be a finite abelian group and let r and k be integers such that $r > k > n = n(G)$. Then any subsequence of r elements of G contains at least $\binom{r}{k} \Big/ \binom{r}{n}$ distinct subsequences of length at most k and at least $k - n$, whose sum is the identity.*

PROOF. See [1, Proposition 1.2]. □

We will use the well-known Siegel–Walfisz theorem in the proof of Theorem 1.2.

THEOREM 2.5 (Siegel–Walfisz). *For any positive number B, there is a constant C_B that depends only on B, such that*

$$\sum_{\substack{p \leq N \\ p \equiv a \bmod k}} \log p = \frac{N}{\phi(k)} + O(N \exp(-C_B (\log N)^{1/2}))$$

whenever $k \leq (\log N)^B$ and a is coprime with k.

PROOF. See [7, Ch. 22]. □

3. Proof of Theorem 1.1

The following proof hinges on Theorem 2.3 and Proposition 2.4, which are key ingredients in the celebrated proof of Alford *et al.* [1] that there are infinitely many Carmichael numbers. In fact it is shown in [1, Theorem 1] that $C(x)$, the number of Carmichael numbers up to x, satisfies $C(x) \geq x^{\beta - \epsilon}$ for any positive ϵ and all sufficiently large x (depending on the choice of ϵ), where

$$\beta = \frac{5}{12}\left(1 - \frac{1}{2\sqrt{e}}\right) = 0.290\,36\ldots.$$

Using a variant of the construction in [1], Harman [12] proved that $\beta = 0.332\,240\,8$ is admissible, and, by combining the ideas of [1, 4, 12], Banks [3] established the following result.

THEOREM 3.1 [3, Theorem 1]. *For every fixed $C < 1$, there is a number $x_0(C)$ such that for all $x \ge x_0(C)$ the inequality*

$$|\{n \le x : n \text{ is Carmichael and } \phi(n) \text{ is an } r\text{th power}\}| \ge x^{\beta-\epsilon}$$

holds, where $\beta = 0.332\,240\,8$ and ϵ is arbitrarily small but positive, for all positive integers $r \le \exp((\log\log x)^C)$.

(Harman [13] subsequently proved that $\beta = 0.7039 \times 0.4736 > 1/3$ is admissible here.) The method of the proof may yield further interesting results.

Theorems 2.1 and 2.2 are also crucial, and it will be manifest that extending the admissible range for y in those theorems will lead to better estimates for $|\mathcal{B}(x; A, r)|$. Explicitly, if $F(X) = \prod_{a\in A}(X + a)$ and

$$\pi_F(x, x^\eta) \asymp_{F,\eta} \frac{x}{\log x},$$

then the following proof yields $|\mathcal{B}(x; A, r)| \ge x^{1-\eta-o(1)}$.

PROOF OF THEOREM 1.1. Fix an integer $r \ge 2$ and a set $A = \{a_1, \ldots, a_s\}$ of nonzero integers. Let x be a large number, and let

$$y = \frac{\log x}{\log\log x}. \tag{3.1}$$

Let $t = \pi(y)$, and let $G = (\mathbb{Z}/r\mathbb{Z})^{st}$, so that by Theorem 2.3,

$$n(G) < r(1 + \log|G|/r) = r(1 + (st - 1)\log r). \tag{3.2}$$

Fix any number $\epsilon \in (0, 1/3s)$, and let

$$u = \begin{cases} 0.2961^{-1} & \text{if } s = 1, \\ \left(1 + \epsilon - \dfrac{1}{2s}\right)^{-1} & \text{if } s \ge 2. \end{cases}$$

Let

$$F(X) = (X + a_1)(X + a_2)\cdots(X + a_s),$$

and let

$$\begin{aligned} S_F(y^u, y) &= \{p \le y^u : F(p) \text{ is } y\text{-smooth}\} \\ &= \{p \le y^u : p + a_1, \ldots, p + a_s \text{ are } y\text{-smooth}\}. \end{aligned}$$

We may suppose that x, and hence y, is large enough so that, by Theorems 2.1 and 2.2,

$$|S_F(y^u, y)| = \pi_F(y^u, y) \gg \frac{y^u}{(\log y^u)^c} \tag{3.3}$$

for some constant c. (We may suppose that $c = 1$ if $s \geq 2$.) Finally, let

$$k = \left[\frac{\log x}{\log y^u}\right], \tag{3.4}$$

where $[\alpha]$ denotes the integer part of a real number α.

By (3.1), (3.3) and (3.4),

$$\frac{\pi_F(y^u, y)}{k} \gg \frac{(\log x)^{u-1}}{(\log\log x)^{u-1+c}},$$

and by (3.1), (3.2) and (3.4),

$$\frac{k}{n(G)} \gg_{r,s} \frac{\log x/\log y^u}{t} \gg \log\log x, \tag{3.5}$$

because $t = \pi(y) \sim y/\log y$ as y tends to infinity, by the prime number theorem. Hence, since $u > 1$, we may assume x is large enough that

$$n(G) < k < \pi_F(y^u, y). \tag{3.6}$$

For primes $p \in S_F(y^u, y)$ and integers $a \in A$, we may write

$$p + a = 2^{\beta_1^{(a)}} 3^{\beta_2^{(a)}} \cdots p_t^{\beta_t^{(a)}},$$

where $\beta_i^{(a)}$ are nonnegative integers when $1 \leq i \leq t$. We define

$$\mathbf{v}_p = (\beta_1^{(a_1)}, \ldots, \beta_t^{(a_1)}, \beta_1^{(a_2)}, \ldots, \beta_t^{(a_2)}, \ldots, \beta_1^{(a_s)}, \ldots, \beta_t^{(a_s)})$$

as the 'exponent vector' for p. For a subset R of $S_F(y^u, y)$, $\prod_{p\in R}(p + a)$ is an rth power for every $a \in A$ if and only if

$$\sum_{p\in R} \mathbf{v}_p \equiv \mathbf{0} \mod r,$$

where $\mathbf{0}$ mod r is the zero element of G. If, moreover, R is of size at most k, then by (3.4),

$$\prod_{p\in R} p \leq y^{uk} \leq x.$$

Thus

$$|\mathcal{B}(x; A, r)| \geq \left|\left\{R \subseteq S_F(y^u, y) : |R| \leq k \text{ and } \sum_{p\in R} \mathbf{v}_p \equiv \mathbf{0} \bmod r\right\}\right|, \tag{3.7}$$

as distinct subsets $R \subseteq S_F(y^u, y)$ give rise to distinct integers n, by the uniqueness of factorization.

Because of (3.6), we may deduce from Proposition 2.4 that the right-hand side of (3.7) is at least

$$\binom{\pi_F(y^u, y)}{k} \Big/ \binom{\pi_F(y^u, y)}{n(G)} \geq \left(\frac{\pi_F(y^u, y)}{k}\right)^k \pi_F(y^u, y)^{-n(G)} = x^{f(x)},$$

where

$$f(x) = (k - n(G))\frac{\log \pi_F(y^u, y)}{\log x} - \frac{k \log k}{\log x}.$$

Letting x tend to infinity and using (3.1), (3.3)–(3.5), we can now see that $f(x) = 1 - 1/u - o(1)$. Therefore, as x tends to infinity,

$$|\mathcal{B}(x; A, r)| \geq x^{1-1/u-o(1)},$$

and Theorem 1.1 follows by our choice of u, and letting ϵ tend to 0 if $s \geq 2$. □

4. Proof of Theorem 1.2

We use a different approach to prove Theorem 1.2. The proof is 'inspired' by the breakthrough results of Goldston *et al.* [11] on short intervals containing primes. Basically, their proof begins with the observation that if $W(n)$ is a nonnegative weight and

$$\sum_{N<n\leq 2N}\Big(\sum_{1\leq h\leq H} \vartheta(n+h) - \log(2N+H)\Big)W(n) \tag{4.1}$$

is positive, then for some $n \in (N, 2N]$, the interval $(n, n+H]$ contains at least two primes. Here and later,

$$\vartheta(n) = \begin{cases} \log n & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Goldston *et al.* were able to obtain a nonnegative weight $W(n)$ for which the sum (4.1), with $H = \epsilon \log N$, is positive for all sufficiently large N. In our problem, we will be led to consider

$$\sum_{1\leq n\leq N}\Big(\sum_{1\leq a\leq H} \vartheta(a^r n+1) - (r-1)\log(H^r N+1)\Big)$$

(see (4.3)). A lower bound for this expression corresponds to a lower bound for the number of positive integers $n \leq N$ for which $\{a^r n + 1 : a \leq H\}$ contains at least r primes. As we do not require H to be 'short' compared to N, we may take $H = r \log N$; then the weight $W(n) = 1$ works, and the problem is much easier.

Proof of Theorem 1.2. Throughout the proof, r is a fixed integer greater than 1, and $n, a, a_1, a_2, \ldots$ are positive integers. Observe that if, for some n, the numbers ℓ_i, given by

$$\ell_i = a_i^r n + 1,$$

are distinct primes (where $i = 1, \ldots, r$), then

$$\phi(\ell_1 \cdots \ell_r) = (a_1 \cdots a_r n)^r.$$

If the primes ℓ_i are of the form $a_i^r n - 1$ then $\sigma(\ell_1 \cdots \ell_r) = (a_1 \cdots a_r n)^r$. We will prove that (1.1) holds for $|\mathcal{B}(x; -1, r)|$, provided that x is sufficiently large, and the same proof applies to $|\mathcal{B}(x; +1, r)|$ if we consider primes of the form $a_i^r n - 1$ rather than $a_i^r n + 1$.

Let N be a parameter tending monotonically to infinity and set $H = r \log N$. Let $\mathcal{A}(N)$ be the set of positive integers $n \le N$ for which

$$C_n := \{a^r n + 1 : a \le H\} \cap \mathcal{P}$$

(where $\mathcal{P}$ is the set of all primes) contains at least r primes. We will show that

$$|\mathcal{A}(N)| \gg \frac{N}{\log N}, \tag{4.2}$$

but first we will describe how this implies a lower bound for $|\mathcal{B}(x; -1, r)|$.

Every $n \in \mathcal{A}(N)$ gives rise, via C_n, to some $\ell_1 \cdots \ell_r \in \mathcal{B}((H^r N + 1)^r; -1, r)$, though different n may give rise to the same r-tuple of primes. On the other hand, given $n \in \mathcal{A}(N)$ and a prime $p = a^r n + 1 \in C_n$, each $m \in \mathcal{A}(N)$ for which $p \in C_m$ corresponds to a solution to $a^r n = b^r m$, $b \le H$. Therefore there can be at most H different integers $n \in \mathcal{A}(N)$ giving rise to the same element of $\mathcal{B}((H^r N + 1)^r; -1, r)$. Consequently,

$$|\mathcal{B}((H^r N + 1)^r; -1, r)| \ge \frac{|\mathcal{A}(N)|}{H} \gg \frac{N}{r(\log N)^2}$$

by (4.2), and (1.1) follows.

We will now establish (4.2). We will show that for all large N,

$$S(N) = \sum_{1 \le n \le N} \Big(\sum_{1 \le a \le H} \vartheta(a^r n + 1) - (r-1) \log(H^r N + 1) \Big) \gg rN \log N. \tag{4.3}$$

Consequently $\mathcal{A}(N)$ is nonempty for large N. Indeed, if (4.3) holds then

$$\begin{aligned} rN \log N \ll S(N) &\le \sum_{n \in \mathcal{A}(N)} \Big(\sum_{1 \le a \le H} \vartheta(a^r n + 1) - (r-1) \log(H^r N + 1) \Big) \\ &\le |\mathcal{A}(N)| H \log(H^r N + 1), \end{aligned}$$

and (4.2) follows because $\log(H^r N + 1) \sim \log N$.

For the evaluation of $S(N)$, first note that

$$\sum_{1 \le n \le N} \sum_{1 \le a \le H} \vartheta(a^r n + 1) = \sum_{1 \le a \le H} \sum_{\substack{p \le a^r N + 1 \\ p \equiv 1 \bmod a^r}} \log p.$$

Since $a^r \ll_r (\log N)^r$ for $a \le H$, we may apply Theorem 2.5 to the last sum. We have

$$\sum_{\substack{p \le a^r N + 1 \\ p \equiv 1 \bmod a^r}} \log p = \frac{a^r N}{\phi(a^r)} + O\Big(\frac{a^r N}{\phi(a^r)(\log N)^2} \Big) \sim \frac{a}{\phi(a)} N.$$

Therefore, from the well-known estimate

$$\sum_{1\le a\le H} \frac{a}{\phi(a)} \sim cH \quad \text{where } c = \prod_p \left(1 + \frac{1}{p(p-1)}\right) = 1.943\,596\ldots,$$

we deduce that

$$\sum_{1\le n\le N} \sum_{1\le a\le H} \vartheta(a^r n + 1) \sim N \sum_{1\le a\le H} \frac{a}{\phi(a)} \sim cNH.$$

Also,

$$\sum_{1\le n\le N} (r-1)\log(H^r N + 1) \sim N(r-1)\log N,$$

so combining all of this yields

$$S(N) \sim N(cH - (r-1)\log N) \gg rN\log N,$$

and (4.3) follows.

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