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SOME RESULTS ON SEMI-PERFECT GROUP RINGS

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The aim of this paper is to find necessary and sufficient conditions on a group G and a ring A for the group ring AG to be semi-perfect. A complete answer is given in the commutative case, in terms of the polynomial ring $A[X]$ (Theorem 5.8). In the general case examples are given which indicate a very strong interaction between the properties of A and those of G. Partial answers to the question are given in Theorem 3.2, Proposition 4.2 and Corollary 4.3.

1. Preliminaries. Given a group G and a ring A (with unit element) the group ring AG is the free left A-module with the elements of G forming a basis. Multiplication is defined by

$$(\sum a_i g_i)(\sum b_j g_j') = \sum\sum (a_i b_j)(g_i g_j').$$

Alternatively AG may be thought of as all functions from G to A with finite support. The function r is identified with the element $\sum_{g \in G} r(g)g$, and the support of r, denoted Supp(r), is the set $\{g \in G : r(g) \neq 0\}$. The fundamental ideal of AG, denoted Δ_{AG} (or simply Δ if no confusion will arise), is the ideal generated by $\{1 - g : g \in G\}$. Then $AG/\Delta \cong A$. If H is a subgroup of G then ωH will denote the right ideal of AG generated by $\{1 - h : h \in H\}$. If H is a normal subgroup of G then ωH is an ideal and $AG/\omega H \cong A(G/H)$. For further details, see [**3**].

If A is any ring then JA will denote the Jacobson radical of A and $\bar{A} = A/JA$. A ring A is semi-perfect if $\bar{A}$ is artinian and every idempotent in $\bar{A}$ is the image of an idempotent in A. Since homomorphic images of semi-perfect rings are semi-perfect [**2**], if AG is semi-perfect then so is A, and so is $A(G/N)$ for every normal subgroup $N \trianglelefteq G$. Moreover if A and B are semi-perfect rings, then their direct sum $A \oplus B$ is semi-perfect.

If E is a division ring the characteristic of E will be denoted char(E).

2. Reduction to the case: A is local. A ring A is called *local* if A has a unique maximal left ideal M. In this case $M = JA$ and $\bar{A}$ is a division ring. If A is any ring, a *local idempotent* in A is an idempotent e such that eAe is a local ring.

THEOREM (Mueller [**4**]). *The following are equivalent for a ring A:*
(1) *A is semi-perfect.*

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(2) *The unit* $1 \in A$ *is a sum of orthogonal local idempotents.*

(3) *Every primitive idempotent is local, and there is no infinite set of orthogonal idempotents in* A.

LEMMA 2.1. *Let* A *be a ring and let* $\{e_1, \ldots, e_n\}$ *be a set of orthogonal idempotents in* A *whose sum is* 1. *Then* A *is semi-perfect if and only if* e_iAe_i *is semi-perfect for each* i.

Proof. Let $A' = \sum_{i=1}^{n} e_iAe_i$. Then A' is a subring of A, and is the direct sum of the rings e_iAe_i. Thus each e_i is central in A', and, it is sufficient to show that A is semi-perfect if and only if A' is semi-perfect.

Suppose A is semi-perfect. Then clearly A' has no infinite set of orthogonal idempotents. Let f be a primitive idempotent in A' and suppose $f = f_1 + f_2$ in A, where f_1, f_2 are orthogonal idempotents. Since $f = \sum_{i=1}^{n} fe_i$ and the fe_i are orthogonal idempotents, $f = fe_i$ for some i. But then $f_1 = ff_1f = e_iff_1fe_i \in A'$. Similarly $f_2 \in A'$. Thus, $f_1 = f$ or $f_2 = f$ and f is primitive in A. Since A is semi-perfect, fAf is a local ring. But $fA'f = fe_iA'e_if = fe_iAe_if = fAf$. Thus A' is semi-perfect.

Conversely suppose A' is semi-perfect. Then $1 \in A'$ can be written $1 = f_1 + \ldots + f_m$ where the f_i are orthogonal local, and hence primitive, idempotents in A'. As above, $f_i = f_ie_j$ for some j, and $f_iAf_i = f_iA'f_i$ is a local ring. Thus A is semi-perfect.

PROPOSITION 2.2. *Let* A *be semi-perfect and let* $\{e_1, \ldots, e_n\}$ *be a set of orthogonal local idempotents in* A *whose sum is* 1. *Let* G *be any group. Then* AG *is semi-perfect if and only if* $(e_iAe_i)G$ *is semi-perfect for each* i.

Proof. $(e_iAe_i)G \cong e_iAGe_i$ and the result follows from Lemma 2.1.

3. Necessary conditions on G. Here we show that if AG is semi-perfect then G is a torsion group and there are no infinite chains of finite subgroups of G whose orders are units in A. In view of the reduction in §2 and the fact that $\bar{A}G$ is semi-perfect whenever AG is, we may assume that A is a division ring.

If p is a prime, a *p'-group* is a group which has no element of order p, and a *p'-element* of a group is an element whose order is not divisible by p. If $p = 0$, every group is a p'-group and every element of a group is a p'-element.

If $p = 0$, by a *p-subgroup* or *Sylow p-subgroup* of G we mean the trivial subgroup.

LEMMA 3.1. *Let* R *be any ring such that* $\bar{R} = R/JR$ *is artinian, and let* $x \in R$. *Let* $\{x_n\}$ *be the sequence:* $x_0 = x$, $x_{i+1} = x_i - x_i^2$ *for* $i \geqq 0$. *Then for some* n, $1 - x_n$ *has a right inverse in* R.

Proof. The chain $x_1R \supseteq x_2R \supseteq \ldots$ of right ideals in R gives rise to a chain

$$(x_1R + JR)/JR \supseteq (x_2R + JR)/JR \supseteq \ldots$$

of right ideals in $\bar{R}$. Thus for some $n \geqq 1$, $(x_nR + JR)/JR = (x_{n+1}R + JR)/JR$, and $x_n \in x_{n+1}R + JR$. For some $r \in R$ and $y \in JR$, $x_n = (x_n - x_n^2)r + y$. Now $1 - y = (1 - x_n)(1 + x_n r)$ has a right inverse in R and so $1 - x_n$ has a right inverse in R.

THEOREM 3.2. *Let A be a division ring of characteristic $p \geqq 0$ and let G be a group. If AG is semi-perfect then G is a torsion group and there is a positive integer n such that no chain of finite p'-subgroups of G has length greater than n.*

Proof. Suppose $x \in G$ has infinite order. Construct a sequence $\{x_m\}$ in AG as in Lemma 3.1, starting with $x_0 = x$. Then for some m, $1 - x_m$ has a right inverse in AG. Since $1 - x_m \in KH$ where K is the prime subfield of A and H is the subgroup of G generated by x, and since KH is a direct summand of AG as left KH-modules, $1 - x_m$ has a right inverse in KH.

Multiplying by a high enough power of x we obtain the factorization $x^r = (1 - x_m)g(x)$ in the polynomial ring $K[x]$. This is impossible since $1 - x_m$ has 2 distinct terms: 1 and $\pm x^{2^m}$. Thus G must be a torsion group.

If $H = \{h_1, \ldots, h_r\}$ is a finite p'-subgroup of G then $r = r \cdot 1$ is a unit in A and $e_H = (1/r)(h_1 + \ldots + h_r)$ is an idempotent in AG. Moreover if $K \leqq H$ then $e_H e_K = e_K e_H = e_H$. Let n be the length of a composition series for the right $\overline{AG}$-module $\overline{AG}$ and suppose

$$\{1\} \subsetneqq H_1 \subsetneqq \ldots \subsetneqq H_{n+1}$$

is a chain of $n + 1$ finite p'-subgroups of G. Let $e_i = e_{H_i}$, $i = 1, \ldots, n + 1$. Then $AG \supseteq e_1 AG \supseteq \ldots \supseteq e_{n+1}AG$. Reducing modulo $J(AG)$ we obtain $\overline{AG} \supseteq \bar{e}_1\overline{AG} \supseteq \ldots \supseteq \bar{e}_{n+1}\overline{AG}$. Thus for some i, $\bar{e}_i\overline{AG} = \bar{e}_{i+1}\overline{AG}$. Then $e_i - e_{i+1}$ is an idempotent in $J(AG)$ and so $e_i = e_{i+1}$. This implies $H_i = H_{i+1}$, a contradiction.

COROLLARY 3.3. *Let A be a division ring of characteristic $p \geqq 0$ and let G be a locally finite group. If AG is semi-perfect then every p'-subgroup of G is finite.*

Remark. It is not known whether AG semi-perfect implies that G is locally finite. If K is a field of characteristic $p > 0$ and G is a non-locally-finite p-group, then KG will be semi-perfect (even local) if $J(KG) = \Delta$. However the problem of determining $J(KG)$ appears to be very difficult. (See [**5**, p. 121].)

From now on we consider only locally finite groups.

4. Some sufficient conditions.

Here we see that if G is locally finite we may consider a suitable subgroup of G rather than all of G.

LEMMA 4.1. *Let A be a ring, G a group and N a normal subgroup of G such that G/N is locally finite. Then $J(AN) \subseteq J(AG)$.*

Proof. Let $x \in J(AN)$, $r \in AG$. We show that $1 - xr$ has a right inverse in AG. Let G' be the subgroup of G generated by N and $\mathrm{Supp}(r)$. Then G'/N is finitely generated, hence finite. Let

$$G'/N = \{g_1N, g_2N, \ldots, g_nN\}$$

where $g_1 = 1$. Then $\{g_1, g_2, \ldots, g_n\}$ is a basis for the free right AN-module AG'. Thus the endomorphism ring of AG' as a module is the matrix ring $AN_{(n)}$. For each $y \in AG'$ let λ_y be the matrix corresponding to left multiplication by y. Then $\lambda : AG' \to AN_{(n)}$ is a ring homomorphism. In particular λ_x is the diagonal matrix with entries $x, g_2^{-1}xg_2, \ldots, g_n^{-1}xg_n$, each of which is in $J(AN)$ since $J(AN)$ is invariant under automorphisms of AN. Thus $\lambda_x \in (JAN)_{(n)} = J(AN_{(n)})$ and for some $f \in AN_{(n)}$, $(1 - \lambda_x\lambda_r)f = 1$. Regarding these as endomorphisms and applying them to $1 \in AG'$ yields $(1 - xr) \cdot f(1) = 1$. Then $f(1) \in AG' \subseteq AG$ is the required inverse of $1 - xr$.

PROPOSITION 4.2. *Let A be a local ring with* $\operatorname{char}(\bar{A}) = p > 0$ *and let G be a locally finite group. Let N be a normal p-subgroup of G and let H be any subgroup of G such that $NH = G$. If AH is semi-perfect, then so is AG.*

Proof. Let $\pi : AG \to \overline{AG}$ be the canonical epimorphism. If $g \in G$ then for some $n \in N$, $h \in H$ we have $g = nh = (n - 1)h + h \in \omega N + AH$. Thus $AG = \omega N + AH$. Since $(JA)G \subseteq J(AG)$, π may be factored into

$$AG \xrightarrow{\pi_1} \bar{A}G \xrightarrow{\pi_2} \overline{AG}$$

where Ker $\pi_2 = J(\bar{A}G)$. Now $\Delta_{\bar{A}N}$ is a nil ideal, hence $\Delta_{\bar{A}N} \subseteq J(\bar{A}N) \subseteq J(\bar{A}G)$. Thus

$$\Delta_{AN} \subseteq \pi_1^{-1}(\Delta_{\bar{A}N}) \subseteq \pi_1^{-1}(J(\bar{A}G)) = J(AG)$$

and $\omega N = \Delta_{AN}AG \subseteq J(AG)$. It follows that $AG = J(AG) + AH$ and $\pi(AH) = \overline{AG}$. By [**3**, Proposition 9], $AH \cap JAG \subseteq JAH$. But $AH/(AH \cap JAG) \cong \overline{AG}$ is semi-simple. Thus $JAH = AH \cap JAG$ and $\overline{AH} \cong \overline{AG}$.

If AH is semi-perfect then $\overline{AG}$ is artinian. Let $\bar{x}^2 = \bar{x}$ in $\overline{AG}$. Then $\bar{x} = \pi(e)$ for some $e^2 = e$ in $AH \subseteq AG$. Thus AG is semi-perfect.

In [**5**] Passman asks: if K is a field when is KG semi-perfect? The next result provides a partial answer in a somewhat more general setting.

COROLLARY 4.3. *Let A be a local perfect ring with* $\operatorname{char}(\bar{A}) = p \geqq 0$ *and let G be a locally finite group. If G has a p-subgroup of finite index then AG is semi-perfect.*

Proof. G has a normal p-subgroup N of finite index and a finite subgroup F such that $NF = G$. Then AF is perfect [**6**], hence semi-perfect and so AG is semi-perfect.

5. Abelian groups. If G is an abelian torsion group and p is a prime then $G \cong G_p \times H$ where G_p is the Sylow p-subgroup of G and H consists of all p'-elements of G. Hence $G/G_p \cong H$ is a p'-group.

LEMMA 5.1. (Burgess [**2**]). *Let A be a local ring with* $\operatorname{char}(\bar{A}) = p \geqq 0$. *Let G be an abelian group and let G_p be the Sylow p-subgroup of G. Then AG is semi-perfect if and only if $A(G/G_p)$ is semi-perfect and in this case G/G_p is finite.*

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Proof. This follows easily from Proposition 4.2 and Corollary 3.3.

We now show that if G is a finite abelian group of exponent n and if C_n is the cyclic group of order n then AG is semi-perfect if and only if AC_n is semi-perfect. Then necessary and sufficient conditions for AC_n to be semi-perfect are given when A is commutative, in terms of the polynomial ring $A[X]$.

We may assume that A is semi-perfect and n is a unit in A. Thus $J(AG) = (JA)G$ and $\overline{AG} = \bar{A}G$, an artinian ring. To prove that AG is semi-perfect it is sufficient to prove either that idempotents lift from $\bar{A}G$ to AG or that every primitive idempotent in AG is local. If e is any idempotent in AG then ne is a unit in $eAGe$ and $\overline{eAGe} = \bar{e}\bar{A}G\bar{e}$.

Let g be an element of order n in an abelian group G, let F be an algebraically closed field whose characteristic does not divide n and let z be a primitive nth root of unity in F. For $i = 0, \ldots, n-1$ let

$$\epsilon_i = \frac{1}{n}\sum_{j=0}^{n-1} z^{ij}g^j.$$

We show that the ϵ_i are orthogonal idempotents whose sum is 1, and that if z^i is a primitive mth root of 1 then $g\epsilon_i$ is a primitive mth root of ϵ_i.

Since $z^i g\epsilon_i = \epsilon_i$, $\epsilon_i^2 = \epsilon_i$. If $i \neq j$ let $\epsilon_i\epsilon_j = (1/n^2)\sum_{t=0}^{n-1} a_t g^t$. Then

$$z^{i-j}a_t = z^{i-j}\sum_{k=0}^{n-1} z^{ik}z^{j(t-k)} = z^{jt}z^{i-j}\sum_{k=0}^{n-1} z^{(i-j)k} = a_t.$$

Since $z^{i-j} \neq 1$, $a_t = 0$ and hence $\epsilon_i\epsilon_j = 0$. Let $\sum_{i=0}^{n-1}\epsilon_i = (1/n)\sum_{t=0}^{n-1} b_t g^t$. Then $z^t b_t = z^t\sum_{i=0}^{n-1} z^{it} = b_t$. If $0 < t < n$, $z^t \neq 1$ and hence $b_t = 0$. Thus

$$\sum_{i=0}^{n-1}\epsilon_i = \frac{1}{n}\cdot n\cdot 1 = 1.$$

If z^i is a primitive mth root of 1 then $g^m\epsilon_i = g^m z^{im}\epsilon_i = \epsilon_i$, but if $0 < r < m$ then $\epsilon_i = g^r z^{ir}\epsilon_i \neq g^r\epsilon_i$ since $z^{ir} \neq 1$ and $\epsilon_i \neq 0$.

For each $m|n$ let $e_m = \sum\epsilon_i$ where the sum is taken over all i such that z^i is a primitive mth root of 1 and let $e_m' = \sum\epsilon_i$ where the sum is taken over all i such that $z^{im} = 1$. Then $\{e_m : m|n\}$ is an orthogonal set of idempotents whose sum is 1. Since $e_m\epsilon_i = \epsilon_i$ whenever z^i is a primitive mth root of unity, ge_m is a primitive mth root of e_m. Clearly $e_m' = \sum_{d|m} e_d$. Since $z^{im} = 1$ if and only if $s|i$ where $s = n/m$, $e_m' = \sum_{j=0}^{m-1}\epsilon_{sj}$. Let

$$e_m' = \frac{1}{n}\sum_{t=0}^{n-1} c_t g^t.$$

Then $c_t = \sum_{j=0}^{m-1} z^{sjt}$. If $m|t$, $z^{sjt} = 1$ and $c_t = m$. If $m \nmid t$, then, since $z^{st}c_t = c_t$ and $z^{st} \neq 1$, $c_t = 0$. Thus

$$e_m' = \frac{m}{n}[1 + g^m + g^{2m} + \ldots + g^{n-m}].$$

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If $F = \mathbf{C}$, the complex numbers, then for each $m|n$, $ne_m' \in \mathbf{Z}G$ where $\mathbf{Z}$ denotes the integers. Since $e_m = e_m' - \sum e_d$ where the sum is taken over all $d|m$, $d < m$, we see by induction that $ne_m \in \mathbf{Z}G$.

Let A be any ring in which n is a unit and let A' be the subring $\{t \cdot 1 : t \in \mathbf{Z}\}$. Then $A' \cong \mathbf{Z}$ or $A' \cong \mathbf{Z}/(r)$ for some r relatively prime to n. In either case, for some $p \nmid n$ there are homomorphisms

$$\mathbf{Z} \rightarrow A' \rightarrow \mathbf{Z}/(p) \rightarrow F$$

where F is the algebraic closure of $\mathbf{Z}/(p)$, which extend to homomorphisms $\mathbf{Z}G \rightarrow A'G \rightarrow FG$. In AG, we may define inductively for each $m|n$, $e_m' = (m/n)[1 + g^m + g^{2m} + \ldots + g^{n-m}]$ and $e_m = e_m' - \sum e_d$ where the sum is taken over all $d|m$, $d < m$. Then $ne_m \in A'G$ for each $m|n$. Using the homomorphisms defined above, $(ne_m)^2 = n(ne_m)$, $(ne_m)(ne_d) = 0$ if $m \neq d$, $\sum_{m|n} ne_m = n$, and $g^m(ne_m) = ne_m$. Hence in AG, $e_m^2 = e_m$, $e_m e_d = 0$ if $m \neq d$, $\sum_{m|n} e_m = 1$ and $g^m e_m = e_m$. If $g^r e_m = e_m$ in AG for some r, $0 < r < m$ then $g^r(ne_m) = ne_m$ in $A'G$, hence in FG. Thus $g^r e_m = e_m$ in FG, a contradiction. It follows that ge_m is a primitive mth root of unity in AGe_m.

LEMMA 5.2. *Let $e \neq 0$ be a primitive idempotent in AG and let $m|n$. Then ge is a primitive mth root of unity in $eAGe$ if and only if $e = e_m e$. In this case $\overline{ge}$ is a primitive mth root of unity in $\overline{eAGe}$.*

Proof. Since $(ge)^n = g^n e = e$, ge is a primitive dth root of unity in $eAGe$ for a unique $d|n$. Since e is primitive and $e = \sum_{m|n} e_m e$, $e = e_m e$ for a unique $m|n$. We show that $d = m$.

Since $(ge_m)^m = e_m$, $(ge)^m = (ge_m e)^m = e_m e = e$. Thus $d|m$. Since $g^d e = e$, $e_d' e = e$. If $d < m$ then $e = e_d' e_m e = 0$, a contradiction. Thus $d = m$.

In this case $\overline{eAGe} = \bar{e}\bar{A}G\bar{e}$ and $\overline{ge} = g\bar{e}$ in $\bar{A}G$. Then $\bar{e} = \bar{e}_m\bar{e}$ and the above argument applied in $\bar{A}G$ shows that $g\bar{e}$ is a primitive mth root of unity in $\bar{e}\bar{A}G\bar{e}$.

LEMMA 5.3. *Let A be a local ring, G a group and e an idempotent in AG such that $eAGe \subseteq eA \cap Ae$ and $e(1)$ is central and not a zero-divisor in A. Let $A' = \{a \in A : ea = ae\}$. Then $eAGe \cong A'$ as rings and A' is local.*

Proof. If $x \in eAGe$ then $x = ea$ for a unique $a \in A$. Define $f : eAGe \rightarrow A$ by $f(ea) = a$. Clearly f preserves sums and $\ker f = 0$. If $ea \in eAGe$ then $eae = ea$. Thus $f(ea \cdot eb) = f(eab) = ab$ and f preserves products. This proves that $eAGe \cong \operatorname{Im} f$.

Clearly $A' \subseteq \operatorname{Im} f$. Let $a \in \operatorname{Im} f$. Then $ea \in eAGe \subseteq eA \cap Ae$ and so $ea = a'e$ for some $a' \in A$. Thus $e(1)a = a'e(1) = e(1)a'$ and $a = a' \in A'$. This completes the proof that $eAGe \cong A'$.

Finally if $a' \in A'$ is a unit in A, then a' is a unit in A'. Thus the set of non-units in A' is precisely $A' \cap JA$, an ideal of A'. It follows that A' is local.

LEMMA 5.4. *Let A be a local ring with $\operatorname{char}(\bar{A}) = p \geqq 0$. Let $G = \langle g \rangle$ be a cyclic group of order n, $p \nmid n$. Let $m|n$ and suppose A has a primitive mth root of* *unity a such that $\bar{a}$ is a primitive* mth *root of unity in $\bar{A}$. Then AGe_m is semi-perfect.*

Proof. Since $AGe_m' = AGe_m \oplus AG(e_m' - e_m)$ it is sufficient to show that AGe_m' is semi-perfect.

For $i = 1, \ldots, m$ let

$$f_i = \left(\frac{1}{m}\right) \sum_{j=0}^{m-1} a^{ij} g^j e_m'.$$

Since $a^i g f_i = f_i$, $f_i^2 = f_i$. If $i \neq k$ then $0 < |i - k| < m$. Thus $\bar{a}^{i-k} \neq \bar{1}$ in $\bar{A}$ and $a^{i-k} - 1$ is a unit in A. Now

$$f_i f_k = \left(\frac{1}{m^2}\right) \sum_{j=0}^{m-1} \sum_{t=0}^{m-1} a^{ij} a^{k(t-j)} g^j g^{t-j} e_m' = \left(\frac{1}{m^2}\right) \sum_{t=0}^{m-1} a^{kt} x g^t e_m'$$

where

$$x = \sum_{j=0}^{m-1} a^{(i-k)j}.$$

But $a^{i-k} x = x$ and so $x = 0$. Thus $f_i f_k = 0$. Moreover

$$\sum_{i=1}^{m} f_i = \left(\frac{1}{m}\right) \sum_{j=0}^{m-1} \left(\sum_{i=1}^{m} a^{ij}\right) g^j e_m' = 1 e_m',$$

the unit element of AGe_m'.

Finally, $f_i AGe_m' f_i = f_i AG f_i$. Since $a^i g f_i = f_i$, $g f_i = a^{-i} f_i \in A f_i$. Thus $AG f_i = A f_i$. Similarly $f_i AG = f_i A$, and so $f_i AG f_i \subseteq f_i A \cap A f_i$. Moreover $f_i(1) = (1/m)(m/n) a^0 = 1/n$, a central unit in A. By Lemma 5.3, $f_i AG f_i$ is local. Thus AGe_m' is semi-perfect.

LEMMA 5.5. *Let g and h be commuting elements in a group G, of orders s and t respectively, and let u = L.C.M.(s, t). Then for some integer r, gh^r has order u.*

Proof. The group $\langle g, h \rangle$ is a finite abelian group of exponent u. Hence $\langle g, h \rangle = Y \times Z$ where $Y = \langle y \rangle$ is a cyclic group of order u and $z^u = 1$ for all $z \in Z$. Let $g = (y^a, z_1)$ and $h = (y^b, z_2)$. Since g and h generate $Y \times Z$, y^a and y^b generate Y. Thus G.C.D. $(a, b, u) = 1$. If $u | a$ let $r = 1$. Otherwise let r be the product of all primes which divide u but not a. A check of possible prime factors reveals that G.C.D. $(a + br, u) = 1$. Thus $gh^r = (y^{a+br}, z_1 z_2^r)$ has order u.

LEMMA 5.6. *Let A be a ring and let $G = C_n$. If AG is semi-perfect then so is $A(G \times G)$.*

Proof. Without loss of generality we may assume that A is local and n is a unit in A. Let g generate G and let $H = \langle h \rangle$ denote the second copy of G. For each $m | n$ define $e_m \in AG$ as at the beginning of this section and define $f_m \in AH$ in a corresponding way using h in place of g.

Let e be a primitive idempotent in $A(G \times H)$. We show that e is local. Now $e = e e_s f_t$ for a unique $s, t | n$. Thus, by Lemma 5.2, in the multiplicative group $\langle ge, he \rangle$, ge has order s and he has order t. Let u = L.C.M.(s, t) and let r be an integer such that $gh^r e$ has order u. The automorphism of $G \times H$ which sends gh^r to g and h to h extends to an automorphism θ of $A(G \times H)$. Since $\theta(e)A(G \times H)\theta(e) \cong eA(G \times H)e$ it is sufficient to show that $\theta(e)$ is a local idempotent.

Since e is a primitive idempotent, so is $\theta(e)$. In $\langle g\theta(e), h\theta(e)\rangle$, $g\theta(e) = \theta(gh^r e)$ has order u and $h\theta(e) = \theta(he)$ has order t. By Lemma 5.2, $\theta(e) = \theta(e)e_u f_t$. Now $A(G \times H)e_u f_t \cong (AGe_u)Hf_t$ in a natural way. Since AGe_u is semi-perfect the unit element e_u is a sum of orthogonal local idempotents. If f is a local idempotent in AGe_u then $f(AGe_u)Hf_t f \cong (fAGe_u f)Hf_t$ is semi-perfect by Lemmas 5.2 and 5.4. Thus $(AGe_u)Hf_t$ is semi-perfect by Lemma 2.1. It follows that

$$\theta(e)A(G \times H)\theta(e) = \theta(e)A(G \times H)e_u f_t \theta(e)$$

is a local ring and $A(G \times H)$ is semi-perfect.

PROPOSITION 5.7. *Let A be a ring and let G be a finite abelian group of exponent n. Then AG is semi-perfect if and only if AC_n is semi-perfect.*

Proof. Since AC_n is a homomorphic image of AG, if AG is semi-perfect then so is AC_n.

Conversely suppose AC_n is semi-perfect. If $r \geqq 2$ then $AC_n{}^r \cong (AC_n{}^{r-2})(C_n \times C_n)$ and $AC_n{}^{r-1} \cong (AC_n{}^{r-2})C_n$. By Lemma 5.6 and induction $AC_n{}^r$ is semi-perfect for all $r > 0$. But AG is a homomorphic image of $AC_n{}^r$ for some r. Thus AG is semi-perfect.

THEOREM 5.8. *Let A be a commutative local ring with* $\operatorname{char}(\bar{A}) = p \geqq 0$ *and let G be an abelian group with Sylow p-subgroup G_p. Then AG is semi-perfect if and only if G/G_p is a finite group of exponent n and every monic factor of $X^n - 1$ in $\bar{A}[X]$ can be lifted to a monic factor of $X^n - 1$ in $A[X]$.*

Proof. By Lemma 5.1 and Proposition 5.7 we may assume $G = C_n$ and n is a unit in A. Then $AG \cong A[X]/(X^n - 1)$ and $\overline{AG} = \bar{A}G \cong \bar{A}[X]/(X^n - 1)$. Since n is a unit in $\bar{A}$, $X^n - 1$ has no multiple roots in any extension of $\bar{A}$. Thus if $X^n - 1 = f(X)g(X)$ in $\bar{A}[X]$ then $f(X)$ and $g(X)$ are relatively prime. By [**1**, Theorem 19] idempotents in $\bar{A}[X]/(X^n - 1)$ lift to idempotents in $A[X]/(X^n - 1)$ if and only if every monic factor of $X^n - 1$ in $\bar{A}[X]$ lifts to a monic factor of $X^n - 1$ in $A[X]$.

6. Examples. In this section it is shown that for a given ring A, the class of groups G for which AG is semi-perfect is not closed under taking subgroups or direct products.

Let g generate C_2, the 2-element group. If A is a local ring and $\operatorname{char}(\bar{A}) \neq 2$ then $(1 + g)/2$ and $(1 - g)/2$ are local idempotents in AC_2 whose sum is 1. Thus AC_2 is semi-perfect. If $\operatorname{char}(\bar{A}) = 2$ then AC_2 is semi-perfect by Proposition 4.2.

LEMMA 6.1. *If A is semi-perfect and S_3 is the symmetric group of degree 3 then AS_3 is semi-perfect.*

Proof. We may assume A is local. If char$(\bar{A}) = 3$ let N be the subgroup of order 3 and let H be a subgroup of order 2 in S_3. Then $S_3 = NH$ and AS_3 is semi-perfect by Proposition 4.2.

If char$(\bar{A}) \neq 3$, let g generate N and h generate H, and let $e = (1 + g + g^2)/3$, a central idempotent. Then

$$AS_3 = AS_3e \oplus AS_3(1 - e).$$

Since $AS_3(1 - e) = \omega N$, $AS_3e \cong AS_3/\omega N \cong A(S_3/N) = AC_2$. Thus AS_3e is semi-perfect.

Let $f_1 = (1 - g)(1 + h)/3$ and let $f_2 = (1 - e) - f_1$. Then f_1 and f_2 are orthogonal idempotents whose sum is $1 - e$. Also for $i = 1, 2, f_iAS_3(1 - e)f_i = f_iAS_3f_i \subseteq f_iA \cap Af_i$ and $f_i(1) = 1/3$. By Lemma 5.3, $f_iAS_3f_i$ is local. Thus $AS_3(1 - e)$ is semi-perfect.

Now we exhibit a local ring A such that AC_3 is not semi-perfect. Let

$$A = \{a/b : a, b \in \mathbf{Z} \text{ and } 7 \nmid b\},$$

a subring of the rationals. Then $\bar{A}$ is the field with 7 elements. In $\bar{A}[X]$, $X^3 - \bar{1} = (X - \bar{1})(X - \bar{2})(X - \bar{4})$ but in $A[X]$, $X^3 - 1 = (X - 1)(X^2 + X + 1)$. Since $X^2 + X + 1$ is irreducible over A, AC_3 is not semi-perfect.

For our second example we let

$$A = \{x/y : x, y \in \mathbf{Z}[i] \text{ and } (2 + i) \nmid y \text{ in } \mathbf{Z}[i]\},$$

a subring of the complex numbers. Then $\bar{A}$ is the field with 5 elements. In $\bar{A}[X]$, $X^3 - 1 = (X - \bar{1})(X^2 + \bar{1}X + \bar{1})$ and $X^8 - 1 = (X - \bar{1})(X + \bar{1})(X - \bar{\imath})(X + \bar{\imath})(X^2 - \bar{\imath})(X^2 + \bar{\imath})$, and the quadratic factors are irreducible. Since these factorizations can be lifted to $A[X]$, AC_3 and AC_8 are semi-perfect.

Now $C_3 \times C_8 = C_{24}$. In $A[X]$, $X^{24} - 1$ has the irreducible factor $X^4 - iX^2 - 1$ but in $\bar{A}[X]$, $X^4 - \bar{\imath}X^2 - \bar{1} = X^4 + \bar{2}X^2 + \bar{9} = (X^2 + \bar{2}X + \bar{3})(X^2 - \bar{2}X + \bar{3})$. Thus AC_{24} is not semi-perfect.

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