CERTAIN REPRESENTATION ALGEBRAS

S. B. CONLON

(received 5 May 1964)

Introduction

Let $\Lambda$ be the set of inequivalent representations of a finite group $G$ over a field $\mathbb{F}$. $\Lambda$ is made the basis of an algebra $\mathcal{A}$ over the complex numbers $\mathbb{C}$, called the representation algebra, in which multiplication corresponds to the tensor product of representations and addition to direct sum. Green [5] has shown that if $\text{char} \mathbb{F} \nmid |G|$ (the non-modular case) or if $G$ is cyclic, then $\mathcal{A}$ is semi-simple, i.e. is a direct sum of copies of $\mathbb{C}$. Here we consider two modular, non-cyclic cases, viz. where $G$ is $V_4 (= \mathbb{Z}_2 \times \mathbb{Z}_2)$ or $A_4$ (alternating group) and $\mathbb{F}$ is of characteristic 2.

Finally we consider the analogous case where the multiplication is changed to the ordinary ring tensor product over the group algebra $\mathbb{F}(G)$.

I would like to thank Dr. G. E. Wall for indicating these problems.

1. Representation algebras of groups

Let $\mathcal{P}$ be an arbitrary commutative ring with a unity, and let $\mathbb{F}(G)$ be the group algebra of a group $G$ over a field $\mathbb{F}$. The representation algebra $\mathcal{A}(\mathcal{P}, \mathcal{F}, G) (= \mathcal{A})$ is defined as follows. It is the $\mathcal{P}$-module generated by the set of all isomorphism classes $\{M\}$ of $\mathcal{F}(G)$-modules $M$, subject to the relations

$$\{M\} = \{M\} + \{M''\},$$

for all $M, M', M''$ such that $M = M' \oplus M''$, and equipped with the bilinear multiplication given by

$$\{M\}\{M'\} = \{M \times M'\}.$$  

Here $M \times M'$ is the module obtained from the tensor (Kronecker) product of the representations afforded by $M, M'$. By the Krull-Schmidt theorem for $\mathbb{F}(G)$-modules, $\mathcal{A}$ is free as a $\mathcal{P}$-module and the $\mathcal{F}(G)$-indecomposable classes form a $\mathcal{P}$-basis. $\mathcal{A}$ is a commutative, associative algebra over $\mathcal{P}$.

---

1 We consider only modules $M$ of finite $\mathcal{F}$-dimension.

and has identity element \( \{F \} \), i.e., the class containing the trivial \( \mathcal{F}(\mathcal{G}) \)-module.

When \( \mathcal{P} \) is taken to be the field \( \mathbb{C} \) of complex numbers, \( \mathcal{A}(\mathbb{C}, \mathcal{F}, \mathcal{G}) \) is semi-simple in the non-modular case, or when \( \mathcal{G} \) is a cyclic group (Green [5]). In these cases there are only a finite number of different indecomposable classes and so \( \mathcal{A} \) is a direct sum of copies of \( \mathbb{C} \). In general the structure of \( \mathcal{A} \) is more complicated, but we may still hope for semi-simplicity.

Green in [5] is more precise. When \( \mathcal{P} = \mathbb{C} \), we define a \( \mathcal{G} \)-character of \( \mathcal{A} \) to be a non-zero algebra homomorphism \( \phi : \mathcal{A} \rightarrow \mathbb{C} \). \( \mathcal{A} \) is then \( \mathcal{G} \)-semisimple if, given any non-zero element \( A \in \mathcal{A} \), there exists some \( \mathcal{G} \)-character \( \phi \) of \( \mathcal{A} \) such that \( \phi(A) \neq 0 \). We may define the \( \mathcal{G} \)-radical of \( \mathcal{A} \) to be the intersection \( \cap \mathcal{M}_a \) of all maximal ideals \( \mathcal{M}_a \) of \( \mathcal{A} \), such that \( \mathcal{A}/\mathcal{M}_a \approx \mathbb{C} \). Then \( \mathcal{A} \) is \( \mathcal{G} \)-semisimple if and only if the \( \mathcal{G} \)-radical = (0).

Let \( \mathbb{F}^* \) be an extension field of \( \mathbb{F} \). Each \( \mathcal{F}(\mathcal{G}) \)-module \( \mathcal{M} \) gives rise to a \( \mathcal{F}^*(\mathcal{G}) \)-module \( \mathcal{M}^* = \mathbb{F}^* \otimes \mathcal{M} \). We have the following

**Proposition 1.**\(^3\) \( \mathcal{M} \approx \mathcal{M}^* \) if and only if \( \mathcal{M}^* \approx \mathcal{M}^* \).

The mapping \( \{\mathcal{M}\} \rightarrow \{\mathcal{M}^*\} \) gives rise to a natural homomorphism

\[
\mathcal{A}(\mathbb{C}, \mathcal{F}, \mathcal{G}) 
\rightarrow 
\mathcal{A}(\mathbb{C}, \mathcal{F}^*, \mathcal{G}).
\]

From proposition 1 it follows that this is actually a monomorphism. In view of this natural embedding we shall use \( \{\mathcal{M}\} \) to denote either \( \{\mathcal{M}\} \) or \( \{\mathcal{M}^*\} \); the interpretation will be clear from the context.

Let \( \mathcal{H} \) be a subgroup of \( \mathcal{G} \), let \( \mathcal{L} \) be a \( \mathcal{F}(\mathcal{H}) \)-module, and let \( \mathcal{M} \) be a \( \mathcal{F}(\mathcal{G}) \)-module. \( \mathcal{L}^\mathcal{G} \) will denote the induced \( \mathcal{F}(\mathcal{G}) \)-module

\[
\mathcal{F}(\mathcal{G}) \otimes \mathcal{F}(\mathcal{H}) \mathcal{L},
\]

while \( \mathcal{M}^\mathcal{G} \) will denote the \( \mathcal{F}(\mathcal{G}) \)-module obtained by restriction of the module multiplications to the subalgebra \( \mathcal{F}(\mathcal{H}) \) of \( \mathcal{F}(\mathcal{G}) \).

**Proposition 2.**\(^5\) \( \mathcal{L}^\mathcal{G} \times \mathcal{M} \approx (\mathcal{L} \times \mathcal{M}^\mathcal{G})^\mathcal{G} \).

Proposition 2 shows that the subspace spanned by all the \( (\mathcal{G}, \mathcal{H}) \)-projective modules\(^6\) is an ideal of \( \mathcal{A} \).

In particular, taking \( \mathcal{H} = \{E\} \), the trivial subgroup of \( \mathcal{G} \), we have that the \( \mathcal{F}(\mathcal{G}) \)-projective modules span an ideal \( \mathcal{D} \) of \( \mathcal{A} \), which we shall call the projective ideal of \( \mathcal{A} \).

**Proposition 3.** The projective ideal \( \mathcal{D} \) is semi-simple and finite dimensional.

**Remark.** In the non-modular case, \( \mathcal{D} = \mathcal{A} \) and the proposition reduces

---

\(^3\) See p. 200 of [3] for the proof.

\(^4\) For further explanations, see [3].

\(^5\) See, for instance, theorem 38.5 (ii), p. 268 of [3].

\(^6\) See definition 63.1 (p. 427), and theorem 63.5 (p. 429) of [3].
to Green's result. We shall therefore assume in the proof that $\mathcal{F}$ is of characteristic $\mathfrak{p} \neq 0$.

**Proof.** In proposition 1 take $\mathcal{F}^*$ to be algebraically closed. Then the restriction of the monomorphism (3) to $\mathcal{D}$, embeds $\mathcal{D}$ in the projective ideal $\mathcal{D}^*_{\mathcal{A}}(\mathcal{E}, \mathcal{F}^*, \mathcal{G})$. It will therefore be sufficient to consider $\mathcal{F}$ algebraically closed.

Let $\mathcal{X}_1, \cdots, \mathcal{X}_r$ be the $\varphi$-regular conjugacy classes of $\mathcal{G}$ and let $X_v \in \mathcal{X}_v$ ($v = 1, \cdots, r$). Any $\mathcal{F}(\mathcal{A})$-module class $\{\mathcal{M}\}$ then defines a Brauer character $\chi$, which is completely determined by the values of $\chi(X_v) \in \mathcal{E}$. Write

$$\beta_v(\mathcal{M}) = \chi(X_v).$$

$\beta_v$ can then be extended linearly over $\mathcal{E}$ to give a map $\beta_v : \mathcal{A} \rightarrow \mathcal{E}$. This is readily verified to be a $\mathcal{E}$-algebra homomorphism and so $\beta_v$ is a $\mathcal{G}$-character of $\mathcal{A}$.

Consider now the restrictions $\gamma_v$ of the $\beta_v$ to $\mathcal{D}$. As $\mathcal{F}$ is algebraically closed, the number of different indecomposable projective modules (i.e. indecomposable summands of the regular module) is equal to the number of $\varphi$-regular conjugacy classes $8$, i.e. $r$. Let $\{\mathcal{P}_1\}, \cdots, \{\mathcal{P}_r\}$ be these different classes. The $\{\mathcal{P}_v\}$ are a basis of $\mathcal{D}$. We prove that $\mathcal{D}$ is semisimple by showing that $\bigcap_v \ker \gamma_v = (0)$. Now this last is so if and only if the matrix $(\gamma_v(\mathcal{P}_v))$ is non-singular. But this matrix is precisely the matrix $H$ on p. 599 of [3], and is non-singular as the Cartan matrix $C$ is non-singular.

**Corollary 4.** If $\mathcal{F}$ is algebraically closed, $\mathcal{D}$ is isomorphic to the direct sum of $r$ copies of $\mathcal{E}$.

**Corollary 5.** $\mathcal{D}$ is an ideal direct summand of $\mathcal{A}$.

**Proof.** This follows directly from the fact that $\mathcal{A}, \mathcal{D}$ have unit elements.

### 2. Representations of $\mathcal{V}_4$ over a field characteristic 2

All representations of the group $\mathcal{V}_4(=Z_2 \times Z_2)$ over a field $\mathcal{F}$ of characteristic 2 have been essentially determined by two authors [1], [6]. Let $\mathcal{V}_4$ have generators $X, Y$ satisfying $X^2 = Y^2 = E$, $XY = YX$, with $E$ the identity element. In the group algebra $\mathcal{F}(\mathcal{V}_4)$ write

$$P = X + E, \quad Q = Y + E.$$

Then $P^2 = Q^2 = 0$, $PQ = QP$ and

$$\mathcal{F}(\mathcal{V}_4) \cong \mathcal{F}[P, Q]/(P^2, Q^2) = \mathcal{R},$$

say.

7 See pages 588, 589 of [3] for the definition of Brauer characters, etc.

8 See page 591 of [3].
where \((P^2, Q^2)\) denotes the ideal generated by \(P^2, Q^2\) in the polynomial ring \(\mathbb{F}[P, Q]\).

The following discussion of indecomposable \(R\)-modules is independent of the characteristic of \(\mathbb{F}\).

Let \(A \rightarrow \lambda(A) (A \in R)\) be a representation of \(R\), where the matrices \(\lambda(A)\) have coefficients in \(\mathbb{F}\). One indecomposable such representation is the regular representation whose \(R\)-module class we denote by \(D\). The underlying \(R\)-module is both \(R\)-projective and \(R\)-injective and so is a direct summand of any module in which it occurs as a submodule. The remaining indecomposable representations all satisfy the condition

\[
\lambda(P)\lambda(Q) = 0.
\]

As \((\lambda(P))^2 = (\lambda(Q))^2 = 0\) in any case, by suitable choice of basis, \(\lambda(P), \lambda(Q)\) can be written in the form

\[
\lambda(P) = \begin{bmatrix} 0 & \cdot & \cdot & \ldots & \cdot \\ \cdot & 0 & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \cdot & \cdot & \cdot & \ldots & 0 \end{bmatrix} \quad \text{and} \quad \lambda(Q) = \begin{bmatrix} 0 & \cdot & \cdot & \ldots & \cdot \\ \cdot & 0 & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \cdot & \cdot & \cdot & \ldots & 0 \end{bmatrix}
\]

and so it is sufficient in describing an indecomposable representation to give \(P, Q\). The following cases \(A_n, B_n, C_n(\pi), C_n(\infty)\) arise.
Both $A_0, B_0$ can be interpreted as the class of the trivial $R$-module.

Let $\pi = T^m - u_{m-1}T^{m-1} - \cdots - u_0$ be an irreducible polynomial in the indeterminate $T$ over $\mathcal{F}$, with degree $\pi = m$. Thus we define

\[ C_n(\pi): \quad \overline{P} = \begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
1 & 0 & & \\
0 & 1 & & \\
\vdots & \vdots & \ddots & \vdots \\
0 & & & 1
\end{array} \quad \overline{Q} = \begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
M & N & & \\
0 & 1 & & \\
\vdots & \vdots & \ddots & \vdots \\
0 & & & M
\end{array} \]

where

\[ I = \begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
1 & 0 & & \\
0 & 1 & & \\
\vdots & \vdots & \ddots & \vdots \\
0 & & & 1
\end{array} \quad \overline{P} = \begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
0 & 1 & & \\
0 & 1 & & \\
\vdots & \vdots & \ddots & \vdots \\
0 & & & 1
\end{array} \quad \overline{Q} = \begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
0 & 1 & & \\
0 & 1 & & \\
\vdots & \vdots & \ddots & \vdots \\
0 & & & 1
\end{array} \]

As a convention we shall say that the degree of $\infty$ is 1.

Here $A_n, B_n, C_n(\pi), D$ denote the module classes associated with the respective representations. With the above convention on $\deg(\infty)$, $\pi$ can be considered to range through all irreducible polynomials over $\mathcal{F}$, together with $\infty$. $\overline{Q}$ for $C_n(\pi)$ ($\pi \neq \infty$) is an indecomposable Jordan block, with invariant factors $\pi^n, 1, 1, \cdots$. Indeed, once the $A_n, B_n, D$ have been removed in the break-up of a given module into indecomposables, the decomposition of the remainder can be determined by elementary divisor techniques, suitable allowance being made for $C_n(\infty)$.

If $\mathcal{F}^*$ is the algebraic closure of $\mathcal{F}$, the representation afforded by

* See § 5 of chapter II of [4].
the class $C_n(\pi)$ may break up further over $\mathcal{F}^*$. Say $\mathcal{F}$ has characteristic $p$; let the degree of inseparability of $\pi$ be $t$, and let the reduced degree of $\pi$ be $s$, i.e. $m = sp^t$; let $a_1, \cdots, a_s$ be the different roots of $\pi$ in $\mathcal{F}^*$. Then

$$\pi = \prod_{a=1}^{s} (T-a_a)^{p^t}. \tag{4}$$

The invariant factors for $\tilde{Q}$ in $\mathcal{F}^*$ are

$$\prod_{a} (T-a_a)^{p^t}, 1, 1, \cdots, \tag{5}$$

and $\tilde{Q}$ splits up into $s$ different blocks each of size $n \cdot p^t$. We write

$$C_n(\pi) = \mathcal{F}^* \sum_{a=1}^{s} C_{n \cdot p^t}(T-a_a), \tag{6}$$

implying that we are considering $\mathcal{F}^*(\mathcal{V}_4)$-module classes.

3. Tensor (Kronecker) products of the $\mathcal{F}(\mathcal{V}_4)$-module classes

Methods for calculating the tensor products of the $\mathcal{F}(\mathcal{V}_4)$-indecomposable modules have been given by Bašev in [1] when the field $\mathcal{F}$ is algebraically closed of characteristic 2. The author has found that Bašev's results are correct except for the following case. Let $a \in \mathcal{F}$, $a \neq 0, 1$ (or $\infty$). Then we have

$$C_1(T+a)C_1(T+a) = C_2(T+a), \tag{7}$$

$$C_n(T+a)C_n(T+a) = n(n-1)D + 2C_n(T+a) \quad (n > 1). \tag{8}$$

Our results can be extended to the case where $\mathcal{F}$ is not algebraically closed by using proposition 1. Let $\mathcal{F}^*$ be the algebraic closure of $\mathcal{F}$. Consider, for instance, $C_n(\pi)C_n(\pi)$ ($n > 1$), where $\pi$ is given by (4) with $p = 2$.

$$C_n(\pi)C_n(\pi) = \mathcal{F}^* \sum_{a, \beta=1}^{s} C_{n \cdot 2^t}(T+a_\alpha)C_{n \cdot 2^t}(T+a_\beta) \quad (by\ 5), \tag{9}$$

$$= \mathcal{F}^* \sum_{a=1}^{s} C_{n \cdot 2^t}(T+a_\alpha)C_{n \cdot 2^t}(T+a_\alpha)$$

$$+ \sum_{a \neq \beta} C_{n \cdot 2^t}(T+a_\alpha)C_{n \cdot 2^t}(T+a_\beta).$$

But

$$C_{n \cdot 2^t}(T+a_\alpha)C_{n \cdot 2^t}(T+a_\beta) = \mathcal{F}^* (n \cdot 2^t)(n \cdot 2^t)D \quad (\alpha \neq \beta), \tag{10}$$

and so
Certain representation algebras

\[ C_n(x)C_n(x) = \sum_{i=1}^{s} [n \cdot 2^i (n \cdot 2^i - 1)D + 2C_{n-2^i}(T + d)] \]
\[ + s(s-1)n^2 2^i D \quad \text{(by (7), (8))}, \]
\[ = mn(nm-1)D + 2C_n(x), \]

where \( m = \text{deg} \, x \). Thus by proposition 1 we have

\[ C_n(x)C_n(x) = mn(nm-1)D + 2C_n(x), \]

this being an equation in \( \mathcal{F}(\mathcal{V}_4) \)-module classes.

Let \( x_1 \) denote either \( T, T + 1, \infty \) or any inseparable irreducible polynomial over \( \mathcal{F} \), let \( x_2 \) denote any other irreducible polynomial; let \( x \) denote the general irreducible polynomial of type \( x_1 \) or \( x_2 \).

The results are summarised in the following multiplication table.

<table>
<thead>
<tr>
<th>( n \leq n' )</th>
<th>( A_n )</th>
<th>( B_n )</th>
<th>( C_n(x), \text{deg} , x = m )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{n'} )</td>
<td>( mn'D + A_{n+n'} )</td>
<td>( nn'(n'+1)D + A_{n'-n} )</td>
<td>( mn'mD + C_n(x) )</td>
<td>( (2n'+1)D )</td>
</tr>
<tr>
<td>( B_{n'} )</td>
<td>( n(n'+1)D + B_{n+n'} )</td>
<td>( mn'D + B_{n+n'} )</td>
<td>( mn'D + C_n(x) )</td>
<td>( (2n'+1)D )</td>
</tr>
<tr>
<td>( C_n(x') )</td>
<td>( nn'm'D + C_n(x') )</td>
<td>( nn'm'D + C_n(x') )</td>
<td>( nn'm'D, \text{if} , x \neq x' )</td>
<td>( 2n'm'D )</td>
</tr>
<tr>
<td>( \text{deg} , x' = m' )</td>
<td></td>
<td></td>
<td>( \frac{mn(m'-1)D + 2C_n(x)}{2} )</td>
<td></td>
</tr>
<tr>
<td>( D )</td>
<td>( (2n+1)D )</td>
<td>( (2n+1)D )</td>
<td>( 2nmD )</td>
<td>( 4D )</td>
</tr>
</tbody>
</table>

4. The representation algebra for \( \mathcal{V}_4 \)

We shall now look at \( \mathcal{A}(\mathcal{P}, \mathcal{F}, \mathcal{V}_4) = \mathcal{A} \), where \( \mathcal{F} \) has characteristic 2.

We require that \( \mathcal{P} \) should contain a subring isomorphic to \( Z[2^{-1}] \).

\( A_0 = B_0 \) is the identity \( I \) of \( \mathcal{A} \). Further \( I_D = \frac{1}{4}D \) is an idempotent. Thus \( I_D \) generates the projective ideal which is an ideal direct summand with complement generated by \( I - I_D \). Write

\[ A_n = A_n(I - I_D) = A_n - \frac{2n+1}{4}D, \]
\[ B_n = B_n(I - I_D) = B_n - \frac{2n+1}{4}D, \]
\[ C_n(x) = C_n(x)(I - I_D) = C_n(x) - \frac{nm}{2}D, \]

where \( \text{deg} \, x = m_x \). The multiplication table in the ideal \( (I - I_D) \) is then as follows:
\[ \begin{array}{|c|c|c|c|} \hline n \leq n' & \mathcal{A}_n & \mathcal{B}_n & \mathcal{C}_n(\pi) \\
\hline \mathcal{A}_{n'} & \mathcal{A}_{n+n'} & \mathcal{A}_{n'-n} & \mathcal{C}_n(\pi) \\
\hline \mathcal{B}_{n'} & \mathcal{B}_{n'-n} & \mathcal{B}_{n+n'} & \mathcal{C}_n(\pi) \\
\hline \mathcal{C}_{n'}(\pi') & \mathcal{C}_{n'}(\pi') & \mathcal{C}_{n'}(\pi') & 0, \text{ if } \pi \neq \pi' \\
\hline \end{array} \]

Let \( X = \mathcal{A}_1 \). Then \( X \) is invertible and

\[
X^n = \begin{cases} 
\mathcal{A}_n, & n \geq 0, \\
\mathcal{B}_n, & n < 0,
\end{cases}
\]

with \( X^n X^m = X^{n+m} \), for all integers \( n, m \).

Clearly

\[
X^n \mathcal{C}_{n'}(\pi) = \mathcal{C}_{n'}(\pi'),
\]

for all \( n, \pi, \text{ and } n' > 0 \). Put

\[
\begin{align*}
I_{1, \pi_1} &= \frac{1}{2} \mathcal{C}_1(\pi_1), \\
I_{n, \pi_2} &= \frac{1}{2} (\mathcal{C}_n(\pi_1) - \mathcal{C}_{n-1}(\pi_1)) \quad (n > 1), \\
I_{1, \pi_3} &= \frac{1}{4} (\mathcal{C}_2(\pi_2) - \sqrt{2} \mathcal{C}_1(\pi_2)), \\
I_{2, \pi_3} &= \frac{1}{4} (\mathcal{C}_2(\pi_2) + \sqrt{2} \mathcal{C}_1(\pi_2)), \\
I_{n, \pi_3} &= \frac{1}{2} (\mathcal{C}_n(\pi_2) - \mathcal{C}_{n-1}(\pi_2)) \quad (n > 2).
\end{align*}
\]

The \( I_{n, \pi} \) are mutually orthogonal idempotents. Hence \( \mathcal{A} \) can be written

\[
\mathcal{A} \simeq (\mathcal{P} \left[ X, \frac{1}{X} \right] + \{ \bigoplus_{n, \pi} \mathcal{P} I_{n, \pi} \}) \oplus \mathcal{P} I_D,
\]

where \( X^m I_{n, \pi} = I_{n, \pi} \) (all integers \( m \)) and where \( \{ \bigoplus_{n, \pi} \mathcal{P} I_{n, \pi} \) is the direct sum of ideals isomorphic to \( \mathcal{P} \).

The structure of \( \mathcal{A} \) is somewhat more complicated if \( \mathcal{P} \) merely contains a subring isomorphic to \( \mathbb{Z}[1/2] \), or if \( \mathcal{P} = \mathbb{Z} \). It can be proved that \( \mathcal{A} \) is semi-simple in the Jacobson sense if \( \mathcal{P} \) is a Jacobson ring (Noetherian ring in which every prime ideal is the intersection of maximal ideals), though the quotients \( \mathcal{A}/\mathcal{M} \) (\( \mathcal{M} \) a maximal ideal) may be very varied in nature.

**Theorem.** \( \mathcal{A}(\mathcal{C}, \mathcal{F}, \mathcal{V}_A) \) is G-semisimple.

**Proof.** \( \mathcal{C}[X, 1/X] \) is a principal ideal domain and the maximal ideals have the form \( (X-a), a \in \mathcal{C}, a \neq 0 \). Clearly \( \mathcal{C}[X, 1/X] \) is G-semisimple and so the G-radical of \( \mathcal{A} \) is contained in
Certain representation algebras

\[ \{ \bigoplus_{n,\pi} CI_{n,\pi} \} \oplus CI_D. \]

Write

\[ \mathcal{M}_D = (I - I_D), \]
\[ \mathcal{M}_{n,\pi} = (I - I_{n,\pi}). \]

Then \( \mathcal{A}/\mathcal{M}_D, \mathcal{A}/\mathcal{M}_{n,\pi} \) are isomorphic to \( \mathbb{C} \) and

\[ \mathcal{M}_D \cap (\bigcap_{n,\pi} \mathcal{M}_{n,\pi}) \cap \left( \{ \bigoplus_{n,\pi} CI_{n,\pi} \} \oplus CI_D \right) = (0), \]

and so \( \mathcal{A} \) is \( G \)-semisimple.

Thus there exists a set of \( G \)-characters \( \phi_a \) on \( \mathcal{A} \). We may think of a set of coordinates \( \{ \phi_a(\mathcal{M}) \} \) of a \( \mathbb{F}(\mathcal{V}_4) \)-module class \( \{ \mathcal{M} \} \), which completely determine \( \{ \mathcal{M} \} \) and which are compatible with direct sum and tensor product of modules.

5. Representations of \( \mathcal{A}_4 \) over a field \( \mathbb{F} \) of characteristic 2

We regard \( \mathcal{A}_4 \) (alternating group of 4 symbols) as being an extension of \( \mathcal{V}_4 \) by a cyclic group of order 3. Thus we take generators \( W, X, Y \) satisfying

\[ W^3 = X^2 = Y^2 = E, \quad XY = YX, \]
\[ W^{-2}XW^2 = W^{-1}YW = XY, \]

where \( E \) is the identity element. \( \mathcal{V}_4 \) is the subgroup generated by \( X, Y \).

Let \( \mathbb{F} \) be an algebraically closed field of characteristic 2. By Higman's theorem 1 in [7], every indecomposable \( \mathbb{F}(\mathcal{A}_4) \)-module is a direct summand of the \( \mathbb{F}(\mathcal{A}_4) \)-module induced from an indecomposable \( \mathbb{F}(\mathcal{V}_4) \)-module. We now look at such induced \( \mathbb{F}(\mathcal{A}_4) \)-modules.

A \( \mathbb{F}(\mathcal{V}_4) \)-module \( \mathcal{L} \) (and the corresponding representation of \( \mathbb{F}(\mathcal{V}_4) \)) will be called stable in \( \mathcal{A}_4 \) if the \( \mathbb{F}(\mathcal{V}_4) \)-submodule

\[ W \otimes \mathbb{F}(\mathcal{V}_4) \mathcal{L} \text{ of } (L^{\mathcal{A}_4})_{\mathcal{V}_4} \]

is isomorphic to \( \mathcal{L} \). We now find which indecomposable \( \mathbb{F}(\mathcal{V}_4) \)-modules are stable in \( \mathcal{A}_4 \).

Let

\[ \mathcal{B} \to \lambda(G), \quad G \to \overline{\lambda}(G) \quad (G \in \mathbb{F}(\mathcal{V}_4)) \]

be the representations afforded by the \( \mathbb{F}(\mathcal{V}_4) \)-modules \( \mathcal{L} \) and \( W \otimes \mathcal{L} \) respectively. Choosing bases appropriately, we can write

\[ \overline{\lambda}(G) = \lambda(W^{-1}GW) \quad (G \in \mathbb{F}(\mathcal{V}_4)). \]

If \( P = X + E \) \( Q = Y + E \), it is readily seen that
\[ \bar{\lambda}(P) = \lambda(Q) \]
\[ \bar{\lambda}(Q) = \lambda(P) + \lambda(Q) + \lambda(PQ), \]

and \( \mathcal{L} \) is stable in \( \mathcal{A}_4 \) if and only if the pair \((\bar{\lambda}(P), \bar{\lambda}(Q))\) is similar to \((\lambda(P), \lambda(Q))\).

Now \( \bar{\lambda}(P)\bar{\lambda}(Q) = \lambda(P)\lambda(Q) \). For the representation afforded by the class \( D \) we have \( \lambda(P)\lambda(Q) \neq 0 \), and so \( \bar{\lambda}(P)\bar{\lambda}(Q) \neq 0 \) and \( D \) is stable in \( \mathcal{A}_4 \).

If \( \mathcal{R} \) is any module in the classes \( A_n, B_n, C_n(\pi) \), then so is \( W \otimes \mathcal{R} \), as \( \lambda(P)\lambda(Q) = \lambda(PQ) \) remains 0. In this latter case we must compare the pair \((\lambda(Q), \lambda(P) + \lambda(Q))\) with \((\lambda(P), \lambda(Q))\) under similarity, or, using the notation of § 2, the pair \((Q, P+Q)\) with \((P, Q)\) under independent non-singular transformations on both sides. This can be done using the invariants in § 5 of chapter II of [4]. Thus it can be shown that \( A_n, B_n \) are stable in \( \mathcal{A}_4 \).

As \( \mathcal{F} \) is algebraically closed, \( \pi \) (irreducible) has the form \( T + a \), for \( a \in \mathcal{F} \), or \( \infty \). We write \( C_n(a) \) for \( C_n(\pi) \), where \( a \in \mathcal{F} \cup \{\infty\} \). By elementary divisors (as mentioned in § 2 for \( Q \)), we see that

\[ \{W \otimes \mathcal{L}\} = C_n(\theta(a)), \]

where \( \mathcal{L} \) is in the class of \( C_n(a) \), and where

\[ \theta(a) = \frac{1+a}{a}, \]

with the obvious interpretation when \( a = \infty \) or 0. Note that \( \theta^3(a) = a \).

Thus \( C_n(a) \) is stable if and only if

\[ \theta(a) = a, \]

i.e.

\[ a^2 + a + 1 = 0, \]

or \( a \) is a primitive cube root \( \omega \) of unity in \( \mathcal{F} \). \( \theta \) is a permutation on \( \mathcal{F} \cup \{\infty\} \).

We denote the typical class of transitivity by \( \mu = \{a, \theta(a), \theta^2(a)\} \). However there are two additional classes, \( \{\omega\} \) and \( \{\omega^2\} \).

To obtain the indecomposable \( \mathcal{F}(\mathcal{A}_4) \)-modules we look at \( \mathcal{L}^{\omega^4} \), where \( \mathcal{L} \) is an indecomposable \( \mathcal{F}(\mathcal{V}_4) \)-module. If \( \mathcal{L} \) is not stable in \( \mathcal{A}_4 \), then \( \mathcal{L}^{\omega^4} \) is indecomposable by the theorem in § 2 of [2]. Thus we obtain indecomposable \( \mathcal{F}(\mathcal{A}_4) \)-modules \( C_n^*(\mu) \) such that

\[ (C_n^*(\mu))_{\omega^4} = C_n(a) + C_n(\theta(a)) + C_n(\theta^2(a)). \]

If \( \mathcal{L} \) is stable in \( \mathcal{A}_4 \), then \( \mathcal{L}^{\omega^4} \) splits up into 3 indecomposable, non-isomorphic \( \mathcal{F}(\mathcal{A}_4) \)-modules \( \mathcal{L}^\omega \) (all superscripts will be considered to be integers modulo 3), such that \( (\mathcal{L}^\omega)_{\omega^4} \approx \mathcal{L} \), as in proposition 3 of [2]. Thus we obtain classes

\[ A_0^\omega, A_n^\omega, B_n^\omega, C_n(\omega), C_n(\omega^2), D^\omega \quad (n > 0). \]
In particular \( A_0^\alpha \) may be taken to be the class corresponding to the 1-dimensional representation
\[
W \rightarrow \omega^\alpha \quad (\alpha = 0, 1, 2),
\]
\[
X, Y \rightarrow 1.
\]
Then we can suppose that \( A_0^\alpha \times \{ \mathcal{L}^\beta \} = \{ \mathcal{L}^{\alpha + \beta} \} \). As the \( \mathcal{L}^\alpha \) are extensions of \( \mathcal{L} \), in the corresponding representations it is only necessary to assign a matrix \( \lambda(W) \), to extend the matrix representations as detailed in § 2. If \( \lambda(W) \) is assigned to the representation afforded by \( \mathcal{L}^0 \), then the corresponding matrix for \( \mathcal{L}^\alpha \) is \( \omega^\alpha \lambda(W) \). The author has constructed suitable matrices \( \lambda(W) \) corresponding to classes \( A_n, B_n, D \) (all \( n > 0 \)), but not for \( C_n(\omega) \), \( C_n(\omega^2) \) in general. However for \( C_1(\omega) \) we take
\[
\lambda(W) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix},
\]
and for \( C_0^\alpha(\omega) \) we take
\[
\lambda(W) = \begin{bmatrix} 1 & \omega^2 & 0 \\ \omega^2 & 0 & \omega \\ 0 & \omega & \omega^2 \end{bmatrix}.
\]
For \( C_1^0(\omega^2), C_2^0(\omega^2) \) we replace \( \omega \) by \( \omega^2 \) in these matrices. For \( A_1^0 \) we take
\[
\lambda(W) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.
\]

It should be noted that in general we still have not chosen which of the 3 extensions \( \mathcal{L}^\alpha \) of \( \mathcal{L} \) will be called \( \mathcal{L}^0 \). This choice will be exercised in the next section.

6. The representation algebra for \( A_4 \)

To obtain the structure of \( \mathcal{A}(\mathcal{C}, \mathcal{F}, \mathcal{A}_4) \), where \( \mathcal{F} \) is algebraically closed of characteristic 2, it is not necessary to find explicitly all tensor (Kronecker) products. By proposition 3 and its corollaries it will only be necessary to obtain the products of the \( \mathcal{F}(\mathcal{A}_4) \)-modules modulo the projective ideal \( \mathcal{D} = (D^0, D^1, D^2) \), and all equations in this section will be taken to be modulo \( \mathcal{D} \). Further by restricting the ring multiplications to \( \mathcal{F}(\mathcal{V}_4) \) and considering the corresponding products of the \( \mathcal{F}(\mathcal{V}_4) \)-modules, we see that the multiplication table (9) must be valid on removing the superscripts \( \alpha \).
Now
\[ C^*_n(\mu) = (C_n(a))^{\mu*} \]
when \( a \neq \omega, \omega^* \) and \( \mu = \{a, \theta(a), \theta^*(a)\} \), and so, using proposition 2, we quickly obtain all products involving \( C^*_n(\mu) \). Thus
\[
A^*_m C^*_n(\mu) = C^*_n(\mu), \quad B^*_m C^*_n(\mu) = C^*_n(\mu),
\]
(11)
\[
C^*_m(\mu)C^*_n(\mu') = \begin{cases} 
0, & \text{if } \mu \neq \mu', \\
2C^*_m(\min(m,n)(\mu)), & \text{if } \mu = \mu',
\end{cases}
\]
except that
\[
C^*_1(\mu)C^*_1(\mu) = C^*_2(\mu) \quad \text{for all } \mu \neq \{1, 0, \infty\}.
\]

Also
\[
C^*_1(1, 0, \infty)C^*_1(1, 0, \infty) = 2C^*_1(1, 0, \infty).
\]

We now choose \( A^*_n(n > 1), B^*_n(n > 0); \) to satisfy
\[
A^*_n = (A^*_1)^n, \quad A^*_1B^*_1 = A^*_0, \quad B^*_n = (B^*_1)^n.
\]

Thus we have
\[
A^*_n B^*_m = \begin{cases} 
A^*_{n-m}, & \text{if } n \geq m, \\
B^*_{m-n}, & \text{if } n < m, \text{ etc.}
\end{cases}
\]

A direct calculation shows that
\[
C^*_1(\omega)C^*_1(\omega) = C^*_2(\omega),
\]
(12i)
\[
C^*_1(\omega)C^*_2(\omega) = 2C^*_1(\omega).
\]
(12ii)

As yet \( C^*(\omega) \ (n > 2) \) have not been specified. Say
\[
C^*_1(\omega)C^*_n(\omega) = C^*_1(\omega) + C^*_1(\omega).
\]

Then \( \beta = \gamma \) or not. Choose \( C^*_n(\omega) \) so that one of the following relations is true
\[
C^*_n(\omega)C^*_n(\omega) = \begin{cases} 
2C^*_1(\omega), & \text{or} \\
C^*_1(\omega) + C^*_1(\omega).
\end{cases}
\]
(13i)

If \( n(> 1) \) is such that (13i) is true then for \( n \geq m \geq 1 \), the associativity of multiplication implies that
\[
C^*_m(\omega)C^*_n(\omega) = 2C^*_m(\omega),
\]
(14i)

while if (13ii) is true, then
\[
C^*_m(\omega)C^*_n(\omega) = C^*_1(\omega) + C^*_m(\omega).
\]
(14ii)

Again a direct calculation shows that
\[
A^*_1 C^*_1(\omega) = C^*_1(\omega).
\]
By associativity of multiplication we prove in succession that

\[
\begin{align*}
A_1^0 C_0^0 (\omega) &= C_1^1 (\omega), \\
A_m^0 C_n^0 (\omega) &= C_m^m (\omega), \\
B_m^0 C_n^0 (\omega) &= C_{n-m}^m (\omega)
\end{align*}
\]  
(superscripts are modulo 3).

Similarly

\[
A_1^0 C_1^0 (\omega^2) = C_1^1 (\omega^2),
\]
and so

\[
\begin{align*}
A_m^0 C_n^0 (\omega^2) &= C_{n+2m}^m (\omega^2), \\
B_m^0 C_n^0 (\omega^2) &= C_{n-2m}^m (\omega^2).
\end{align*}
\]  

We now look at the structure of \( \mathcal{A} = \mathcal{A} (\mathbb{C}, \mathcal{F}, \mathcal{A}) \). The projective ideal \( \mathcal{D} \) is isomorphic to \( \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \). The complement to \( \mathcal{D} \) in \( \mathcal{A} \) is isomorphic to \( \mathcal{B} = \mathcal{A} / \mathcal{D} \), and so to find \( \mathcal{B} \) we continue as above modulo \( \mathcal{D} \).

\( A_0^0 \) is the identity element of \( \mathcal{B} \). Let \( \mu \) be a primitive cube root of unity in \( \mathbb{C} \), and write

\[
J_\beta = \frac{1}{3} (A_0^0 + \mu^\beta A_1^0 + \mu^{2\beta} A_2^0) \quad (\beta = 0, 1, 2).
\]

Then

\[
A_0^0 = J_0 + J_1 + J_2,
\]
and the \( J_\beta \) are mutually orthogonal idempotents.

Write

\[
\begin{align*}
A_{n\beta} &= A_n^0 J_\beta, & B_{n\beta} &= B_n^0 J_\beta \\
C_{n\beta} (\omega) &= C_n^0 (\omega) J_\beta, & C_{n\beta} (\omega^2) &= C_n^0 (\omega^2) J_\beta.
\end{align*}
\]

Then

\[
A_n^\alpha J_\beta = \mu^{-\alpha \beta} A_{n\beta}, \quad \text{etc.}
\]

and

\[
A_{n\alpha} A_{m\beta} = \begin{cases} 0, & \text{if } \alpha \neq \beta, \\ A_{(m+n)\beta}, & \text{if } \alpha = \beta, \text{ etc.} \end{cases}
\]

Further

\[
C_n^* (\mu) J_\beta = \begin{cases} C_n^* (\mu), & \text{if } \beta = 0, \\ 0, & \text{if } \beta \neq 0. \end{cases}
\]

Finally the elements (17) and \( C_n^* (\mu) \) together form a basis of \( \mathcal{B} \) over \( \mathbb{C} \).

We now look at the 3 ideal direct summands of \( \mathcal{B} \) generated by the \( J_\beta \).

Set \( Y_\beta = A_{1\beta}, 1/ Y_\beta = B_{1\beta}; \) then \( Y_\beta^m = A_{m\beta} \) etc., and the subalgebra of \( \mathcal{B} J_\beta \) generated by \( A_{n\beta}, B_{n\beta} \) may be written \( \mathbb{C} [Y_\beta, 1/Y_\beta] \), \( Y_\beta \) being regarded as an indeterminate over \( \mathbb{C} \). From (15) and (16)

\[
\begin{align*}
Y_\beta^m C_{n\beta} (\omega) &= \mu^{-m} C_{n\beta} (\omega), \\
Y_\beta^m C_{n\beta} (\omega^2) &= \mu^m C_{n\beta} (\omega^2),
\end{align*}
\]
for any integer, and

\[ C_{n\beta}(\omega)C_{n'\beta}(\omega^2) = 0, \]

for all positive \( n, n' \). From (12i), (12ii), (14i),

\[
\begin{align*}
C_{1\beta}(\omega)C_{1\beta}(\omega) &= C_{2\beta}(\omega), \\
C_{1\beta}(\omega)C_{2\beta}(\omega) &= 2C_{1\beta}(\omega) \\
C_{2\beta}(\omega)C_{2\beta}(\omega) &= 2C_{2\beta}(\omega).
\end{align*}
\]

As in § 4, set

\[
\begin{align*}
I_{1\beta}(\omega) &= \frac{1}{3}(C_{2\beta}(\omega) + \sqrt{2}C_{1\beta}(\omega)) \\
I_{2\beta}(\omega) &= \frac{1}{3}(C_{2\beta}(\omega) - \sqrt{2}C_{1\beta}(\omega)),
\end{align*}
\]

and these are mutually orthogonal idempotents. For \( n > 2 \), if we have the situation of (13i), then

\[ C_{n\beta}(\omega)C_{n\beta}(\omega) = 2C_{n\beta}(\omega), \]

and we write

\[ C_{n\beta}(\omega) = \frac{1}{2}C_{n\beta}(\omega). \]

In case (13ii), we have

\[ C_{n\beta}(\omega)C_{n\beta}(\omega) = (u^{-\beta} + u^{-2\beta})C_{n\beta}(\omega), \]

and we write

\[ C_{n\beta}(\omega) = \frac{1}{u^{-\beta} + u^{-2\beta}}C_{n\beta}(\omega). \]

Then the \( C_{n\beta}(\omega) \) are idempotents. To obtain orthogonal idempotents we put

\[ I_{3\beta} = C_{3\beta}(\omega) - I_{1\beta}(\omega) - I_{2\beta}(\omega), \]

and for \( n > 3 \)

\[ I_{n\beta}(\omega) = C_{n\beta}(\omega) - C_{(n-1)\beta}(\omega). \]

Then all the \( I_{n\beta}(\omega) \) are mutually orthogonal idempotents. \( I_{n\beta}(\omega^2) \) are similarly defined. From (11), (18), we can proceed as in § 4 and \( I_{n0}(\mu) \) are defined.

Hence \( \mathcal{A} \) has the following structure.

\[
\begin{align*}
\begin{cases}
\mathcal{C} \left[ Y_0, \frac{1}{Y_0} \right] + \left\{ \bigoplus_{\phi = \omega, \omega^2, \mu} \mathcal{C}I_{n0}(\phi) \right\} \\
\bigoplus_{\beta = 1, 2} \mathcal{C} \left[ Y_\beta, \frac{1}{Y_\beta} \right] + \left\{ \bigoplus_{\phi = \omega, \omega^2} \mathcal{C}I_{n\beta}(\phi) \right\} \\
\bigoplus \left( \mathcal{C} \oplus \mathcal{C} \oplus \mathcal{C} \right),
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
Y_\beta^m I_n(\omega^2) &= u^{-2\beta m} I_{n\beta}(\omega^2), \\
Y_0^m I_{n0}(\mu) &= I_{n0}(\mu);
\end{align*}
\]

the last term is the projective ideal \( \mathcal{D} \).
As in § 4 this is $G$-semisimple. As far as $G$-semisimplicity is concerned we may now drop the restriction that $F$ is algebraically closed. For, if not, let $F^*$ be the algebraic closure of $F$. Then, by (3), $A(C, F, A_4)$ can be regarded as embedded in $A(C, F^*, A_4)$. Thus the restriction of the $G$-characters to the subalgebra will ensure the $G$-semisimplicity of $A(C, F, A_4)$.

**Theorem.** $A(C, F, A_4)$ is $G$-semisimple for all fields $F$ of characteristic 2.

7. Ring-tensor-product representation algebras

Given a commutative ring $R$ and two $R$-modules $M, M'$ then the tensor product

$$M \otimes_R M'$$

can also be defined to be an $R$-module. This product is then commutative, associative and distributes over direct sum $\oplus$. If we now take the set of $R$-modules which satisfy the ascending and descending chain conditions, this set is closed under $\oplus$, $\otimes$ and the Krull-Schmidt theorem is applicable. If $P$ is any commutative ring with an identity element, then, as in § 1, we can define the representation algebra $A(P, R)$ to be the free $P$-module generated by the set of all $R$-indecomposable isomorphic classes $\{M\}$, equipped this time with the multiplication

$$\{M\} \{M'\} = \{M \otimes_R M'\}.$$

If $R$ is a Dedekind domain, then the indecomposable $R$-modules of finite length have the form

$$R/\mathcal{D}_n,$$

where $\mathcal{D}_n$ is any non-zero prime ideal of $R$. Further it is readily seen that

$$R/\mathcal{D}_\alpha \otimes_R R/\mathcal{D}_\beta = \begin{cases} (0), & \text{if } \alpha \neq \beta, \\ R/\mathcal{D}_\alpha^{\min(n,m)}, & \text{if } \alpha = \beta. \end{cases}$$

Write then

$$I_{\alpha 1} = \{R/\mathcal{D}_\alpha\},$$

$$I_{\alpha n} = \{R/\mathcal{D}_\alpha^n\} - \{R/\mathcal{D}_\alpha^{n-1}\} \quad (n > 1).$$

Then

$$A(P, R) = \bigoplus_{\alpha, n \geq 1} PI_{\alpha n}.$$

This algebra does not have an identity.

Another case which can readily be deduced from the above is that of the quotient of the Dedekind domain $R$ by an ideal $I = \mathcal{D}_n^\bullet$, where only a finite number of $n_\alpha$ are strictly positive ($n_\alpha > 0$). Then the indecomposable $R/I$-modules of finite length have the form
$$\mathcal{A}(\mathcal{P}, \mathcal{R}|\mathcal{S}) = \bigoplus_{a=1}^{n_a} \mathcal{P} I_{A_m}.$$  

This algebra has finite rank over $\mathcal{P}$ and has an identity.

We now take $\mathcal{R} = \mathcal{F}[P, Q]/(P^2, Q^2)$, as in § 2 (of arbitrary characteristic). We assume for simplicity that $\mathcal{F}$ is algebraically closed. Then the different classes are $A_n$, $B_n$, $C_n(a)$, $D$, where $a \in \mathcal{F} \cup \{\infty\}$.

The multiplication table under $\otimes_{\mathcal{A}}$ is as follows.

<table>
<thead>
<tr>
<th>$n \leq m$</th>
<th>$A_n$</th>
<th>$B_n$</th>
<th>$C_n(a)$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_m$</td>
<td>$(n+1)(m+1)A_0$</td>
<td>$(m+2)(n-1)A_0 + A_{m-n+1}$</td>
<td>$(m+1)nA_0$</td>
<td>$A_m$</td>
</tr>
<tr>
<td>$B_m$</td>
<td>$m(n+1)A_0$ $(m &gt; n)$</td>
<td>$(n-1)(m-1)A_0 + B_{m+n-1}$</td>
<td>$n(m-1)A_0 + C_n(a)$</td>
<td>$B_m$</td>
</tr>
<tr>
<td>$C_{m'}(a')$</td>
<td>$(n+1)mA_0$</td>
<td>$m(n-1)A_0 + C_{m'}(a')$</td>
<td>$n(m-1)A_0 + C_n(a)$ $(a = a')$</td>
<td>$C_{m'}(a')$</td>
</tr>
<tr>
<td>$D$</td>
<td>$A_n$</td>
<td>$B_n$</td>
<td>$C_n(a)$</td>
<td>$D$</td>
</tr>
</tbody>
</table>

$D$ is the identity element in $\mathcal{A} = \mathcal{A}(\mathcal{G}, \mathcal{R})$. $A_0$, $B_1$ are obvious idempotents and

$$D = A_0 + [B_1 - A_0] + [D - B_1]$$

is a splitting of the identity into mutually orthogonal idempotents. The elements $A_0$, $D - B_1$ generate ideal direct summands each isomorphic to $\mathcal{G}$. Write

$$\tilde{A}_n = (B_1 - A_0)A_n,$$
$$\tilde{B}_n = (B_1 - A_0)B_n,$$
$$\tilde{C}_n(a) = (B_1 - A_0)C_n(a).$$

Then the multiplication table in the ideal $(B_1 - A_0)$ generated by $B_1 - A_0$ is as follows.

<table>
<thead>
<tr>
<th>$n \leq m$</th>
<th>$A_n$</th>
<th>$B_n$</th>
<th>$C_n(a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_m$</td>
<td>0</td>
<td>$A_{m-n+1}$</td>
<td>0</td>
</tr>
<tr>
<td>$B_m$</td>
<td>0 $(n &gt; m)$</td>
<td>$B_{m+n-1}$</td>
<td>$C_n(a)$</td>
</tr>
<tr>
<td>$C_{m'}(a')$</td>
<td>0</td>
<td>$C_{m'}(a')$</td>
<td>$C_n(a)$ $(a = a')$</td>
</tr>
</tbody>
</table>
Certain representation algebras

Place $T = B_2$. Then $B_{n+1} = T^n$. Place $I_{1a} = C_1(a)$, $I_{na} = C_n(a) - C_{n-1}(a)$ ($n > 1$). Then the ideal generated by the $\{C_n(a)\}$ is $\oplus_{a,n>0} C_{I_{na}}$. The subalgebra generated by the $\{B_n\}$ may be written $\mathcal{C}[T]$, where the identity element is $B_1$. Write $U_n = A_n$. Then the structure of the ideal $(B_1 - A_0)$ may be written

$$\mathcal{C}[T] + \left( \bigoplus_{n>0} \mathcal{C}U_n \right) + \left( \bigoplus_{a,n>0} C_{I_{na}} \right),$$

where

$$U_n I_{ma} = 0, \quad U_n U_m = 0,$$

$$TU_{m+1} = U_m, \quad TI_{ma} = I_{ma},$$

and the $I_{ma}$ are mutually orthogonal idempotents.

The Jacobson radical of this algebra is nonzero as it contains $U_1(U_1^2 = 0)$. Hence, a fortiori, $\mathcal{A}(\mathcal{C}, \mathcal{R})$ is not $G$-semisimple. When the characteristic of $\mathcal{F}$ is 2, we get a direct comparison between the two kinds of representation algebras that can be formed from $\mathcal{F}(\mathcal{V}_4)$-modules.

References


University of Sydney.