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Uniform Embeddings into Hilbert Space and a Question of Gromov

A. N. Dranishnikov, G. Gong, V. Lafforgue and G. Yu

Abstract. Gromov introduced the concept of uniform embedding into Hilbert space and asked if every separable metric space admits a uniform embedding into Hilbert space. In this paper, we study uniform embedding into Hilbert space and answer Gromov's question negatively.

1 Introduction

Gromov introduced the following concept of uniform embeddings in [7]:

Definition 1.1 A map f from a metric space X to another metric space Y is said to be a uniform embedding if there exist two non-decreasing functions ρ_1 and ρ_2 on $[0, +\infty)$ such that

(1) $\rho_1\big(d(x, y)\big) \leq d\big(f(x), f(y)\big) \leq \rho_2\big(d(x, y)\big)$ for all $x, y \in X$;
(2) $\lim_{r\to+\infty} \rho_1(r) = +\infty$.

Gromov raised the question whether every separable metric space admits a uniform embedding into Hilbert space [7, p. 218]. A positive answer to this question would imply the Novikov conjecture on homotopy invariance of higher signatures and Gromov's conjecture that a uniformly contractible Riemannian manifold can not have uniformly positive scalar curvature [17]. For the purpose of Novikov higher signature conjecture and Gromov's positive scalar curvature conjecture, it is enough to consider uniform embeddings of locally finite metric spaces into Hilbert space (recall that a metric space is called locally finite if every ball has finitely many elements).

In this note, we study uniform embeddings of locally finite metric spaces into Hilbert space. We shall first give an intrinsic characterization of locally finite metric spaces which admit a uniform embedding into Hilbert space in terms of negative type functions in Section 2. This is used to show that the question whether a locally finite metric space admits a uniform embedding into Hilbert space is a local problem in Section 3. In Section 4, we construct a universal metric space with bounded geometry for all bounded geometry spaces satisfying given growth condition (recall that a locally finite metric space is said to have bounded geometry if, for every $r > 0$, there exists N such that every ball with radius r has at most N number of elements). Every bounded geometry space satisfying the given growth condition admits a uniform embedding into Hilbert space if and only if the universal space admits a uniform embedding into Hilbert space. In Section 5, we show that a locally finite metric space

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with bounded geometry admits a uniform embedding into Hilbert space if an associated locally finite metric space with exponential growth admits a uniform embedding into Hilbert space. Based on ideas of Enflo [5], we construct a locally finite metric space which does not admit a uniform embedding into Hilbert space in Section 6. This answers negatively Gromov's question. However our example of locally finite metric space does not have bounded geometry.

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2 Uniform Embeddings into Hilbert Space and Negative Type Functions

In this section we characterize uniform embeddings into Hilbert space in terms of negative type functions.

Definition 2.1 Let X be a locally finite metric space. A function $h\colon X \times X \to \mathbb{R}$, is called a negative type function on X if

(0) $h(x, x) = 0$ for all $x \in X$;
(1) $h(x, y) = h(y, x)$ for all x and y in X;
(2) $\sum_{i,j=1}^{n} t_i t_j h(x_i, x_j) \leq 0$ for all $\{t_i\}_{i=1}^{n} \subseteq \mathbb{R}$ satisfying $\sum_{i=1}^{n} t_i = 0$, and all $\{x_i\}_{i=1}^{n} \subseteq X$.

The following result is inspired by Schoenberg's result characterizing metric spaces which admit an isometric embedding into Hilbert space [15].

Proposition 2.2 *A locally finite metric space X has a uniform embedding into Hilbert space if and only if there exists a negative type function h on X such that there exist two non-decreasing functions ρ_1 and ρ_2 on $[0, +\infty)$ satisfying*

(1) $\rho_1\big(d(x, y)\big) \leq h(x, y) \leq \rho_2\big(d(x, y)\big)$ *for all* $x, y \in X$;
(2) $\lim_{r\to+\infty} \rho_1(r) = +\infty$.

Proof Assume that X admits a uniform embedding $f\colon X \to H$, where H is a Hilbert space. Let $h(x, y) = \|f(x) - f(y)\|^2$ for all x and y in X. If $\{t_i\}_{i=1}^{n} \subseteq \mathbb{R}$ and

$\sum_{i=1}^{n} t_i = 0$, we have

$$\begin{aligned}\sum_{i,j=1}^{n} t_i t_j h(x_i, x_j) &= \sum_{i,j=1}^{n} t_i t_j \langle f(x_i) - f(x_j), f(x_i) - f(x_j)\rangle \\ &= -\sum_{i,j=1}^{n} t_i t_j \langle f(x_i), f(x_j)\rangle - \sum_{i,j=1}^{n} t_i t_j \langle f(x_j), f(x_i)\rangle \\ &= -2\Big\langle \sum_{i=1}^{n} t_i f(x_i), \sum_{i=1}^{n} t_i f(x_i)\Big\rangle \\ &\leq 0.\end{aligned}$$

It is also not difficult to see that h satisfies the required conditions in Proposition 2.2.

Conversely assume that h is a negative type function satisfying the conditions in Proposition 2.2. Let V be the real vector space of all formal finite sums $\sum_i t_i x_i$, where $t_i \in \mathbb{R}$, $\sum_i t_i = 0$, and $x_i \in X$. We define a semi-norm on V by:

$$\Big\| \sum_i t_i x_i \Big\| = \sqrt{-\frac{1}{2} \sum_{i,j} t_i t_j h(x_i, x_j)}.$$

We define an equivalence relation $\sim$ on V by: $u \sim v$ if $\|u - v\| = 0$. Clearly the semi-norm descends to a norm on the quotient vector space $W = V/\sim$. Let H be the norm completion of W. By the conditions in Definition 2.1, it is not difficult to verify the norm on H is induced by a Hilbert space structure. Fix $x_0 \in X$. Define $f\colon X \to H$ by sending $x \in X$ to $[x - x_0] \in H$, where $[x - x_0]$ is the equivalence class of $x - x_0$ in $V/\sim$. By the conditions of h, it is easy to see that f is a uniform embedding. ■

3 Local Uniform Embeddings into Hilbert Space

In this section we show that existence of a local uniform embedding of a locally finite metric space into Hilbert space implies existence of a uniform embedding into Hilbert space.

Definition 3.1 A locally finite metric space X is said to admit a local uniform embedding into Hilbert space H if there exist non-decreasing functions ρ_1 and ρ_2 on $[0, \infty)$ satisfying $\lim_{r\to\infty} \rho_1(r) = \infty$ such that, for every finite subspace $F \subseteq X$, there exists a map $f\colon F \to H$ satisfying

$$\rho_1\big(d(x, y)\big) \leq \|f(x) - f(y)\| \leq \rho_2\big(d(x, y)\big)$$

for all x and y in F.

Proposition 3.2 *If a locally finite metric space X admits a local uniform embedding into Hilbert space, then X admits a uniform embedding into Hilbert space.*

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Proof Fix $x_0 \in X$. Let $X_i = \{x \in X : d(x, x_0) \leq i\}$. By assumption there exist non-decreasing functions ρ_1 and ρ_2 on $[0, \infty)$ such that $\lim_{r \to \infty} \rho_1(r) = \infty$, and for every i there exists $f_i \colon X_i \to H$ satisfying

$$\rho_1\big(d(x, y)\big) \leq \|f_i(x) - f_i(y)\| \leq \rho_2\big(d(x, y)\big)$$

for all x and y in X_i. Define $h_i \colon X_i \times X_i \to \mathbb{R}$ by:

$$h_i(x, y) = \|f_i(x) - f_i(y)\|^2.$$

By a standard diagonal argument, there exists a subsequence $\{h_{i_k}\}_{k=1}^{\infty}$ such that h_{i_k} converges to a function $h \colon X \times X \to \mathbb{R}$ in the following sense: for every $(x, y) \in X \times X$, there exists k_0 for which $(x, y) \in X_{i_{k_0}} \times X_{i_{k_0}}$ and $\lim_{k \geq k_0, k \to \infty} h_{i_k}(x, y) = h(x, y)$. It is not difficult to see that h is a negative type function on X satisfying

$$\rho_1^2\big(d(x, y)\big) \leq h(x, y) \leq \rho_2^2\big(d(x, y)\big)$$

for all x and y in X. Hence by Proposition 2.2, X admits a uniform embedding into Hilbert space. ∎

Corollary 3.3 *Let $\mathbb{N}$ be the set of all natural numbers; let l^1 be the Banach space $l^1(\mathbb{N})$. If a locally finite metric space X admits a uniform embedding into l^1, then X admits a uniform embedding into Hilbert space.*

Proof Without loss of generality, we can assume that X is a subspace of l^1. By Proposition 3.2, it is enough to show that X admits a local uniform embedding into Hilbert space. Let $\rho_1(r) = \sqrt{\max\{r - 10, 0\}}$ and $\rho_2(r) = \sqrt{r + 10}$ for all $r \geq 0$. Let F be a finite subspace of X. By finiteness of F there exists a large natural number N such that the projection P from l^1 to $l^1_N = l^1(\{1, \ldots, N\})$ satisfies the following inequality:

$$d(x, y) - 1 \leq d(Px, Py) \leq d(x, y) + 1$$

for all x and y in F. Let $Y = PF \subseteq l^1_N$. We define $g \colon \mathbb{R} \to l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ by:

$$g(x) = \Big(\underbrace{\frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}}}_{k}, 0, \ldots\Big) \oplus 0$$

if $x \geq 0$ and $\frac{k}{N} \leq x < \frac{k+1}{N}$ for some integer k, and

$$g(x) = 0 \oplus \Big(\underbrace{-\frac{1}{\sqrt{N}}, \ldots, -\frac{1}{\sqrt{N}}}_{-k}, 0, \ldots\Big)$$

if $x < 0$ and $\frac{k-1}{N} \leq x < \frac{k}{N}$ for some integer k. Let H be the Hilbert space $\{(v_1, \ldots, v_N) : v_i \in l^2(\mathbb{N}) \oplus l^2(\mathbb{N})\}$. We define a map $f_1 \colon Y \to H$ by:

$$f_1(x_1, \ldots, x_N) = \big(g(x_1), \ldots, g(x_N)\big)$$

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for all $(x_1, \dots, x_N) \in Y \subseteq l^1_N$. Finally we define $f\colon F \to H$ by: $f(v) = f_1(Pv)$ for all $v \in F$. It is not difficult to verify that

$$\rho_1\big(d(x,y)\big) \le \|f(x) - f(y)\| \le \rho_2\big(d(x,y)\big)$$

for all x and y in F. ■

The fourth author introduced a geometric property, called property A, and proved that any locally finite metric space with property A admits a unform embedding into a Hilbert space (*cf.* Theorem 2.2 in [17]). The proof of Theorem 2.2 in [17] can be used to show that a discrete metric space X with property A admits a uniform embedding in $l^1(\mathbb{N})$.

4 Universal Metric Spaces

In this section we construct a universal metric space with bounded geometry for all bounded geometry spaces satisfying given growth condition. Every bounded geometry space satisfying the given growth condition admits a uniform embedding into Hilbert space if and only if the universal space admits a uniform embedding into Hilbert space. This is used to show that the question whether every locally finite metric space with bounded geometry admits a uniform embedding into Hilbert space is essentially a problem in finite mathematics.

Proposition 4.1 *Given a non-decreasing function ρ on $[0, \infty)$, there exists a (universal) metric space Y_ρ such that*

(1) *$\#B_{Y_\rho}(x,r) \le \rho(r)$ for all $x \in Y_\rho$ and $r > 0$, where $B_{Y_\rho}(x,r) = \{a \in Y_\rho : d(a,x) \le r\}$ and $\#B_{Y_\rho}(x,r)$ is the number of elements in $B_{Y_\rho}(x,r)$;*
(2) *for every finite metric space F satisfying $\#B_F(x,r) \le \rho(r)$ for all x in F and $r > 0$, there is a map $f\colon F \to Y_\rho$, for which $d(x,y) \le d\big(f(x), f(y)\big) \le d(x,y) + 1$ for all x and y in F;*
(3) *every metric space X satisfying $\#B_X(x,r) \le \rho(r)$ for all $x \in X$ and $r \ge 0$ admits a uniform embedding into Hilbert space if and only if Y_ρ admits a uniform embedding into Hilbert space.*

Claim For every finite metric space F with a metric d, there exists a new metric d' on F such that (1) d' is integer valued, *i.e.*, $d'(x,y)$ is an integer for all x and y in F; (2) $d(x,y) \le d'(x,y) \le d(x,y) + 1$ for all x and y in F.

Proof of Claim Define $d'(x,y)$ to be the smallest integer greater than or equal to $d(x,y)$. It is not difficult to verify that d' satisfies the desired conditions.

It is not difficult to see that, for every pair of natural numbers n and m, there exist only finitely many finite metric spaces F with integer valued distance functions (up to isometry) satisfying $\#F \le n$ and diameter$(F) \le m$. Hence there exist only countably many finite metric spaces with integer valued distance functions. Let $\{F_n\}_{n=1}^\infty$ be the collection of all finite metric spaces with integer valued distance functions satisfying

$\#\big(B_{F_n}(x,r)\big) \le \rho(r)$ for all n, $x \in F_n$, and $r \ge 0$. Let $\{c_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of non-negative numbers satisfying $c_1 = 0$ and $\rho(\frac{c_n}{2}) > \#(\bigcup_{i=1}^{n} F_i)$ for every natural number $n > 1$. Let Y_ρ be the disjoint union of all F_n. Endow Y_ρ with a metric d such that the restriction of d to F_n is the original metric on F_n, and $d(F_n, F_m) \ge c_{n+m}$ if $n \ne m$. For any $x \in Y_\rho$ and $r > 0$, assume that $x \in F_m$ and $\frac{c_n}{2} \le r < \frac{c_{n+1}}{2}$ for some n. If $m > n$, by the property of d, we have that $B_{Y_\rho}(x,r) \subseteq F_m$. Hence $\#B_{Y_\rho}(x,r) \le \#B_{F_m}(x,r) \le \rho(r)$. If $m \le n$, then we have that $B(x,r) \subseteq \bigcup_{i=1}^{n} F_i$, and $B(x,r) \subseteq F_1$ if $n = 1$. By the property of c_n, we have that $\#B(x,r) \le \rho(\frac{c_n}{2}) \le \rho(r)$ if $n > 1$, and $\#B(x,r) \le B_{F_1}(x,r) \le \rho(r)$ if $n = 1$. This, together with Proposition 3.2 and the above claim, implies that Y_ρ satisfies the desired properties. ■

Corollary 4.2 *Every locally finite metric space with bounded geometry admits a uniform embedding into Hilbert space H if and only if, for every nondecreasing function ρ on $[0,+\infty)$ satisfying $\lim_{r\to\infty} \rho(r) = \infty$, there exist non-decreasing functions ρ_i $(i = 1, 2)$ on $[0,+\infty)$ satisfying $\lim_{r\to\infty} \rho_1(r) = \infty$ such that, for every finite metric space F satisfying $\#\big(B_F(x,r)\big) \le \rho(r)$, there exists $f\colon F \to H$ satisfying $\rho_1\big(d(x,y)\big) \le \|f(x) - f(y)\| \le \rho_2\big(d(x,y)\big)$ for all $x, y \in F$, where ρ_1 and ρ_2 depend only on ρ.*

Proof The if part of Corollary 4.2 follows from the proof of Proposition 3.2. The only if part follows from Proposition 4.1. ■

The case when ρ has subexponential growth is solved in [6]. However, in the case that ρ has exponential growth, Gromov recently showed non-existence of ρ_1 and ρ_2 satisfying conditions in Corollary 4.2 (*cf.* Gromov's preprint "Spaces and questions").

We also observe that if a finitely generated group Γ with a word length metric admits a uniform embedding into Hilbert space, then ρ_2 can be chosen to be $\rho_2(r) = r$. This is because finitely generated groups are geodesic in the sense that, for every pair of x and y in Γ, there exists $\{x_i\}_{i=0}^{n} \subseteq \Gamma$ such that $x_0 = x$, $x_n = y$, $d(x_i, x_{i+1}) = 1$ for all $0 \le i \le n-1$, and $d(x,y) = \sum_{i=0}^{n-1} d(x_i, x_{i+1})$.

We also remark that, given two computable non-decreasing functions ρ_1 and ρ_2 on $[0,\infty)$ satisfying $\rho_1(r) \le \rho_2(r)$ for all $r \ge 0$ and $\lim_{r\to\infty} \rho_1(r) = \infty$, Proposition 2.2 can be used to construct an algorithm to decide if a finite metric space F admits a map f from F into a Hilbert space H satisfying $\rho_1\big(d(x,y)\big) \le \|f(x) - f(y)\| \le \rho_2\big(d(x,y)\big)$ for all x and y in F.

5 Reduction to the Exponential Growth Case

In this section, we show that a locally finite metric space with bounded geometry admits a uniform embedding into Hilbert space if an associated locally finite metric space with exponential growth admits a uniform embedding into Hilbert space. This is a consequence of the following result.

Proposition 5.1 *Every locally finite metric space X with bounded geometry admits a uniform embedding into the vertex set of a connected graph G_X such that every vertex of G_X has at most three adjacent vertices, where G_X is endowed with the path metric and the metric on the vertex set is the restriction of the path metric.*

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Proof Without loss of generality, we can assume that X is an infinite set with an integer valued distance function. Let $\rho(r)$ be a non-decreasing function on $[0,\infty)$ such that $\#B_X(x,r) \le \rho(r)$ for all $x \in X$ and $r \ge 0$. Let $\mathbb{R}_+ = [0,\infty)$ be the tree such that its set of vertices is $\mathbb{Z}_+$, the set of all non-negative integers, and its set of edges is $\{[n,n+1] : n \in \mathbb{Z}_+\}$. For each $x \in X$, let $p_x \colon X \to \mathbb{Z}_+$, be a bijective map satisfying

(1) $p_x(x) = 0$;
(2) if $d(x_1,x) > d(x_2,x)$ for some x_1 and x_2 in X, then $p_x(x_1) > p_x(x_2)$.

Note that

$$p_x(y) \le \rho\big(d(y,x)\big)$$

for every $y \in X$.

Define G_X to be the smallest connected graph such that (1) G_X contains the disjoint union $\bigsqcup_{x\in X} \mathbb{R}_{+,x}$, where $\mathbb{R}_{+,x} = \mathbb{R}_+$ for every $x \in X$; (2) $p_x(y)$ in $\mathbb{R}_{+,x}$ is connected to $p_y(x)$ in $\mathbb{R}_{+,y}$ by a path with length $d(x,y)$ for all x and y in X.

Let $f \colon X \to G_X$ be the map defined by: $x \to 0$ in $\mathbb{R}_{+,x}$ for every $x \in X$. Clearly we have

$$d\big(f(x), f(y)\big) \le d(x,y) + 2\rho\big(d(x,y)\big)$$

for all x and y in X.

For every pair x and y in X, let $\{z_0 = f(x), z_1, \dots, z_{n-1}, z_n = f(y)\}$ be a chain of vertices in G_X such that $d(z_i, z_{i+1}) = 1$ for all $0 \le i \le n-1$ and $d\big(f(x), f(y)\big) = \sum_{i=0}^{n-1} d(z_i, z_{i+1}) = n$. Let $i_0 = 0$, $x_{i_0} = x$. Let i_1 be the smallest positive integer such that $z_{i_1} \in \mathbb{R}_{+,x_{i_1}}$ for some $x_{i_1} \in X$ satisfying $x_{i_1} \ne x_{i_0}$. By induction we define i_k to be the smallest integer such that $i_k > i_{k-1}$ and $z_{i_k} \in \mathbb{R}_{+,x_{i_k}}$ for some $x_{i_k} \in X$ satisfying $x_{i_k} \ne x_{i_{k-1}}$. Let k_0 be the smallest integer such that $x_{i_{k_0}} = y$ (*i.e.*, k_0 is the smallest integer such that $z_{i_{k_0}} \in \mathbb{R}_{+,y}$). We have

$$\begin{aligned} d\big(f(x), f(y)\big) &= \sum_{k=0}^{k_0-1} d(z_{i_k}, z_{i_{k+1}}) \\ &\ge \sum_{k=0}^{k_0-1} d(x_{i_k}, x_{i_{k+1}}) \\ &\ge d(x,y). \end{aligned}$$

Hence we have

$$d(x,y) \le d\big(f(x), f(y)\big) \le d(x,y) + 2\rho\big(d(x,y)\big)$$

for all x and y in X. ■

Note that if every vertex of a connected graph G has at most three adjacent vertices, then $\#B_G(x,r) \le 3^r$ for all $x \in G$ and $r \ge 0$. By Propositions 5.1 and 4.1, every locally finite metric with bounded geometry admits a uniform embedding into Hilbert space if and only the universal metric space Y_ρ associated to $\rho(r) = 3^r$ admits a uniform embedding into Hilbert space.

6 A Counter Example to Gromov's Question

In this section, we construct a locally finite metric space which does not admit a uniform embedding into Hilbert space. The argument is based on beautiful ideas of Enflo [5]. A different construction is given in [3].

Lemma 6.1 (**[5]**) *Let H be a Hilbert space. If $\{x_1,\dots,x_k,y_1,\dots,y_k\}\subseteq H$, then we have*

$$\sum_{i,j=1}^{k} d^2(x_i,y_j) \ge \sum_{i>j,i,j=1}^{k} d^2(x_i,x_j) + \sum_{i>j,i,j=1}^{k} d^2(y_i,y_j).$$

Proof Let $x_j = y_{j-k}$ for all $k+1 \le j \le 2k$. We have

$$\sum_{i,j=1}^{2k} t_i t_j d^2(x_i,x_j) \le 0$$

if $\sum_{i=1}^{2k} t_i = 0$. Now Lemma 6.1 follows from the above inequality if we take $t_i = 1$ for all $1 \le i \le k$, and $t_i = -1$ for all $k+1 \le i \le 2k$. ■

Let $Z_n = \mathbb{Z}/n\mathbb{Z}$ be given the metric:

$$d([k],[l]) = \min_{[k'-l']=[k-l]} |k'-l'|$$

for all $[k]$ and $[l]$ in Z_n. Let $Z_{n,m} = \{([k_1],\dots,[k_m]) : [k_i] \in Z_n\}$ be given the metric:

$$d\big(([k_1],\dots,[k_m]),([l_1],\dots,[l_m])\big) = \max_{1\le i\le m} d([k_i],[l_i]).$$

Let p be a non-negative integer, let k be a positive even number. Assume that $m = k^q$ for some $q \ge p+2$, and $n \ge 2^{p+2}$. A pair of points $\{x,y\} \subseteq Z_{n,m}$ is called a (k,p)-pair if

(1) $d([k_i],[l_i]) = 2^p$ whenever $[k_i] \ne [l_i]$, where $x = ([k_1],\dots,[k_m])$ and $y = ([l_1],\dots,[l_m])$;
(2) the number of elements in $\{i : [k_i] \ne [l_i]\}$ is $\frac{2m}{k^{p-1}}$.

Lemma 6.2 *Let p, k, n and m be as above. There exists a pair of subsets $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_k\}$ of $Z_{n,m}$ such that $\{x_i,x_j\}$ and $\{y_i,y_j\}$ are $(k,p+1)-$pairs if $i \ne j$, and $\{x_i,y_j\}$ are (k,p)-pairs for all i and j.*

Proof We choose a pair of subsets $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_k\}$ of $Z_{n,m}$ by:

$$x_1 = (\underbrace{[2^p],\dots,[2^p]}_{\frac{m}{k^{p-1}}},\underbrace{[2^{p+1}],\dots,[2^{p+1}]}_{\frac{m}{k^p}},[0],\dots,[0]),$$

$$x_2 = (\underbrace{[2^p],\dots,[2^p]}_{\frac{m}{k^{p-1}}},\underbrace{[0],\dots,[0]}_{\frac{m}{k^p}},\underbrace{[2^{p+1}],\dots,[2^{p+1}]}_{\frac{m}{k^p}},[0],\dots,[0]),$$

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$$\ldots\ldots,$$

$$x_k = (\underbrace{[2^p], \ldots, [2^p]}_{\frac{m}{k^{p-1}}}, \underbrace{[0], \ldots, [0]}_{\frac{(k-1)m}{k^p}}, \underbrace{[2^{p+1}], \ldots, [2^{p+1}]}_{\frac{m}{k^p}}, [0], \ldots, [0]),$$

$$y_1 = (\underbrace{[2^{p+1}], \ldots, [2^{p+1}]}_{\frac{m}{k^p}}, \underbrace{[0], \ldots, [0]}_{\frac{(k-1)m}{k^p}}, \underbrace{[2^p], \ldots, [2^p]}_{\frac{m}{k^{p-1}}}, [0], \ldots, [0]),$$

$$y_2 = (\underbrace{[0], \ldots, [0]}_{\frac{m}{k^p}}, \underbrace{[2^{p+1}], \ldots, [2^{p+1}]}_{\frac{m}{k^p}}, \underbrace{[0], \ldots, [0]}_{\frac{(k-2)m}{k^p}}, \underbrace{[2^p], \ldots, [2^p]}_{\frac{m}{k^{p-1}}}, [0], \ldots, [0]),$$

$$\ldots\ldots,$$

$$y_k = (\underbrace{[0], \ldots, [0]}_{\frac{(k-1)m}{k^p}}, \underbrace{[2^{p+1}], \ldots, [2^{p+1}]}_{\frac{m}{k^p}}, \underbrace{[2^p], \ldots, [2^p]}_{\frac{m}{k^{p-1}}}, [0], \ldots, [0]).$$

It is easy to see that $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ satisfy the required conditions. ■

Let X be the disjoint union of $Z_{n,m}$ for all $n \geq 1$ and $m \geq 1$. We endow a metric d on X such that

(1) the restriction of d to each $Z_{n,m}$ coincides with the original metric on $Z_{n,m}$;
(2) $d(x, y) \geq n_1 + n_2 + m_1 + m_2$ if $x \in Z_{n_1,m_1}$ and $y \in Z_{n_2,m_2}$, and $(n_1, m_1) \neq (n_2, m_2)$.

Remark X has the universal property that every finite metric space with an integer valued distance function admits an isometric embedding into X.

Proof of Remark Every finite metric space F with an integer valued distance function admits an isometric embedding $f\colon F \to l^\infty(F)$, defined by: $\bigl(f(x)\bigr)(y) = d(x, y)$ for $x, y \in F$. Note that $f(x)$ is an integer valued function for every $x \in F$. Now the Remark follows from the fact that every finite subset of integer valued functions in $l^\infty(F)$ admits an isometric embedding into $Z_{n,m}$, where m is the number of elements in F, and n is large enough.

By the above Remark and the Claim in the proof of Proposition 4.1, it follows that every finite metric space F admits a map f from F into X such that $d(x, y) \leq d\bigl(f(x), f(y)\bigr) \leq d(x, y) + 1$ for all x and y in F.

Proposition 6.3 *Let X be the locally finite metric space defined as above. X does not admit a uniform embedding into Hilbert space.*

Proof Assume by contradiction that there exists a map f from X to a Hilbert space H for which there exist two non-decreasing functions ρ_1 and ρ_2 on $[0, +\infty)$ such that

(1) $\rho_1\bigl(d(x, y)\bigr) \leq \|f(x) - f(y)\| \leq \rho_2\bigl(d(x, y)\bigr)$ for all $x, y \in X$;
(2) $\lim_{r\to+\infty} \rho_1(r) = +\infty$.

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Let p_0 be a positive even number such that $\rho_1(2^{p_0}) \geq 10\rho_2(1)$ and $(1-\frac{1}{p_0})^{p_0} \geq \frac{1}{3}$. Let $k = p_0$, $m = k^{p_0+2}$ and $n = 2^{p_0+2}$.

Notice that the isometry group of $Z_{n,m}$ acts transitively on the set of all (k,p)-pairs. This, together with Lemma 6.2, implies that, for every $p \leq p_0$, there exists a natural number L such that, for every (k,p)-pair $\{x,y\}$ in $Z_{n,m}$, there are exactly L number of pairs of subsets $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_k\}$ of $Z_{n,m}$ for which $\{x,y\} = \{x_{i_0}, y_{j_0}\}$ for some i_0 and j_0, $\{x_i,x_j\}$ and $\{y_i,y_j\}$ are $(k,p+1)$-pairs for $i \neq j$, and $\{x_i,y_j\}$ is a (k,p)-pair for all i and j. Similarly for every $p \leq p_0$, there exists a natural number L' such that, for every $(k,p+1)$-pair $\{x,y\}$ in $Z_{n,m}$, there are exactly L' number of pairs of subsets $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_k\}$ of $Z_{n,m}$ for which $\{x,y\} = \{x_{i_1},x_{i_2}\}$ or $\{x,y\} = \{y_{i_1},y_{i_2}\}$ for some $i_1 \neq i_2$, $\{x_i,x_j\}$ and $\{y_i,y_j\}$ are $(k,p+1)$-pairs for $i \neq j$, and $\{x_i,y_j\}$ is a (k,p)-pair for all i and j. Let N be the number of all pairs of subsets $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_k\}$ of $Z_{n,m}$ such that $\{x_i,x_j\}$ and $\{y_i,y_j\}$ are $(k,p+1)$-pairs for $i \neq j$, and $\{x_i,y_j\}$ is a $(k,p)-$pair for all i and j. We have

$$Nk^2 = LN_p, \quad Nk(k-1) = L'N_{p+1},$$

where N_p is the number of all (k,p)-pairs in $Z_{n,m}$. It follows that

$$\frac{L}{L'} = \frac{k}{k-1}\frac{N_{p+1}}{N_p}.$$

By Lemma 6.1, for each pair of subsets $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_k\}$, we have

$$\sum_{i,j=1}^{k} d^2\big(f(x_i),f(y_j)\big) \geq \sum_{i>j,i,j=1}^{k} d^2\big(f(x_i),f(x_j)\big) + \sum_{i>j,i,j=1}^{k} d^2\big(f(y_i),f(y_j)\big).$$

Summing over all pairs of subsets $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_k\}$ of $Z_{n,m}$ such that $\{x_i,x_j\}$ and $\{y_i,y_j\}$ are $(k,p+1)$-pairs for $i \neq j$, and $\{x_i,y_j\}$ is a (k,p)-pair for all i and j, we obtain

$$\frac{1}{N_p}\sum_{\{x,y\}:(k,p)\text{-pair}} d^2\big(f(x),f(y)\big) \geq \Big(1-\frac{1}{k}\Big)\frac{1}{N_{p+1}}\sum_{\{u,v\}:(k,p+1)\text{-pair}} d^2\big(f(u),f(v)\big).$$

Iterating the above inequality we have

$$\begin{aligned}\frac{1}{N_0}\sum_{\{x,y\}:(k,0)\text{-pair}} d^2\big(f(x),f(y)\big) &\geq \Big(1-\frac{1}{k}\Big)^{p_0}\frac{1}{N_{p_0}}\sum_{\{u,v\}:(k,p_0)\text{-pair}} d^2\big(f(u),f(v)\big)\\ &\geq \Big(1-\frac{1}{p_0}\Big)^{p_0}\rho_1^2(2^{p_0})\\ &\geq \frac{100}{3}\rho_2^2(1).\end{aligned}$$

But this contradicts with the following

$$\frac{1}{N_0}\sum_{\{x,y\}:(k,0)\text{-pair}} d^2\big(f(x),f(y)\big) \leq \rho_2^2(1). \qquad \blacksquare$$

We remark that the proof of Proposition 6.3 is essentially due to Enflo [5].

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