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MORE ON COMPACT HAUSDORFF SPACES AND FINITARY DUALITY

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It is an old conjecture by P. Bankston that the category **CompHaus** of compact Hausdorff spaces and their continuous maps is not dually equivalent to any elementary **P**-class of finitary algebras (taken as a category with all homomorphisms between its members as maps), where *elementary* means defined by first order axioms, and a **P**-class is one closed under arbitrary (cartesian) products. One motivation for this conjecture is the fact that such a dual equivalence would make ultracopowers of compact Hausdorff spaces correspond to ultrapowers of finitary algebras, and one might expect this to have contradictory consequences.

As a possible step towards proving his conjecture, Bankston [**2**] showed that no elementary **SP**-class of finitary algebras can be dually equivalent to **CompHaus**. However, it was subsequently proved in [**1**] that the same holds for any **SP**-class of finitary algebras, using an argument independent of ultrapowers.

The present note, at last, provides a proof of the original conjecture. In actual fact, we show here, somewhat more generally, that **CompHaus** has no full subcategory strictly larger than the category **BooS** of Boolean spaces which is dually equivalent to an elementary **P**-class of finitary algebras. Given that **BooS** itself has such a dual equivalence, namely classical Stone Duality, this may be considered an interesting comment on Boolean spaces besides settling the conjecture.

To put our result in the proper perspective, one should take note of the following additional facts concerning the category **CompHaus**. On the one hand, this is indeed dually equivalent to a **P**-class of finitary algebras, as is implicit in the lattice-theoretical duality of **CompHaus** recently given by Banaschewski [**2**], and on the other hand, it is actually dually equivalent to an equationally defined class, albeit of infinitary algebras ([**5**]), which can be described by four finitary and a single ω-ary operation ([**6**]). This shows how narrowly Bankston's conjecture misses to be false.

In the following, **C** is any full subcategory of **CompHaus** containing **BooS**, **K** any elementary **P**-class of finitary algebras, and $T:\mathbf{C} \rightarrow \mathbf{K}$ a dual equivalence. Further, $P = T\mathbf{1}$ is the algebra in **K** corresponding to the

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one-point space **1** in **C**. P and its powers, which belong to **K** by hypothesis, play a crucial role in our arguments.

The first result we need is due to Bankston [**2**].

LEMMA 1. *P is finite.*

Proof. Because **BooS** $\subseteq$ **C**, the usual copowers of **1** in **CompHaus** belong to **C**: for any set S, the S-fold copower of **1** in **CompHaus** is the Stone-Čech compactification βS of S taken as a discrete space, and βS is Boolean. Now, if $\mathfrak{U}$ is any ultrafilter on a set I then the ultracopower of **1** relative to $\mathfrak{U}$ in **CompHaus**, i.e., the projective limit $\underset{\longleftarrow}{\lim}\ \beta S(S \in \mathfrak{U})$, is just the intersection of the βS, $S \in \mathfrak{U}$, viewed as subspaces of βI. Since this is the one-point space $\{\mathfrak{U}\}$, the ultracopower is again **1** ([**2**]), and hence the same holds in **C**. From this and the fact that **K** is a **P**-class it follows for the algebra P that the map

$$P \rightarrow \underset{\longrightarrow}{\lim}\ P^S(S \in \mathfrak{U})$$

induced by the diagonal maps $P \rightarrow P^S$ is an isomorphism. Moreover, since **K** is elementary the colimit $\underset{\longrightarrow}{\lim}\ P^S(S \in \mathfrak{U})$ in **K** is actually the usual ultrapower of P relative to $\mathfrak{U}$. Hence the natural map of P into any of its ultrapowers is an isomorphism, and this shows P is finite ([**4**]).

Lemma 1 has the important consequence that P is compact as a discrete space and hence, for any power P^I of P, each element $s \in P^I$ has a unique continuous extension $\hat{s}:\beta I \rightarrow P$. Next we describe the effect of homomorphisms between powers of P based on this observation.

LEMMA 2. *For any continuous map $\phi:\beta J \rightarrow \beta I$ with discrete I and J, the homomorphism $T\phi:P^I \rightarrow P^J$ is the map $s \rightsquigarrow \hat{s}\phi|J$.*

Proof. First consider the case $J = 1 = \beta J$ and let $u \in \beta I$ be the element picked by ϕ. Then, we have the homomorphism $h:P^I \rightarrow P$ such that $h(s) = \hat{s}(u)$, and the claim is that $T\phi = h$. Now, by the finiteness of P, each $s \in P^I$ occurs in the image of some homomorphism $P^n \rightarrow P^I$ with natural number n, and hence it is sufficient to show that $T\phi$ and h have equal composites with any such $P^n \rightarrow P^I$. For this, take any continuous map $\sigma:\beta I \rightarrow n$ and let $k \in n$ be the number picked by $\sigma\phi:1 \rightarrow n$, i.e., $k = \sigma(u)$. Then, $T\phi T\sigma = T(\sigma\phi)$ is the kth projection $P^n \rightarrow P$. On the other hand, $T\sigma:P^n \rightarrow P^I$ is the unique homomorphism making all triangles

$$\begin{array}{ccc} P^n & \longrightarrow & P^I \\ & \searrow \quad \swarrow & \\ \mathrm{pr}_{\sigma(i)} & P & \mathrm{pr}_i \end{array} \qquad (i \in I)$$

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commute (pr for projection) and therefore

$$T\sigma(c) = c\sigma|I \quad \text{for each } c \in P^n.$$

Now, by the definition of h and the obvious fact that $c\sigma = (c\sigma|I)^\wedge$, we have

$$h(c\sigma|I) = (c\sigma|I)^\wedge (u) = c\sigma(u) = c(k)$$

and hence $hT\sigma$ is also the kth projection $P^n \rightarrow P$. Since $\sigma:\beta I \rightarrow n$ was arbitrary, this proves $T\phi = h$.

Now take J arbitrary and, for any $j \in J$, let $\tau:1 \rightarrow \beta J$ be the map which picks out j. Then

$$T(\phi\tau)(s) = \hat{s}(\phi(j))$$

by the preceding discussion, and since $T\tau$ is the projection $P^J \rightarrow P$ at j this implies

$$T\phi(s)(j) = \hat{s}\phi(j) \quad \text{for each } j \in J,$$

i.e., $T\phi(s) = \hat{s}\phi|J$ as claimed.

In the following, we use the relation $a \equiv b$ on any $A \in \mathbf{K}$ to mean that each homomorphism $A \rightarrow P$ maps a and b equally. Obviously, for any homomorphism $h:A \rightarrow P^I$, if $a \equiv b$ then $h(a) = h(b)$ for all $a, b \in A$. On the other hand, if $i:A \rightarrow P^I$ is a homomorphism such that $i = T\phi$ where ϕ is an onto map then $i(a) = i(b)$ implies $a \equiv b$: for any $h:A \rightarrow P$, one then has a $g:P^I \rightarrow P$ for which $h = gi$ because $\mathbf{1}$ is obviously projective in $\mathbf{C}$ relative to onto maps and this makes P injective in $\mathbf{K}$ relative to the corresponding homomorphisms.

An important power of P is $Q = P^{|P|}$ where $|P|$ is the underlying set of P. Below, $e \in Q$ will always be the identity map on P. For the motivation of the following result, one should bear in mind that in the case of Stone Duality, P is the two-element Boolean algebra and Q the Boolean algebra free on $\{e\}$, the point being that, in general, Q still has somewhat similar properties.

LEMMA 3. *For any $A \in \mathbf{K}$ and $a \in A$, there exists a homomorphism $h:Q \rightarrow A$ such that $h(e) \equiv a$.*

Proof. Any compact Hausdorff space X appears in a coequalizer diagramme in **CompHaus** of the form

$$\beta J \underset{\psi}{\overset{\phi}{\rightrightarrows}} \beta I \underset{\text{onto}}{\overset{\sigma}{\rightarrow}} X$$

with discrete spaces I and J, and since βI and βJ belong to $\mathbf{C}$ this is also a coequalizer diagramme in $\mathbf{C}$ if $X \in \mathbf{C}$. As a result, any $A \in \mathbf{K}$ appears in an equalizer diagramme

$$A \xrightarrow{i} P^I \underset{T\psi}{\overset{T\phi}{\rightrightarrows}} P^J$$

where $i = T\sigma$ comes from an onto map σ. Then, for any $a \in A$, let $c = i(a)$ and $k:Q \to P^I$ be the homomorphism such that $k(x) = xc$, the composite $I \xrightarrow{c} P \xrightarrow{x} P$, i.e., $k = T\hat{c}$ by Lemma 3. Since i is the equalizer of $T\phi$ and $T\psi$ in **K**, we have $T\phi(c) = T\psi(c)$ and hence, by Lemma 2,

$$T\phi(xc) = (xc)^{\wedge}\phi|J = x\hat{c}\phi|J = xT\phi(c) = xT\psi(c) = T\psi(xc),$$

where the important step $(xc)^{\wedge} = x\hat{c}$ results from the fact that $x\hat{c}$ is obviously a continuous extension of xc to βI. It follows that $(T\phi)k = (T\psi)k$, and therefore k factors through i by a homomorphism $h:Q \to A$. Now, for $e \in Q$, we then have

$$ih(e) = k(e) = c = i(a),$$

and as has already been noted this implies $h(e) \equiv a$ because i comes from an onto map.

The homomorphisms $Q \to A$, for a given $A \in$ **K**, may not, for all one can tell, cover A but at least they do so up to the relation $\equiv$, and this is sufficient for the following.

LEMMA 4. *Q is a generator in* **K**.

Proof. Consider any distinct homomorphisms $f, g:A \to B$ in **K** and let $j:B \to P^J$ be any monomorphism in **K**. Then one also has $jf \neq jg$ and hence $jf(a) \neq jg(a)$ for some $a \in A$. Now, if $h:Q \to A$ is a homomorphism such that $h(e) \equiv a$ as provided by Lemma 3 then

$$jfh(e) = jf(a) \neq jg(a) = jgh(e),$$

hence $jfh \neq jgh$ and therefore $fh \neq gh$. Hence the homomorphisms $Q \to A$ distinguish, by composition, the homomorphisms $A \to B$ for any A, $B \in$ **K**, and this says Q is a generator in **K**.

The last lemma supplies the decisive tool for the proof of our main result:

PROPOSITION. *The only full subcategory* **C** $\supseteq$ **BooS** *of* **CompHaus** *dually equivalent to an elementary* **P***-class* **K** *of finitary algebras is* **BooS**.

Proof. Since Stone Duality provides a dual equivalence of the type in question for **BooS**, we only have to show that any dual equivalence T:**C** $\to$ **K** as stated implies that each $X \in$ **C** is Boolean. Now, by Lemma 4, the space $S \in \mathbf{C}$ corresponding to Q by T is a cogenerator of $\mathbf{C}$, and as $\mathbf{1} \in \mathbf{C}$ this obviously implies, for any $X \in \mathbf{C}$, that the continuous maps $X \to S$ separate the points of X. The latter, however, makes X Boolean because S is Boolean, being a copower of $\mathbf{1}$ since Q is a power of P.

Remark 1. On the algebraic side, it is clear from the above arguments that **K** could also be an elementary **P**-class of finitary structures, allowing relations as well as operations. In the same vein, **K** need not really be a **P**-class; it is already sufficient if it contains all powers for each of its members. Actually, in the final analysis, **K** need only be a concrete category with concrete powers and ultrapowers, i.e., there is a faithful functor from **K** to the category of sets and **K** has powers and ultrapowers which are preserved by it.

Remark 2. On the space side, it is obviously enough that βS belongs to **C** for any discrete S to make all the lemmas work, and the conclusion then is that $\mathbf{C} \subseteq \mathbf{BooS}$. Taking Stone Duality into account, this raises the question which full subcategories of the category of Boolean algebras containing all complete atomic Boolean algebras can be category equivalent to an elementary **P**-class. It did not seem in the spirit of this note to pursue ramifications of this type but we observe that the elementary **P**-class of all atomic Boolean algebras is an obvious example.

Remark 3. Stone Duality furnishes examples of full subcategories of **CompHaus** dually equivalent to elementary **P**-classes, in fact of Boolean algebras, which differ very much from the **K** appearing in the above consideration. A typical case is the category of all Boolean spaces without isolated points which is dually equivalent to the elementary **P**-class of all atomless Boolean algebras: the latter neither has an initial object (which is P in the above **K**) nor any member with only finitely many endomorphisms.

Remark 4. If **K** satisfies the stronger condition that it is closed under arbitrary direct limits rather than just ultraproducts then the duality T becomes entirely like Stone Duality even if **C** is only assumed to contain all Stone-Čech compactifications of discrete spaces. In this case, any $X \in \mathbf{C}$ is the projective limit $\varprojlim X/\Theta$ in **C** where X/Θ ranges over the finite quotient spaces of X, and hence

$$TX = \varinjlim T(X/\Theta) = \varinjlim P^{X/\Theta}$$

in **K**. On the other hand, for the algebra $C(X, P)$ of continuous P-valued functions on X one has, in the category of all algebras of the type of P,

$$C(X, P) = \lim_{\rightarrow} C(X/\Theta, P) = \lim_{\rightarrow} P^{X/\Theta}$$

and the present hypothesis on **K** then implies that $C(X, P) \in \mathbf{K}$. Therefore $TX \cong C(X, P)$ for all $X \in \mathbf{C}$, necessarily natural in X, so that T is equivalent to the functor $C(-, P)$. This raises the obvious question: for which **C** and (finite) P is the class of all $C(X, P)$, $X \in \mathbf{C}$, elementary? Note that, in the case where $\mathbf{C} = \mathbf{BooS}$ and P is the two-element Boolean algebra, $C(-, P)$ is exactly the functor of Stone Duality from Boolean spaces to Boolean algebras.

Remark 5. An alternative, entirely different proof of Bankston's conjecture, involving rather more extensive background from model and category theory but which does not seem to yield the complete result of the present Proposition, has been given by J. Rosicky [7].

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