REMARK ON THE TRICOMI EQUATION

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§ 1. As an application of the Carleman-type estimation Hörmander [4], p. 221, has proved the following:
A solution (distribution) of the Tricomi equation
\[ \frac{\partial^2 u}{\partial t^2} + t \frac{\partial^2 u}{\partial x^2} = 0 \]
in an open set \( \Omega \) in \( \mathbb{R}^{2} \) belongs to \( C^{\infty}(\Omega) \) if it is in \( C^{\infty}(\Omega_{-}) \) where
\( \Omega_{-} = \{(x, t); (x, t) \in \Omega, t < 0\} \).

In this note we shall consider the same problem for the inhomogeneous Tricomi equation
\[ \frac{\partial^2 u}{\partial t^2} + t \frac{\partial^2 u}{\partial x^2} = f(x, t) \]
in a different manner. The existence of the solution in the generalized sense is well known. Furthermore we shall consider the propagation of analyticity. More precisely, the solution \( u \) is analytic in \( \Omega \) if it is analytic in \( \Omega_{-} \) and if \( f(x, t) \) is analytic in \( \Omega \) (Theorem 3.1). We shall use the results of [2] and [5] in the proof.

§ 2. The following theorem is obtained from the results of Berezin [2].

**Theorem 2.1.** Consider the following (backward) Cauchy problem:
\[ (2.1) \quad u_{tt} + tu_{xx} = f(x, t) \quad \text{in } \mathcal{D}, \]
\[ (2.2) \quad u(x, 0) = \varphi(x), \quad u_{t}(x, 0) = \psi(x) \quad \text{in } a \leq x \leq b \]
where \( \mathcal{D} \) denotes a domain in the region \( t < 0 \) bounded by characteristics passing through \((a, 0)\) and \((b, 0)\), \((a < b)\). Assume \( f(x, t) \) and \( f_{x}(x, t) \) are continuous in \( \mathcal{D} \) and the initial data \( \varphi(x), \psi(x) \) are thrice continuously differentiable in \([a, b]\). Then there exists one and only one solution

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$u(x,t)$ of the problem (2.1), (2.2) having continuous second derivatives in $\bar{D}$. Furthermore, if $f(x,t)$ and $\psi(x), \varphi(x)$ are infinitely differentiable in $\bar{D}$ and in $[a, b]$ respectively, then the solution $u(x,t)$ is an infinitely differentiable function in $\bar{D}$.

By virtue of Theorem 2.1 it is shown that there exists a fundamental solution $E(x,t)$ for the backward Cauchy problem for the equation $Lu = u_{tt} + tu_{xx} = 0$. That is, there exists a distribution $E(x,t)$ in the region $t \leq 0$ such that

\begin{align}
LE &= E_{tt} + tE_{xx} = 0 \quad \text{for } t < 0, \\
E(x,0) &= 0, \quad E_t(x,0) = \delta_x.
\end{align}

In fact, take $f(x,t) = 0, \varphi(x) = 0$ and

$$
\psi(x) = \begin{cases} 0 & x < 0 \\ x^t/4! & x \geq 0 \end{cases}
$$

in Theorem 2.1. Then there exists a solution $v(x,t)$ for the problem (2.1), (2.2) with these data having second continuous derivatives in the region $t \leq 0$. The desired fundamental solution is given by

$$
E(x,t) = \frac{\partial}{\partial x^t}v(x,t) \quad t \leq 0,
$$

where differentiation in $x$ is interpreted in the sense of distributions. By Theorem 2.1 and (2.5) we have

\begin{align}
\text{supp. } E(x,t) &\subset \{(x,t); -\frac{3}{2}(-t)^{3/2} \leq x \leq \frac{3}{2}(-t)^{3/2}, t \leq 0\}, \\
E(\cdot, t) &\in C([-T, 0]; \mathcal{D}'(R_x)), \\
E_t(\cdot, t) &\in C([-T, 0]; \mathcal{D}'(R_x))
\end{align}

for any $T > 0$, where $\mathcal{D}'(R_x)$ denotes the space of distributions in $R_x$.

Furthermore, by using the partial hypoellipticity of the Tricomi operator $L$ in $t$ (cf. [4], §§2.2, 4.3), we have the following.

**Corollary 2.1.** Let $\Omega$ be an open set in $R^3_x$, such that $\{(x,0); a < x < b\} \subset \Omega$. If $u \in \mathcal{D}'(\Omega)$ satisfies

\begin{align}
Lu &= u_{tt} + tu_{xx} = 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega_+ = \{(x,t); (x,t) \in \Omega, t > 0\}.
\end{align}
Then \( u = 0 \) in \( \Omega_+ \cap (\bar{D} \cup \Omega) \) where \( D \) denotes a domain in the region \( t < 0 \) bounded by characteristics passing through \((a,0)\) and \((b,0)\).

For the proof we apply Theorem 2.1 by regularizing \( u \) with respect to \( x \).

§3. Let \( \Omega \) be an open set in \( \mathbb{R}^2 \) which intersects \( x \)-axis.

**Theorem 3.1.** Let \( u = u(x,t) \in \mathcal{D}'(\Omega) \) be a solution of the equation

\[
Lu = u_{tt} + tu_{xx} = f(x,t) \quad \text{in } \Omega
\]

with \( f \in C^\infty(\Omega) \). Then \( u \in C^\infty(\Omega) \) if it is in \( C^\infty(\Omega_-) \) where \( \Omega_- = \{(x,t); (x,t) \in \Omega, t < 0\} \). Furthermore, \( u \) is an analytic function in \( \Omega \) if it is analytic in \( \Omega_- \) and if \( f(x,t) \) is analytic in \( \Omega \).

We shall prove this theorem in several steps. First we shall show that \( u(x,0) \in C^\infty \{x; (x,0) \in \Omega\} \).

Assume \( \{(x,0); 0 \leq x \leq b\} \subset \Omega, (0 < b) \). If we take \( T > 0 \) sufficiently small then the closed domain \( \bar{D} \) bounded by \( \{(x,0); 0 \leq x \leq b\} \), characteristics passing through \((0,0)\) and \((b,0)\) and \( \{(x,-T); -\infty < x < +\infty\} \) is contained in \( \Omega \cap \{(x,t); t \leq 0\} \). Let \( u(x,t) \) and \( f(x,t) \) be functions given in Theorem 3.1 and \( b, T \) be sufficiently small, then by the usual way (cf. [3]) we have

\[
u(x,0) = \int E_t(x-y, -T)u(y, -T)dy - \int E(x-y, -T)u_t(y, -T)dy
\]

\[-\int_{-T \leq \tau \leq 0} E(x-y, \tau)f(y, \tau)d\tau, \quad 0 < x < b,
\]

where the integral is taken in the sense of distributions. We note that there exists \( u(x,0) = \lim_{t \to 0} u(\cdot, t) \) in \( \mathcal{D}'(0 < x < b) \) by the partial hypoellipticity of \( L \) in \( t \) (cf. [4], §4). The formula (3.2) is justified because of the assumptions for \( u, f \) and the properties of \( E(x,t) \): (2.6), (2.7), (2.8). Thus we have proved that \( u(x,0) \in C^\infty(0, b) \), and hence

\[
u(x,0) \in C^\infty \{x; (x,0) \in \Omega\}.
\]

Similarly, if \( u \) and \( f \) are analytic in \( \Omega_- \) and \( \Omega \) respectively, then we see that \( u(x,0) \) is analytic in \( \{x; (x,0) \in \Omega\} \). We omit the detail.

In the next section we shall show that

\[
u \in C^\infty(\Omega \cap \{(x,t); t \geq 0\})
\]
from which we see that \( u(x,0) \) and \( u_t(x,0) \) are in \( C^\infty(x; (x,0) \in \Omega) \). Then, applying Theorem 2.1 and Corollary 2.1, we have

\[
(3.4) \quad u \in C^\infty(\Omega \cap \{(x,t); t \leq 0\}).
\]

By (3.3), (3.4) and noting that the form of the equation is \( u_{tt} + tu_{xx} = f \) in \( \Omega \) we have \( u \in C^\infty(\Omega) \) by the usual method of calculation (cf. § 4).

In the analytic case, from the assumption the \( u(x,0) \) is analytic in \( \{x; (x,0) \in \Omega\} \) we shall show, in the next section, \( u = u(x,t) \) is analytic in \( \Omega \cap \{(x,t); t \geq 0\} \) from where we have \( u(x,0), u_t(x,0) \) are analytic in \( \{x; (x,0) \in \Omega\} \). Then by Cauchy-Kowalevski theorem and Corollary 2.1, \( u \) is analytic in a neighbourhood of the \( x \)-axis contained in \( \Omega \). On the other hand, \( u \) is analytic in \( \Omega_+ = \{(x,t) \in \Omega, t > 0\} \) because it is a solution of an elliptic equation in \( \Omega_+ \). Thus \( u \) is analytic in \( \Omega \).

§ 4. It remains for us to prove the regularity property of the solution \( u \) in \( \Omega \cap \{(x,t); t \geq 0\} \).

**Theorem 4.1.** Let \( f \in C^\infty(\Omega) \) (\( \in C^\omega(\Omega) \)) and \( u \in \mathcal{D}'(\Omega) \) such that

\[
(4.1) \quad Lu = u_{tt} + tu_{xx} = f(x,t) \quad \text{in } \Omega,
\]

\[
(4.2) \quad u(x,0) = \psi(x) \in C^\infty(x; (x,0) \in \Omega) \quad (\in C^\omega(x; (x,0) \in \Omega)).
\]

Then we have \( u \in C^\infty(\Omega \cap \{(x,t); t \geq 0\}) \) (\( \in C^\omega(\Omega \cap \{(x,t); t \geq 0\}) \)). Here \( C^\omega \) denotes the set of analytic functions.

To prove this theorem we use the method employed in [5], §§ 5, 6. We note that it is sufficient to prove the case \( u(x,0) = \psi(x) = 0 \). First we prepare the following theorem which is derived by a direct computation. Take \( G = (a < x < b) \times [0, T) \) such that \( G \subset \Omega \) and introduce the notation:

\[
(4.3) \quad \|v\|_{\mathcal{S}(G)}^2 = \sum_{j=0}^2 \|D^jv\|_{L^2(G)}^2 + \|t^{1/2}v_{xt}\|_{L^2(G)} + \|t^{1/2}v_{xx}\|_{L^2(G)}^2 + \|tv_{xx}\|_{L^2(G)}^2.
\]

(\( \mathcal{S}(G) \)) is a Hilbert space with the norm \( \|\cdot\|_{\mathcal{S}(G)} \).

**Theorem 4.2** (cf. [5], Theorem 4.2). There exists a constant \( C > 0 \) such that

\[
(4.4) \quad \|v\|_{\mathcal{S}(G)} \leq C\|Lv\|_{L^2(G)}
\]

for all \( v \in \mathcal{S}(G) \) with \( \text{supp. } v \subset G \) and \( v(x,0) = 0 \).
Suppose \( f(x,t) \in C^\infty(\Omega) \), then by the partial hypoellipticity of \( L \) in \( t \) (cf. [4], § 4.3) we conclude that for any \( r \geq 2 \) there exists a number \( \beta = \beta(u,r) \) such that
\[
\zeta u \in H_{(r,\beta)}(G) = H_{(r,\beta)}(R^2)|_G
\]
for any \( \zeta = \zeta(x,t) \in C^\infty(G) \). For the notation \( H_{(r,\beta)}(R^2) \), we refer to [4], § 2.5.

For a real number \( s \) we define an operator \( T_s \):
\[
\tilde{T}_s \psi(x,t) = (1 + |\xi|^2)^{\alpha} \hat{\psi}(\xi, t),
\]
where \( \psi \in \mathscr{S}'(R^2_{x,t} \cap \{ t \geq 0 \}) \) and \( \hat{\psi}(\xi, t) \) denotes the partial Fourier transformation of \( \psi \) with respect to \( x \). (cf. [4], § 1.7.)

For any \( x_0 \in (a,b) \) take \( \zeta \in C^\infty(G) \) such that \( \zeta(x_0,0) \neq 0 \) and
\[
\frac{\partial \zeta}{\partial t}(x,t) = 0 \quad \text{if} \quad (x,t) \in G, \quad 0 \leq t \leq \frac{T}{2}.
\]

Then by (4.5) we have
\[
\varphi T_s \zeta u \in \mathcal{S}(G)
\]
for any \( \varphi \in C^\infty(G) \). Starting with (4.6), by using the estimate (4.4) we can easily show that \( \varphi T_s \zeta u \in \mathcal{S}(G) \) for any \( s \) and \( \varphi \in C^\infty(G) \) from where we have \( \varphi D^s_x u \in \mathcal{S}(G), j = 0,1,2, \ldots \). And rewriting the form of the equation \( u_{tt} = -tu_{xx} + f \), we have \( \varphi D^j_x D^s_t u \in L^2(G), 0 \leq r, j < \infty \). Then we have \( u \in C^\infty(G) \), from where we have \( u \in C^\infty(G) \).

Next we consider the case where \( f \in C^\infty(G) \) and \( u(x,0) = 0 \). In this case we have \( u \in C^\infty(G) \) by the above result. To obtain the analyticity of \( u \) in \( \Omega \cap \{ (x,t); t \geq 0 \} \), we have to estimate precisely the successive derivatives of \( u \). We can pursue the manner employed in [6], § 6 where the analyticity of the solutions of the equations \( u_{tt} + t^{2k}u_{xx} = f, k = 0,1,2, \ldots \), was proved. In the following we shall give an outline of the reasoning.

Introduce the notations:
\[
G_* = (a + \varepsilon < x < b - \varepsilon) \times [0 \leq t < T], \quad 0 < \varepsilon < \min \left( \frac{b - a}{2}, \frac{T}{2} \right),
\]
\[
G_*^* = G_* \setminus (a + \varepsilon < x < b - \varepsilon) \times \left[ 0 \leq t < \frac{T}{2} \right],
\]
\[
N_\psi(v) = \| v \|_{L^2(G_\psi)}, \quad N^*\psi(v) = \| v \|_{L^2(G_*^*)}.
\]
LEMMA 4.1 (cf. [4], ch. 1). Let \( \varepsilon, \varepsilon_1 \) be positive numbers with \( 0 < \varepsilon + \varepsilon_1 < \text{Min} ((b - a)/2, T/2) \). Then there exists functions \( \psi = \psi_{\varepsilon, \varepsilon_1} \in C^*(G_{\varepsilon}) \) such that \( \psi = \psi_{\varepsilon, \varepsilon_1} \equiv 1 \) on \( G_{\varepsilon, \varepsilon_1} \) and

\[
\text{Max} |D_{x}D_{\tau}^{j}\psi| \leq C_{j+r, \varepsilon} \varepsilon^{-(j+r)} \quad 0 \leq j + r \leq 2
\]

\[
D_{\tau}\psi \equiv 0 \quad \text{on} \quad (a + \varepsilon_1, b - \varepsilon_1) \times \left[ 0, \frac{T}{2} \right].
\]

LEMMA 4.2 (cf. [6], Lemma 6.2). There exists a constant \( C > 0 \) such that

\[
\sum_{j=0}^{3} \varepsilon^j N_{\varepsilon, \varepsilon_1}(D_{x}^j v) + \sum_{j=0}^{3} \varepsilon^j N_{\varepsilon, \varepsilon_1}(tD_{x}^j v) + N_{\varepsilon, \varepsilon_1}(v)
\]

\[
+ \varepsilon N_{\varepsilon, \varepsilon_1}(D_x v) + \varepsilon^2 N_{\varepsilon, \varepsilon_1}(D_x D_x v)
\]

\[
\leq C(\varepsilon^2 N_{\varepsilon_1}(Lv) + \sum_{j=0}^{3} \varepsilon^j N_{\varepsilon_1}(tD_{x}^j v) + N_{\varepsilon_1}(v) + \varepsilon N_{\varepsilon_1}(D_x v))
\]

for all \( v \in C^\infty(G) \) and \( v(x, 0) = 0 \). The constant \( C \) does not depend on \( \varepsilon, \varepsilon_1 \) under the condition mentioned previously.

This lemma is obtained by substituting \( \psi_{\varepsilon, \varepsilon_1} v \) in (4.4).

LEMMA 4.3 (cf. [4], ch. 7). Let \( w \) be an analytic function in \( G \). Then there exists a constant \( C > 0 \) such that

\[
\varepsilon^{j+k} N_k(D_x^j D_{\tau}^r w) \leq C^{j+k+1} \quad \text{if} \quad j + r < k,
\]

for all integer \( k \geq 0 \). Conversely, if \( w \in C^\infty(G) \) satisfies (4.9), then \( w \) is analytic in \( G \).

Proof of the analyticity of \( u \) in \( \Omega \cap \{ (x, t); t \geq 0 \} \).

First we shall show that there exists a constant \( B > 0 \) such that, for any \( \varepsilon > 0 \) and for any integer \( l \geq 0 \),

\[
\left( \sum_{r=0}^{3} \varepsilon^{r+j} N_{t_k}(D_{x}^r D_{\tau}^j u) \right) \leq B^{l+1}
\]

\[
\left( \sum_{r=0}^{3} \varepsilon^{r+j} N_{t_k}(t^{2k} D_{x}^r D_{\tau}^j u) \right) \leq B^{l+1}
\]

\[
\left( \sum_{r=0}^{3} \varepsilon^{r+j} N_{t_k}(D_{x}^{r+j} u) \right) \leq B^{l+1}
\]

if \( j < l \).

It we take \( B \) sufficiently large, we have (4.10) for \( l = 1 \) by Lemma 4.2. Next, since \( f(x, t) \) is analytic in \( G \), there exists a constant \( C_0 > 0 \) such that
for \( j = 1, 2, \ldots \) and \( 0 < \varepsilon < (b - a)/2 \).

Assuming that (4.10) have been proved for an \( l > 0 \), we shall prove (4.10) for \( l + 1 \). Replacing \( v \) by \( \varepsilon \partial_2 u \) and \( \varepsilon \) by \( \varepsilon \varepsilon \) in (4.8), we see that the terms in the left hand side of (4.10) for the case \( l + 1 \) are smaller than \( 5C_2B^{l+1} \) if \( j < l + 1 \). Hence we have (4.10) for \( l + 1 \) if \( 5C_2B^{l+1} \leq B^{l+2} \).

This condition is satisfied for all \( l \) if \( B > \max (5C_2, 1) \).

From (4.10) (cf. Lemma 4.3) we obtain

\[
\sum_{r=0}^{j} ||D_iD^j_u||_{L^r(G_{\varepsilon})} \leq C^{l+1}j, \quad j = 0, 1, 2, \ldots
\]

for some constant \( C_1 > 0 \) where \( G_{\varepsilon} = (a + \varepsilon, b - \varepsilon) \times [0, T/2] \) with \( \varepsilon > 0 \) sufficiently small.

To obtain the successive estimates including the derivatives in both \( x \) and \( t \), we rewrite the equation \( Lu = f \) in the form \( D^2_t u = -tD^2_x u + f \).

And using (4.11) by the usual way (cf. [6] for example) we have

\[
||D^2_t D^j u||_{L^r(G_{\varepsilon})} \leq C^{l+r+1}(j + r)^{l+r} \quad 0 \leq j, r < \infty
\]

for some constant \( C_2 > 0 \), from which we have the analyticity of \( u \) in \( G_{\varepsilon} \) by the Sobolev lemma.

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