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Abstract

Let S be a complex smooth projective surface and L be a line bundle on S. For any given collection of isolated topological or analytic singularity types, we show the number of curves in the linear system $|L|$ with prescribed singularities is a universal polynomial of Chern numbers of L and S, assuming L is sufficiently ample. More generally, we show for vector bundles of any rank and smooth varieties of any dimension, similar universal polynomials also exist and equal the number of singular subvarieties cutting out by sections of the vector bundle. This work is a generalization of Göttsche's conjecture.

1. Introduction

For a pair of a smooth projective surface and a line bundle (S, L), it is a classical problem to find the number of r-nodal curves in a generic r-dimensional linear subsystem of $|L|$. Göttsche conjectured that for any $r \geqslant 0$, there exists a universal polynomial T_r of degree r, such that $T_r(L^2, LK_S, c_1(S)^2, c_2(S))$ equals the number of r-nodal curves in a general linear subsystem, provided that L is $(5r-1)$-very ample. Moreover, the generating series of T_r has a multiplicative structure and satisfies the Göttsche–Yau–Zaslow formula [Göt98, Tze12]. Recently, Göttsche's universality conjecture was proven by the second named author [Tze12] using degeneration methods, and a different proof was given by Kool–Shende–Thomas [KST11] using BPS calculus and computation of tautological integrals on Hilbert schemes (see also the approach of Liu [Liu00, Liu04]). In this paper, we address the question of whether a similar phenomenon is true for curves with higher singularities.

The goal of this article is to generalize Göttsche's universality conjecture to curves with arbitrary isolated (analytic or topological) singularities. Consider a collection of isolated singularity type $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{l(\underline{\alpha})})$. We say a curve C has singularity type $\underline{\alpha}$ if there exists $l(\underline{\alpha})$ points $x_1, x_2, \ldots, x_{l(\underline{\alpha})}$, such that the singularity type of C at x_i is exactly α_i and C has no more singular points. We prove the following theorem concerning curves with singularity type $\underline{\alpha}$.

Theorem 1.1. *For every collection of (analytic or topological) isolated singularity type $\underline{\alpha}$, there exists a universal polynomial $T_{\underline{\alpha}}(x, y, z, t)$ of degree $l(\underline{\alpha})$ with the following property: given a smooth projective surface S and an $(N(\underline{\alpha}) + 2)$-very ample line bundle L on S, a general $\operatorname{codim}(\underline{\alpha})$-dimensional sublinear system of $|L|$ contains exactly $T_{\underline{\alpha}}(L^2, LK, c_1(S)^2, c_2(S))$ curves with singularity type $\underline{\alpha}$.*

The constant $N(\underline{\alpha}) := \sum N(\alpha_i)$ can be computed explicitly using the defining equation of each singularity α_i, see Definition 2.2.

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For instance, there is a universal polynomial which counts curves with one triple point, two E_8 singularities, and one 5-fold point analytically equivalent to $x^5 - y^5 = 0$. The coefficients of this universal polynomial can be determined if we know the numbers of such curves on finite pairs of surfaces and $(N(\underline{\alpha}) + 2)$-very ample line bundles. However, for most singularity types, such numbers are not known even for the projective plane. At the end of § 3 we will discuss the current results on computing universal polynomials.

To characterize the conditions of given singularity type, we study the locus of zero-dimensional subschemes of a special shape on the surface. The shape is determined by the singularity type such that if a curve has a prescribed singularity then it must contain a zero-dimensional closed subscheme of the corresponding shape; moreover, the converse is true for generic curves. In the case of a node, the locus is the collection of subschemes isomorphic to $\operatorname{Spec} \mathbb{C}\{x, y\}/(x, y)^2$ because a curve C on S is singular at p if and only if C contains $\operatorname{Spec} \mathcal{O}_{S,p}/\mathfrak{m}^2_{S,p}$. This technique was developed by [Göt98] and [HP95] to show that the enumeration of nodal curves can be achieved by computing a certain intersection number on Hilbert schemes of points (called tautological integrals). In this paper, we find a uniform way of defining such a correspondence from isolated singularity types to the isomorphism types of zero-dimensional subschemes. As a result, the number of curves with given singularity types can be expressed again as tautological integrals on Hilbert schemes of points on S. This allows us to apply a degeneration argument developed in [Tze12] to show the existence of universal polynomials.

We illustrate our method below, by outlining how it can be applied to enumerate cuspidal curves.[1]

(i) If a reduced curve C has a cusp at a point p, then local coordinates at p can be chosen such that C is locally defined by $y^2 - x^3 = 0$. Therefore, C contains the subscheme $\operatorname{Spec} \mathbb{C}\{x, y\}/\langle y^2 - x^3\rangle$ supported at p. Since we aim to characterize isolated singularities by subschemes of finite length, we observe that C must contain $\operatorname{Spec} \mathbb{C}\{x, y\}/\langle y^2 - x^3, \mathfrak{m}^4\rangle$. The converse is true for generic curves. So counting cuspidal curves is equivalent to counting curves having a subscheme isomorphic to $\operatorname{Spec} \mathbb{C}\{x, y\}/\langle y^2 - x^3, \mathfrak{m}^4\rangle$.

(ii) Define $S^0(\text{cusp})$ to be all subschemes of S isomorphic to $\operatorname{Spec} \mathbb{C}\{x, y\}/\langle y^2 - x^3, \mathfrak{m}^4\rangle$. Then $S^0(\text{cusp})$ is a subset of the Hilbert scheme of seven points $S^{[7]}$. So we can define $S(\text{cusp})$ to be the closure of $S^0(\text{cusp})$ (with induced reduced structure). It is elementary to see that the dimensions of $S^0(\text{cusp})$ and $S(\text{cusp})$ are both 5.

(iii) Since the locus of cuspidal curves is of codimension two in $|L|$, we aim to compute the (finite) number of cuspidal curves in a general linear subsystem $V \cong \mathbb{P}^2 \subset |L|$. Suppose that L is sufficiently ample and let $L^{[7]}$ be the tautological bundle of L in $S^{[7]}$ (Definition 2.3), the number of cuspidal curves is equal to the number of points in the locus cut out by the three sections in $H^0(L^{[7]})$ which define V in $S(\text{cusp})$; when the intersection is discrete, this number is represented by the tautological integral $d_{\text{cusp}}(S, L) := \int_{S(\text{cusp})} c_{7-3+1}(L^{[7]})$ (the Thom–Porteous formula).

(iv) The degree of the zero cycle $d_{\text{cusp}}(S, L)$ does not have any contribution from nonreduced curves, curves with more than two singular points or with other singularities. Because if a curve in $d_{\text{cusp}}(S, L)$ is nonreduced or has more than two singular points, then it must contain $\operatorname{Spec} \mathcal{O}_{S,p}/\langle y^2 - x^3, {\mathfrak{m}_{S,p}}^4\rangle \cup \operatorname{Spec} \mathcal{O}_{S,q}/\mathfrak{m}^2_{S,q}$ for some points $p \neq q$ in S. Similarly a curve in $d_{\text{cusp}}(S, L)$ with singularity different from cusp must contain $\operatorname{Spec} \mathbb{C}\{x, y\}/\langle (y^2 - x^3)\mathfrak{m}, \mathfrak{m}^4\rangle$.

[1] Since it is impossible to explain every detail in a short paragraph, please refer to the corresponding sections.

Dimension count shows that this is impossible for general two-dimensional V and sufficiently ample L (see Proposition 2.4).

(v) In the last step, we apply the degeneration technique developed in [Tze12]. We show $d_{\mathrm{cusp}}(S, L)$ only depends on the class of $[S, L]$ in the algebraic cobordism group, which only depends on L^2, LK, $c_1(S)^2$ and $c_2(S)$. Hence $d_{\mathrm{cusp}}(S, L)$ is precisely the universal polynomial we are looking for.

We will define a generating series containing all universal polynomials as coefficients and show it has a compact exponential description (Corollary 3.3) and is multiplicative (Theorem 3.2), generalizing the case of nodal curves discussed in [Göt98] and [KP99]. These properties of generating series impose strong restrictions on universal polynomials and therefore greatly reduce the computation complexity. After proving the existence of universal polynomials on surfaces, we also discuss how Göttsche's conjecture can be generalized to higher dimensional varieties. At the end of § 3, we discuss a few known cases and the relation with the Thom polynomials. Moreover, we will discuss the irreducibility and smoothness of the locus of curves with fixed singularity type $\underline{\alpha}$ in § 4.

Shortly after our paper was written, we learned that similar results were obtained independently by Rennemo [Ren12]. Both we and Rennemo compute the number of curves containing certain types of zero-dimensional subschemes and express the number in tautological integrals. The main difference between these two papers is the method to prove that tautological integrals are universal: Rennemo uses Chern–Schwartz–MacPherson class and Hilbert schemes of ordered points, while we use algebraic cobordism. We also associate topological singularities with different isomorphism types of zero-dimensional subschemes, but that is minor.

Göttsche's conjecture also holds for higher dimensional varieties. Consider hypersurfaces or in general the common zero of sections from several vector bundles E_i on a smooth projective variety X. If E_i are all sufficiently ample, the numbers of such subvarieties with prescribed analytic singularities should be universal polynomials of the Chern numbers of X and E_i. The proofs in Rennemo's and our papers can be carried to this more general setting without any change. The discussion about the higher dimensional case is in § 3.

2. Tautological integrals

In this section we construct a tautological integral $d_{\underline{\alpha}}(S, L)$ on Hilbert schemes of points on S, and prove that this number can be used to count the number of curves with singularity type $\underline{\alpha}$ in $|L|$. First, we recall some results in singularity theory.

Let $\mathbb{C}\{x, y\}$ be the ring of convergent power series in x and y with maximal ideal $\mathfrak{m} = (x, y)$, f be a germ in $\mathbb{C}\{x, y\}$, and let the Jacobian ideal $\langle \partial f/\partial x, \partial f/\partial y\rangle$ be $J(f)$, then the Milnor number $\mu(f)$ and Tjurina number $\tau(f)$ are defined as:

$$\mu(f) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/J(f); \quad \tau(f) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/\langle f + J(f)\rangle.$$

Two planar curves C_1 and C_2 have *analytically equivalent* singularities at the origin if their defining germs f_1 and f_2 in $\mathbb{C}\{x, y\}$ are *contact equivalent*; i.e. there exists an automorphism ϕ of $\mathbb{C}\{x, y\}$ and a unit $u \in \mathbb{C}\{x, y\}$ such that $f = u \cdot \phi(g)$. We say they have *topologically equivalent* (or *equisingular*) singularities if the following equivalent conditions are satisfied [GLS07]:

(i) there exist balls B_1 and B_2 with center 0 such that $(B_1, B_1 \cap C_1, 0)$ is homeomorphic to $(B_2, B_2 \cap C_2, 0)$;

(ii) C_1 and C_2 have the same number of branches, Puiseux pairs of their branches C_{1i} and C_{2i} coincide, and intersection multiplicities $i(C_{1i}, C_{1j}) = i(C_{2i}, C_{2j})$ for any i, j;

(iii) the systems of multiplicity sequences of an embedded resolution coincide.

It is easy to see that analytic equivalence implies topological equivalence. However, the converse is true only for ADE singularities. The Milnor number is both analytic and topological invariant (by a result of Milnor), while the Tjurina number is only an analytic invariant. Therefore $\tau(\alpha)$ and $\mu(\alpha)$ are well defined when they are invariants of the singularity α.

Recall that the tangent space of the miniversal deformation space Def of the singular curve $C := \{f = 0\}$ at the origin can be naturally identified with $\mathbb{C}\{x, y\}/\langle f + J(f)\rangle$, and its dimension is the Tjurina number $\tau(f)$. The dimension of the miniversal deformation space also has an important geometric meaning: it is the expected codimension of the locus of curves in a linear system with the same analytic singularity α of C at the origin, and is denoted by $\operatorname{codim}(\alpha)$. Roughly speaking, $\operatorname{codim}(\alpha)$ is the number of conditions imposed by the singularity α. If α is a topological singularity type, then inside Def there is the equisingular locus ES, which parametrizes equisingular or topologically trivial deformations. In this case, $\operatorname{codim}(\alpha)$ is $\dim_{\mathbb{C}}\mathbb{C}\{x, y\}/\langle f + J(f)\rangle - \dim_{\mathbb{C}} ES$ and is also the expected codimension of curves with singularity α in a linear system.

A natural question is, in order to determine the singularity type of $f = 0$ at the origin, is it sufficient to look at the first several terms of f? For example, does the curve $y^2 = x^3 + xy^{99}$ have a cusp because the term x^{99} can be ignored? The answer is yes, provided the terms being ignored are of sufficiently high degrees, according to the finite-determinacy theorem.

Definition 2.1. We say that a germ f is (analytically) k-determined if $f \equiv g \pmod{\mathfrak{m}^{k+1}}$ implies they are contact equivalent; i.e. there exists an automorphism ϕ of $\mathbb{C}\{x, y\}$ and a unit $u \in \mathbb{C}\{x, y\}$ such that $f = u \cdot \phi(g)$. In other words, if f and g differ by an element in $\mathfrak{m}^{k+1}$, then the curves $f = 0$ and $g = 0$ have analytically equivalent singularities at the origin.

Theorem 2.1 [GLS07, Theorem 2.23]. *If f is (analytically) k-determined, then*

$$\mathfrak{m}^{k+1} \subseteq \mathfrak{m}J(f) + \langle f\rangle.$$

Conversely, if

$$\mathfrak{m}^{k} \subseteq \mathfrak{m}J(f) + \langle f\rangle,$$

then f is k-determined.

Corollary 2.2 [GLS07, Corollary 2.24]. *For any germ $f \in \mathfrak{m} \subset \mathbb{C}\{x, y\}$, f is $\tau(f)$-determined.*

As a result, all representatives of an analytical (respectively topological) singularity α are $\tau(\alpha)$-determined (respectively $\mu(\alpha)$-determined). Therefore we can define $k(\alpha)$ to be the smallest k such that all representatives of α are k-determined.

Definition 2.2. For any isolated planar (analytic or topological) singularity α, pick a representative $f_\alpha \in \mathbb{C}\{x, y\}$ and let $N(\alpha)$ be the length of the zero-dimensional closed subscheme $\xi_\alpha = \operatorname{Spec} \mathbb{C}\{x, y\}/\langle f_\alpha, \mathfrak{m}^{k(\alpha)+1}\rangle$.

The number $N(\alpha)$ only depends on $k(\alpha)$ and the multiplicity of f_α at the origin, which is an invariant of the singularity.

Example 2.1. If α is a simple node, we can choose $f_\alpha = xy$. Then $\tau(\alpha) = \operatorname{codim}(\alpha) = 1$, $k(\alpha) = 2$, $\xi_\alpha = \operatorname{Spec} \mathbb{C}\{x, y\}/\langle xy, \mathfrak{m}^3\rangle$ and $N(\alpha) = 5$.

Example 2.2. If α is an ordinary cusp, we can choose $f_\alpha = y^2 - x^3$. Then $\tau(\alpha) = \mathrm{codim}(\alpha) = 2$, $k(\alpha) = 3$, $\xi_\alpha = \mathrm{Spec}\,\mathbb{C}\{x, y\}/\langle y^2 - x^3, \mathfrak{m}^4\rangle$ and $N(\alpha) = 7$.

Example 2.3. If α is an analytical n-fold point ($n \geqslant 3$) which is defined by $f_\alpha = x^n - y^n$, then $\tau(\alpha) = \mathrm{codim}(\alpha) = (n-1)^2 = \dim_{\mathbb{C}}\mathbb{C}[x^i y^j \,|\, 0 \leqslant i, j \leqslant n-2]$, $k(\alpha) = 2n - 4$, $\xi_\alpha = \mathrm{Spec}\,\mathbb{C}\{x, y\}/\langle x^n - y^n, \mathfrak{m}^{2n-3}\rangle$.

Example 2.4. If α is a topological n-fold point ($n \geqslant 3$), we can choose $f_\alpha = x^n - y^n$ again. Then $\tau(f_\alpha) = (n-1)^2$, $\mathrm{codim}(\alpha) = n(n+1)/2 - 2$ (all terms of degree less than n have to vanish), it is not easy to compute $k(\alpha)$ but it satisfies $2n - 4 \leqslant k(\alpha) \leqslant \mu(\alpha) = (n-1)^2$.

Recall that if $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{l(\underline{\alpha})})$ is a collection of isolated planar (analytic or topological) singularity types, a curve C has singularity type $\underline{\alpha}$ if C is singular at exactly $l(\underline{\alpha})$ distinct points $\{x_1, x_2, \ldots, x_{l(\underline{\alpha})}\}$ and the singularity type at x_i is α_i. Note we consider the topological and analytic singularities defined by the same germ to be different singularities.

Define

$$N(\underline{\alpha}) = \sum_{i=1}^{l(\underline{\alpha})} N(\alpha_i) \quad \text{and} \quad \mathrm{codim}(\underline{\alpha}) = \sum_{i=1}^{l(\underline{\alpha})} \mathrm{codim}(\alpha_i).$$

If the line bundle is sufficiently ample, the conditions imposed by the singularities are independent, and $\mathrm{codim}(\underline{\alpha})$ is the expected codimension of the locus of curves with singularity types $\underline{\alpha}$ in $|L|$.

For a smooth surface S, let $S^{[N(\underline{\alpha})]}$ be the Hilbert scheme of $N(\underline{\alpha})$ points on S. Define $S^0(\underline{\alpha}) \subset S^{[N(\underline{\alpha})]}$ to be the set of points $\coprod_{i=1}^{l(\underline{\alpha})} \eta_i$ satisfying the following conditions:

(i) the η_i are supported on distinct points of S;
(ii) every η_i is isomorphic to $\mathrm{Spec}\,\mathbb{C}\{x, y\}/\langle g_i, \mathfrak{m}^{k(\alpha_i)+1}\rangle$, for some germ g_i such that $g_i = 0$ has singularity type α_i at the origin.

Consider the closure $S(\underline{\alpha}) = \overline{S^0(\underline{\alpha})}$ as a closed subscheme in $S^{[N(\underline{\alpha})]}$ with reduced induced scheme structure. For every $n \in \mathbb{N}$, let $Z_n \subset S \times S^{[n]}$ be the universal closed subscheme with projections $p_n : Z_n \to S$, $q_n : Z_n \to S^{[n]}$.

DEFINITION 2.3. If L is a line bundle on S, define $L^{[n]} = (q_n)_*(p_n)^*L$. Because q_n is finite and flat, $L^{[n]}$ is a vector bundle of rank n on $S^{[n]}$ and it is called the *tautological bundle* of L.

To count curves with singularity type $\underline{\alpha}$, we use the cycle

$$\Lambda_{\underline{\alpha}}(S, L) := c_{N(\underline{\alpha}) - \mathrm{codim}(\underline{\alpha})}(L^{[N(\underline{\alpha})]}) \cap [S(\underline{\alpha})]$$

in the Chow group of $S^{[N(\underline{\alpha})]}$.

LEMMA 2.3. *The cycle $\Lambda_{\underline{\alpha}}(S, L)$ is a zero cycle; i.e.* $\dim S(\underline{\alpha}) = N(\underline{\alpha}) - \mathrm{codim}(\underline{\alpha})$.

Proof. It suffices to prove that for every isolated singularity α, the dimension of $S^0(\alpha)$ is equal to $N(\alpha) - \mathrm{codim}(\alpha)$.

Suppose that α is an analytic singularity, by definition every closed subscheme in $S^0(\alpha)$ is only supported at one point on S. Define the projection $p : S^0(\alpha) \to S$ to be the map sending a closed subscheme to its support, then the fiber over a point is the collection of closed subschemes $\mathrm{Spec}\,\mathbb{C}\{x, y\}/\langle g, \mathfrak{m}^{k(\alpha)+1}\rangle$ supported at that point such that $g = 0$ has singularity α. If we pick a representative f, all such g are in the orbit of f under the group action by $\mathcal{K} = \mathbb{C}\{x, y\}^* \ltimes \mathrm{Aut}(\mathbb{C}\{x, y\})$. Let $\mathrm{orb}(f)$ be the orbit of f in $\mathbb{C}\{x, y\}/\mathfrak{m}^{k(\alpha)+1}$ under the action of the restriction

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of $\mathcal{K}$ on the $\mathbb{C}\{x,y\}/\mathfrak{m}^{k(\alpha)+1}$. According to [GLS07], $\operatorname{orb}(f)$ is smooth and its tangent space at f is

$$(\mathfrak{m}\cdot J(f)+\langle f\rangle+\mathfrak{m}^{k(\alpha)+1})/\mathfrak{m}^{k(\alpha)+1}.$$

But since $\operatorname{Spec}\mathbb{C}\{x,y\}/\langle f,\mathfrak{m}^{k(\alpha)+1}\rangle$ and $\operatorname{Spec}\mathbb{C}\{x,y\}/\langle u\cdot f,\mathfrak{m}^{k(\alpha)+1}\rangle$ describe the same closed subscheme if u is a unit in $\mathbb{C}\{x,y\}/\mathfrak{m}^{k(\alpha)+1-m(f)}$ (where $m(f)$ is the multiplicity of f at the origin), $\operatorname{orb}(f)/(\mathbb{C}\{x,y\}/\mathfrak{m}^{k(\alpha)+1-m(f)})^*$ is isomorphic to the fiber of p over every point on S.

The discussion above and Theorem 2.1 imply

$$\begin{aligned}
&\dim_{\mathbb{C}}S^0(\alpha)\\
&\quad=2+\dim_{\mathbb{C}}(\mathfrak{m}\cdot J(f)+\langle f\rangle+\mathfrak{m}^{k(\alpha)+1})/\mathfrak{m}^{k(\alpha)+1}-\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/\mathfrak{m}^{k(\alpha)+1-m(f)}\\
&\quad=\dim_{\mathbb{C}}(J(f)+\langle f\rangle)/\mathfrak{m}^{k(\alpha)+1}-\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/\mathfrak{m}^{k(\alpha)+1-m(f)}\\
&\quad=\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/\mathfrak{m}^{k(\alpha)+1}-\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/(J(f)+\langle f\rangle)-\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/\mathfrak{m}^{k(\alpha)+1-m(f)}\\
&\quad=(\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/\langle f,\mathfrak{m}^{k(\alpha)+1}\rangle+\dim_{\mathbb{C}}\langle f,\mathfrak{m}^{k(\alpha)+1}\rangle/\mathfrak{m}^{k(\alpha)+1})\\
&\qquad-\operatorname{codim}(\alpha)-\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/\mathfrak{m}^{k(\alpha)+1-m(f)}\\
&\quad=N(\alpha)+\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/\mathfrak{m}^{k(\alpha)+1-m(f)}-\operatorname{codim}(\alpha)-\dim_{\mathbb{C}}\mathbb{C}\{x,y\}/\mathfrak{m}^{k(\alpha)+1-m(f)}\\
&\quad=N(\alpha)-\operatorname{codim}(\alpha).
\end{aligned}$$

If $f=0$ defines an analytic singularity α and a topological singularity β at the origin, it follows from definition that $\dim_{\mathbb{C}}S^0(\beta)=\dim_{\mathbb{C}}S^0(\beta)+\dim_{\mathbb{C}}ES$, $N(\alpha)=N(\beta)$, and $\operatorname{codim}(\beta)=\operatorname{codim}(\alpha)-\dim_{\mathbb{C}}ES$. Therefore the desired equality is established for topological singularity types. □

By the previous lemma, we can define $d_{\underline{\alpha}}(S,L)$ to be the degree of the zero cycle $\Lambda_{\underline{\alpha}}(S,L)$.

Definition 2.4. We call a line bundle L k-very ample if for every zero-dimensional subscheme $\xi\subset S$ of length $k+1$, the natural restriction map $H^0(S,L)\to H^0(\xi,L\otimes\mathcal{O}_\xi)$ is surjective.

If L and M are very ample, $L^{\otimes k}\otimes M^{\otimes l}$ is $(k+l)$-very ample. In particular, very ampleness implies 1-very ampleness.

Proposition 2.4. *Assume L is $(N(\underline{\alpha})+2)$-very ample, then a general linear subsystem $V\subset|L|$ of dimension* $\operatorname{codim}(\underline{\alpha})$ *contains precisely $d_{\underline{\alpha}}(S,L)$ curves whose singularity types are $\underline{\alpha}$.*

Proof. The structure of the proof is very similar to [Göt98, Proposition 5.2]. However, we have to make a generalization to deal with all singularity types.

If $\{s_i\}$ is a basis of V, then $\{(q_n)_*(p_n)^*s_i\}$ are global sections in $H^0(L^{[N(\underline{\alpha})]})$. By the ampleness assumption, $H^0(L)\to L|_\xi$ is surjective for every $\xi\in S^{[N(\underline{\alpha})]}$. Therefore, for general V the locus W where $\{(q_n)_*(p_n)^*s_i\}$ are linearly dependent is of the expected dimension $N(\underline{\alpha})-\operatorname{codim}(\underline{\alpha})$, and W is Poincare dual to $c_{N(\underline{\alpha})-\operatorname{codim}(\underline{\alpha})}(L^{[N(\underline{\alpha})]})$ by the Thom–Porteous formula, see [Ful98, Example 14.4.2]. Consequently,

$$[W\cap S(\underline{\alpha})]=c_{N(\underline{\alpha})-\operatorname{codim}(\underline{\alpha})}(L^{[N(\underline{\alpha})]})\cap[S(\underline{\alpha})]=\Lambda_{\underline{\alpha}}(S,L).$$

Because the dimension of $S(\underline{\alpha})\backslash S^0(\underline{\alpha})$ is less than $N(\underline{\alpha})-\operatorname{codim}(\underline{\alpha})$, $\Lambda_{\underline{\alpha}}(S,L)$ is only supported on $S^0(\underline{\alpha})$ for general V. By an argument similar to [Göt98, Proposition 5.2], W is smooth for general V. Therefore $d_{\underline{\alpha}}(S,L)$ is the number of curves in a general $\operatorname{codim}(\underline{\alpha})$-dimensional linear subsystem in $|L|$ which contain a point in $S^0(\underline{\alpha})$.

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Next, we show curves in $\Lambda_{\underline{\alpha}}(S,L)$ can not have more than $l(\underline{\alpha})$ singular points. If C is a curve in $\Lambda_{\underline{\alpha}}(S,L)$ with $l(\underline{\alpha})+1$ singular points, then C must contain a point in

$$S'(\underline{\alpha}) = \left\{ \coprod_{i=1}^{l(\underline{\alpha})+1} \eta_i \,\middle|\, \coprod_{i=1}^{l(\underline{\alpha})} \eta_i \in S^0(\underline{\alpha}) \text{ and } \eta_{l(\underline{\alpha})+1} \cong \operatorname{Spec} \mathbb{C}\{x,y\}/\mathfrak{m}^2 \right\} \subset S^{[N(\underline{\alpha})+3]}.$$

In a general linear subsystem of dimension $\operatorname{codim}(\underline{\alpha})$ in $|L|$, the number of such curves can be computed by

$$c_{N(\underline{\alpha})+3-\operatorname{codim}(\underline{\alpha})}(L^{[N(\underline{\alpha})+3]}) \cap [\overline{S'(\underline{\alpha})}].$$

But it vanishes because

$$\dim \overline{S'(\underline{\alpha})} = \dim S(\underline{\alpha}) + 1 < N(\underline{\alpha}) + 3 - \operatorname{codim}(\underline{\alpha}).$$

It also follows that all curves in $\Lambda_{\underline{\alpha}}(S,L)$ must be reduced.

Finally, we prove that curves in $\Lambda_{\underline{\alpha}}(S,L)$ must have singularity type precisely $\underline{\alpha}$. Let C be a curve in $\Lambda_{\underline{\alpha}}(S,L)$, then it must contain some point $\coprod_{i=1}^{l(\underline{\alpha})} \eta_i$ in $S^0(\underline{\alpha})$ and hence has singularity 'at least' α_i at x_i, where x_i is the support of η_i. If the singularity type of C is not $\underline{\alpha}$, without loss of generality we can assume the singularity type at x_1 is not α_1. Let $\eta_1 = \operatorname{Spec} \mathbb{C}\{x,y\}/\langle g_1, \mathfrak{m}^{k(\alpha_1)+1}\rangle$ and the germ of C at x_1 be f. Since C contains η_1, $f \in \langle g_1, \mathfrak{m}^{k(\alpha_1)+1}\rangle$ so $f \equiv u \cdot g_1 \pmod{\mathfrak{m}^{k(\alpha_1)+1}}$. If u is a unit, then $u^{-1}f$ also defines C and the finite-determinacy theorem implies C must have singularity precisely α_1 and this is a contradiction. Otherwise, u is not a unit and f is in the ideal $\langle g_1\mathfrak{m}, \mathfrak{m}^{k(\alpha_1)+1}\rangle$. Let $\tilde{S}^0(\underline{\alpha})$ be the set of $\operatorname{Spec} \mathbb{C}\{x,y\}/\langle g_1\mathfrak{m}, \mathfrak{m}^{k(\alpha_1)+1}\rangle \cup (\coprod_{i=2}^{l(\underline{\alpha})} \eta_i)$ such that there exists $\eta_1 = \operatorname{Spec} \mathbb{C}\{x,y\}/\langle g_1, \mathfrak{m}^{k(\alpha_1)+1}\rangle$ and $\coprod_{i=1}^{l(\underline{\alpha})} \eta_i \in S^0(\underline{\alpha})$. The natural map from $S^0(\underline{\alpha})$ to $\tilde{S}^0(\underline{\alpha})$, which sends $\coprod_{i=1}^{l(\underline{\alpha})} \eta_i$ to $\operatorname{Spec} \mathbb{C}\{x,y\}/\langle g_1\mathfrak{m}, \mathfrak{m}^{k(\alpha_1)+1}\rangle \cup (\coprod_{i=2}^{l(\underline{\alpha})} \eta_i)$, is surjective and therefore $\dim S^0(\underline{\alpha}) \geqslant \dim \tilde{S}^0(\underline{\alpha})$. Let $\tilde{S}(\underline{\alpha})$ be the closure of $\tilde{S}^0(\underline{\alpha})$ in $S^{[N(\underline{\alpha})+1]}$ and apply the Thom–Porteous formula to $\tilde{S}(\underline{\alpha})$, we see C contributes a positive number in the counting $\int_{\tilde{S}(\underline{\alpha})} c_{N(\underline{\alpha})+1-\operatorname{codim}(\underline{\alpha})}(L^{[N(\underline{\alpha})+1]})$. On the other hand, this intersection is empty because

$$N(\underline{\alpha}) + 1 - \operatorname{codim}(\underline{\alpha}) > N(\underline{\alpha}) - \operatorname{codim}(\underline{\alpha}) = \dim S(\underline{\alpha}) \geqslant \dim \tilde{S}(\underline{\alpha}).$$

This is a contradiction and C must have singularity type precisely $\underline{\alpha}$. □

Remark. Kleiman pointed out to us that one can associate every topological singularity α with a complete ideal I_α, such that a general element in I_α defines a curve with singularity α. Therefore we can also associate α to the isomorphism type of $\operatorname{Spec} \mathbb{C}\{x,y\}/I_\alpha$. This approach is taken by Rennemo [Ren12], and it may weaken the ampleness condition needed in Proposition 2.4 and in Theorem 3.4.

3. Universal polynomials and generating series

In this section, we prove the existence of a universal polynomial that counts the number of curves of singularity type $\underline{\alpha}$. In particular, we will prove a degeneration formula for the tautological integrals $d_{\underline{\alpha}}(S,L)$, and show $d_{\underline{\alpha}}(S,L)$ is the universal polynomial we are looking for. Moreover, we construct the generating series of $d_{\underline{\alpha}}(S,L)$ and show the series has a compact exponential description.

Let U be a curve and $\infty \in U$ be a specialized point, consider a flat projective family of schemes $\pi : \mathcal{X} \to U$ that satisfies:

(i) $\mathcal{X}$ is smooth and π is smooth away from the fiber $\pi^{-1}(\infty)$;

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(ii) $\pi^{-1}(\infty) =: X_1 \bigcup_D X_2$ is a union of two irreducible smooth components X_1 and X_2 which intersect transversally along a smooth divisor D.

In [LW11], Li and Wu constructed a family of Hilbert schemes of n points $\pi^{[n]} : \mathcal{X}^{[n]} \to U$, whose smooth fiber over $t \neq \infty$ is $X_t^{[n]}$, the Hilbert scheme of n points on the smooth fiber X_t of π. To compactify this moduli space, one has to replace $\mathcal{X}$ by a new space $\mathcal{X}[n]$ so that $\mathcal{X}$ and $\mathcal{X}[n]$ have the same smooth fibers over $t \neq \infty$, but over ∞ the fiber of $\mathcal{X}[n]$ is a semistable model

$$\mathcal{X}[n]_\infty = X_1 \cup \Delta_1 \cup \Delta_2 \cup \ldots \Delta_{n-1} \cup X_2,$$

where $\Delta_i \cong \mathbb{P}_D(\mathcal{O}_D \oplus N_{X_1/D})$. The fiber of $\mathcal{X}^{[n]}$ over ∞ is the union of products

$$\bigcup_{k=0}^{n} (X_1/D)^{[k]} \times (X_2/D)^{[n-k]}$$

for all possible $n \geqslant k \geqslant 0$, where points in $(X_1/D)^{[k]}$ are subschemes of length k supported on the smooth locus of $X_1 \cup \Delta_1 \cup \cdots \cup \Delta_i$.

Li and Wu proved that the moduli stack $\mathcal{X}^{[n]}$ is a separated and proper Deligne–Mumford stack of finite type over U. If $\mathcal{L}$ is a line bundle on $\mathcal{X}$, there is a tautological bundle $\mathcal{L}^{[n]}$ whose restriction on smooth fibers $X_t^{[n]}$ is the tautological bundle of $\mathcal{L}|_{X_t}$ and its restriction on $(X_1/D)^{[k]} \times (X_2/D)^{[n-k]}$ is the direct sum of tautological bundles of $L_1 := \mathcal{L}|_{X_1}$ and $L_2 := \mathcal{L}|_{X_2}$. Recall $S(\underline{\alpha}) \subset S^{[N(\underline{\alpha})]}$ is the closure of the set of points $\coprod_{i=1}^{l(\underline{\alpha})} \eta_i$ so that the η_i are supported on distinct points of S and of the isomorphism type associated to α_i. Similarly we can define $(X_1/D)(\underline{\alpha})$ (respectively $(X_2/D)(\underline{\alpha})$) by taking the closure of the set of points $\coprod_{i=1}^{l(\underline{\alpha})} \eta_i$ in $(X_1/D)^{[N(\underline{\alpha})]}$ (respectively $(X_2/D)^{[N(\underline{\alpha})]}$) so that the η_i are supported on distinct points and of the isomorphism type associated to α_i. Tautological integrals on relative Hilbert schemes can now be defined by

$$d_{\underline{\alpha}}(X_i/D, L_i) = \int_{(X_i/D)(\underline{\alpha})} c_{N(\underline{\alpha})-\mathrm{codim}(\underline{\alpha})}(L_i^{N(\underline{\alpha})}).$$

Lemma 3.1. *There is a family of cycles $\mathcal{X}(\underline{\alpha}) \subset \mathcal{X}^{[N(\underline{\alpha})]}$ such that*

$$\mathcal{X}(\underline{\alpha}) \cap X_t^{[N(\underline{\alpha})]} = X_t(\underline{\alpha}) \quad \text{for } t \neq \infty$$

and

$$\mathcal{X}(\underline{\alpha}) \cap ((X_1/D)^{[m]} \times (X_2/D)^{[N(\underline{\alpha})-m]}) = \bigcup (X_1/D)(\underline{\alpha}_1) \times (X_2/D)(\underline{\alpha}_2),$$

where the sum is over all $\underline{\alpha}_1$ and $\underline{\alpha}_2$ satisfying $\underline{\alpha} = \underline{\alpha}_1 \cup \underline{\alpha}_2$, $N(\underline{\alpha}_1) = m$, $N(\underline{\alpha}_2) = N(\underline{\alpha}) - m$. Furthermore, $\mathcal{X}(\underline{\alpha})$ is flat over U via the composition $\mathcal{X}(\underline{\alpha}) \hookrightarrow \mathcal{X}^{[N(\underline{\alpha})]} \to U$.

Proof. Let $\mathcal{X}^0(\underline{\alpha})$ be the union of all $\mathcal{X}_t(\underline{\alpha})$, for all smooth fibers $\mathcal{X}_t$ over $t \in U$. Define $\mathcal{X}(\underline{\alpha})$ to be the closure of $\mathcal{X}^0(\underline{\alpha})$ in $\mathcal{X}^{[N(\underline{\alpha})]}$, then by definition $\mathcal{X}(\underline{\alpha}) \cap X_t^{[N(\underline{\alpha})]} = X_t(\underline{\alpha})$. Since $\mathcal{X}$ is a smooth family of complex varieties, for every point p in $\mathcal{X}$ one can choose an analytic neighborhood V_p around p such that V_p is a trivial fibration over its image in U.

Suppose b is a point in $\mathcal{X}(\underline{\alpha})$ which lies over ∞. Write b as the union of disjoint schemes b_i, so that b_i is only supported at one point p_i. Then p_i must belong to the smooth locus of a component Y of some semistable model $\mathcal{X}[n]_\infty = X_1 \cup \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_{n-1} \cup X_2$ for some n. Since b is in the closure of $\mathcal{X}^0(\underline{\alpha})$, there is a sequence of points $\{b_j\}$ in $\mathcal{X}^0(\underline{\alpha})$ approaching b. By shrinking V_{p_i}, we can assume $\{V_{p_i}\}$ are pairwise disjoint and every b_j is in $\bigcup_i V_{p_i}$, thus every b_j is the disjoint union of closed subschemes b_{ji} of V_{p_i}.

Since $V_{p_i} \subset \mathcal{X} \to U$ is a trivial fibration over its image, there is a projection $q_i : V_{p_i} \to Y$ whose image is a small open neighborhood of p_i on Y. Since q_i is a trivial projection which does not change the isomorphism type of zero-dimensional schemes, $\coprod_i q_i(b_{ji})$ is a sequence in $\mathcal{X}(\underline{\alpha})$ that approaches $\coprod_i b_i = b$. It follows that b_i is in $Y(\beta_i)$, where β_i is the union of several α_i, because if we take the limit of a family of a union of closed subschemes of types $\alpha_1, \ldots, \alpha_{l(\underline{\alpha})}$, some closed subschemes would approach each other and create a degenerated isomorphism type. Therefore the splitting $b = \coprod_i b_i$ corresponds to a partition $\underline{\alpha} = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_{n+1}$ such that b is in

$$(X_1/D)(\beta_1) \times (\Delta_1/D)(\beta_2) \times \cdots \times (\Delta_{n-1}/D)(\beta_n) \times (X_2/D)(\beta_{n+1})$$

and therefore belongs to $(X_1/D)(\underline{\alpha}_1) \times (X_2/D)(\underline{\alpha}_2)$ if there exists m such that

$$\underline{\alpha}_1 = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_m, \quad \underline{\alpha}_2 = \beta_{m+1} \cup \cdots \cup \beta_{n+1}.$$

On the other hand, if $\underline{\alpha} = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_{n+1}$ then

$$(X_1/D)(\beta_1) \times (\Delta_1/D)(\beta_2) \times \cdots \times (\Delta_{n-1}/D)(\beta_n) \times (X_2/D)(\beta_{n+1})$$

is a subset of $\mathcal{X}(\underline{\alpha})$. This is because the trivialized neighborhood allows us to translate subschemes away from the fiber of ∞ by following sections of $V_{p_i} \to U$ and this translation does not change isomorphism types. We conclude that

$$\mathcal{X}(\underline{\alpha}) \cap ((X_1/D)^{[m]} \times (X_2/D)^{[N(\underline{\alpha})-m]}) = \bigcup (X_1/D)(\underline{\alpha}_1) \times (X_2/D)(\underline{\alpha}_2)$$

as desired.

The open part $\mathcal{X}^0(\underline{\alpha})$ is irreducible and it dominates the curve U. Since the scheme structure of $\mathcal{X}(\underline{\alpha})$ is the induced reduced structure, $\mathcal{X}(\underline{\alpha})$ is flat over U. □

Assign a formal variable x_α to every isolated singularity type α and define $x_{\underline{\alpha}} = \prod_{i=1}^{l(\underline{\alpha})} x_{\alpha_i}$ if $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{l(\underline{\alpha})})$. The multiplication $x_{\underline{\alpha}'} \cdot x_{\underline{\alpha}''}$ is equal to $x_{\underline{\alpha}}$ if and only if $\underline{\alpha}$ is the union of $\underline{\alpha}'$ and $\underline{\alpha}''$. The multiplication is commutative because permutations of α_i do not change $\underline{\alpha}$.

Definition 3.1. For a line bundle L on S (L does not need to be ample), let $d_{\underline{\alpha}}(S, L) = 1$ if $\underline{\alpha}$ is the empty set (it corresponds to the number of smooth curves in $|L|$ through $\dim |L|$ general points). Define the generating series

$$T(S, L) = \sum_{\underline{\alpha}} d_{\underline{\alpha}}(S, L) x_{\underline{\alpha}}$$

and similarly

$$T(S/D, L) = \sum_{\underline{\alpha}} d_{\underline{\alpha}}(S/D, L) x_{\underline{\alpha}}$$

if D is a smooth divisor of S.

Theorem 3.2. *There exist universal power series A_1, A_2, A_3, A_4 in $\mathbb{Q}[[x_\alpha]]$ such that the generating series $T(S, L)$ has the form*

$$T(S, L) = A_1^{L^2} A_2^{LK_S} A_3^{c_1(S)^2} A_4^{c_2(S)}. \tag{3.1}$$

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Proof. The proof uses algebraic cobordism of pairs of smooth schemes and vector bundles, and degeneration of Quot schemes, which are developed in [LP12, LP04] and [LW11]. We only need to use the special case of pairs of surfaces and line bundles and degeneration of Hilbert schemes of points, which is summarized in [Tze12, §§ 2 and 3].

Let $[X_i, L_i]$ be pairs of surfaces and line bundles. Suppose

$$[X_0, L_0] = [X_1, L_1] + [X_2, L_2] - [X_3, L_3]$$

is the double point relation obtained from a flat morphism $\mathcal{X} \to \mathbb{P}^1$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$. That means $\mathcal{X}$ is a smooth 3-fold, X_0 is the smooth fiber over 0 and the fiber over ∞ is $X_1 \cup X_2$, intersecting transversally along a smooth divisor D. Moreover, $X_3 = \mathbb{P}(\mathcal{O}_D \oplus N_{X_1/D}) \cong \mathbb{P}(N_{X_2/D} \oplus \mathcal{O}_D)$ is a $\mathbb{P}^1$ bundle over D, $L_i = \mathcal{L}|_{X_i}$ for $i = 0, 1, 2$ and L_3 is the pullback of $\mathcal{L}|_D$ to X_3. The algebraic cobordism group of pairs of surfaces and line bundles $\omega_{2,1}$ is defined to be the formal sum of all pairs modulo double point relations. The class of $[S, L]$ in $\omega_{2,1}$ is uniquely determined by all Chern numbers of L and S; i.e. L^2, LK, $c_1(S)^2$ and $c_2(S)$ (see [LP12, Tze12]).

For every double point relation induced by $\mathcal{X} \to \mathbb{P}^1$, because some fiber X_t could be singular, we can find an open subset U of $\mathbb{P}^1$ such that $0, \infty \in U$ and all other fibers X_t over U are smooth. Therefore Li and Wu's construction of relative Hilbert schemes [LW11] can be applied to the family

$$\pi : \mathcal{X}_U := \mathcal{X} \underset{\mathbb{P}^1}{\times} U \to U.$$

The line bundle $\mathcal{L}$ on $\mathcal{X}$ induces the tautological bundle ${\mathcal{L}_U}^{[n]}$ on $\mathcal{X}_U^{[n]}$. By Lemma 3.1 the restrictions of the flat 1-cycle $\int_{\mathcal{X}_{U(\underline{\alpha})}} c_{N(\underline{\alpha})-\mathrm{codim}(\underline{\alpha})}({\mathcal{L}_U}^{[N(\underline{\alpha})]})$ on fibers over 0 and ∞ are zero cycles of the same degree, which implies

$$d_{\underline{\alpha}}(X_0, L_0) = \sum_{\underline{\alpha} = \underline{\alpha}_1 \cup \underline{\alpha}_2} d_{\underline{\alpha}_1}(X_1/D, L_1) d_{\underline{\alpha}_2}(X_2/D, L_2)$$

and

$$T(X_0, L_0) = T(X_1/D, L_1) T(X_2/D, L_2). \tag{3.2}$$

To derive a relation of generating series without relative series, we apply (3.2) to four families: $\mathcal{X}$, the blowup of $X_1 \times \mathbb{P}^1$ along $D \times \{\infty\}$, the blowup of $X_2 \times \mathbb{P}^1$ along $D \times \{\infty\}$, the blowup of $X_3 \times \mathbb{P}^1$ along $D \times \{\infty\}$, and multiply all equalities. Therefore, a double point relation

$$[X_0, L_0] = [X_1, L_1] + [X_2, L_2] - [X_3, L_3]$$

implies

$$T(X_0, L_0) = \frac{T(X_1, L_1) T(X_2, L_2)}{T(X_3, L_3)}. \tag{3.3}$$

Equation (3.3) implies that T induces a homomorphism from the algebraic cobordism group $\omega_{2,1}$ to $\mathbb{C}[[x_\alpha]]$. So (3.1) can be proved by quoting the theorem that the algebraic cobordism group $\omega_{2,1}$ is isomorphic to $\mathbb{Q}^4$ by the morphism $[S, L]$ to $(L^2, LK, c_1(S)^2, c_2(S))$ (see [LP12, Tze12]). □

Corollary 3.3. *The generating series $T(S, L)$ has an exponential description*

$$T(S, L) = \exp\left(\sum_{\underline{\alpha}} \frac{a_{\underline{\alpha}}(L^2, LK, c_1(S)^2, c_2(S)) x_{\underline{\alpha}}}{\# \operatorname{Aut}(\underline{\alpha})}\right)$$

where every $a_{\underline{\alpha}}(L^2, LK, c_1(S)^2, c_2(S))$ is a linear polynomial in L^2, LK, $c_1(S)^2$ and $c_2(S)$.

Proof. Since $\ln T$ is an additive homomorphism from $\omega_{2.1}$ to $\mathbb{C}[[x_\alpha]]$, so is every coefficient of $x_{\underline{\alpha}}$ in $\ln T$. □

The following theorem shows Göttsche's conjecture holds for curves on smooth surfaces with any given topological or analytic singularities.

Theorem 3.4. *For every collection of isolated singularity type $\underline{\alpha}$, there exists a universal polynomial $T_{\underline{\alpha}}(x, y, z, t)$ of degree $l(\underline{\alpha})$ with the following property: given a smooth projective surface S and an $(N(\underline{\alpha}) + 2)$-very ample line bundle L on S, a general* $\operatorname{codim}(\underline{\alpha})$*-dimensional sublinear system of $|L|$ contains exactly $T_{\underline{\alpha}}(L^2, LK, c_1(S)^2, c_2(S))$ curves with singularity type precisely $\underline{\alpha}$.*

Proof. We compare the coefficient of $x_{\underline{\alpha}}$ in (3.1). The coefficient of $x_{\underline{\alpha}}$ in $T(S, L)$ is $d_{\underline{\alpha}}(S, L)$; on the right-hand side the coefficient of $x_{\underline{\alpha}}$ can be computed by binomial expansion (note L^2, LK, $c_1(S)^2$ and $c_2(S)$ are integers) and it is a polynomial of degree $l(\underline{\alpha})$. Therefore $d_{\underline{\alpha}}(S, L)$ is always a universal degree d polynomial of L^2, LK, $c_1(S)^2$ and $c_2(S)$. Moreover, Proposition 2.4 implies that $d_{\underline{\alpha}}(S, L)$ is the universal polynomial $T_{\underline{\alpha}}(L^2, LK, c_1(S)^2, c_2(S))$ counting the number of curves with singularity type $\underline{\alpha}$. □

Recently both we and Rennemo [Ren12] realized Göttsche's conjecture can also be generalized to the higher dimension case. Here we only state the theorem for analytic singularities but in fact the same statement should hold for any isolated complete intersection singularity satisfying:

(i) if the singularity type is k-determined, then for any variety Y which has the singularity type at p, all $\operatorname{Spec} \mathcal{O}_{Y,p}/\mathfrak{m}_{Y,p}^{k+1}$ are of the same length (every isolated complete intersection singularity is finitely determined);

(ii) the locus of points in the punctual Hilbert scheme isomorphic to those in (i) is a constructible set.

Definition 3.2. If E is a vector bundle on X, E is called k-very ample if for every closed subscheme ξ of length $k+1$, the natural restriction map $H^0(X, E) \to H^0(\xi, E \otimes \mathcal{O}_\xi)$ is surjective.

Theorem 3.5. *Let X be a smooth projective complex variety of dimension n and E_i be vector bundles of rank r_i. Suppose $\underline{\alpha}$ is a collection of analytic isolated complete intersection singularities, every vector bundle E_i is $(N(\underline{\alpha}) + n)$-very ample, V_i are general dimension m_i linear subspaces of $\mathbb{P}(H^0(E_i))$ and $\sum m_i =$* $\operatorname{codim}(\underline{\alpha})$*. For each i we take a section s_i in V_i and consider the common zero of s_i, then the number of such subvarieties of X with singularity type $\underline{\alpha}$ is a universal polynomial of Chern numbers of X and E_i.*

Proof. The tautological bundle of E_i on the Hilbert scheme of n points $X^{[n]}$ can be defined by pulling back E_i to the incidence scheme, then pushing forward the result to $X^{[n]}$. The final result $E_i^{[n]}$ is a vector bundle of rank nr_i. The argument in § 2 shows the number of subvarieties in question is given by the tautological integral

$$\int_{X(\underline{\alpha})} \prod_i c_{N(\underline{\alpha}) r_i - m_i}(E_i^{N(\underline{\alpha})}).$$

By [Ren12, Theorem 1.1], the tautological integral above is a universal polynomial in the Chern numbers of X and E_i. Therefore the theorem is proved. We note that the proof of Theorem 3.4 can also be used to show the tautological integral is universal, because the results in algebraic cobordism and degeneration of Hilbert schemes are established for any dimension. □

Although the universal polynomial $T_{\underline{\alpha}}(S,L)$ is equal to $d_{\underline{\alpha}}(S,L)$, the degree of a explicit zero cycle on Hilbert schemes, there is no known way to compute $d_{\underline{\alpha}}(S,L)$ directly. Usually the explicit formula is obtained by other methods and here are some cases where the universal polynomials are known.

- If $\underline{\alpha}$ is r nodes for any $r \in \mathbb{N}$, the universal polynomial of r-nodal curves on surfaces can be determined by combining any two of [BL00, CH98] and [Vak00], or by [KST11]. The explicit formula for $r \leqslant 8$ was proved and listed in [KP99] and [Vai95].
- If $\underline{\alpha}$ is r nodes, Vainsencher [Vai03] computed the number of r-nodal hypersurfaces when $r \leqslant 6$.
- Kleiman and Piene [KP99] also computed $T_{\underline{\alpha}}$ in many cases when the codimension is low, such as $\underline{\alpha} = (D_4), (D_4, A_1)$, (D_4, A_1, A_1), (D_4, A_1, A_1, A_1), (D_6), (D_6, A_1) and (E_7).
- When there is only one singular point, Kerner [Ker06] found an algorithm to enumerate the number of plane curves with one fixed topological type singularity, provided that the normal form is known.
- If $\underline{\alpha}$ is a collection of hypersurface singularities, then we believe the Legendre characteristic classes [Kaz03, §§ 8–10] should give universal polynomials for singular hypersurfaces in any dimension. The explicit formula of Legendre characteristic classes when the codimension is at most four is listed in [Kaz03, § 8] and it agrees with Kleiman and Piene's results on surfaces. However, we do not know how to prove it using algebro-geometric methods.

Another possible way to compute universal polynomials is through the $a_{\underline{\alpha}}$ $(L^2, LK, c_1(S)^2, c_2(S))$ in Corollary 3.3. Since every $a_{\underline{\alpha}}$ has only four terms and they determine $T_{\underline{\alpha}}$, it might be easier to compute the $a_{\underline{\alpha}}$ directly. When $\underline{\alpha}$ is r nodes and $r \leqslant 8$, the $a_{\underline{\alpha}}$ were realized as algebraic cycles in [Qvi11].

4. Irreducibility of Severi strata of singular curves

The locus of irreducible reduced degree d plane curves with r nodes is a locally closed subset in $|\mathcal{O}(d)|$ on $\mathbb{P}^2$. Its closure is called the Severi variety and has been studied extensively, especially its degree. In particular, the irreducibility of Severi varieties for every d and r was proved by Harris [Har86]. In this section we will prove the irreducibility for the Severi strata of curves with arbitrary analytic singularities in the linear system of a sufficiently ample line bundle on all smooth surfaces. A similar method is used in [Kem13] for the nodal case.

Theorem 4.1. *Let $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{l(\underline{\alpha})})$ be a collection of analytic singularity types, S be a complex smooth projective surface and L be an $(N(\underline{\alpha})+2)$-very ample line bundle on S. Define $V_0^{d,\underline{\alpha}}(S,L)$ to be the locally closed subset of $|L|$ parametrizing curves with singularity type exactly $\underline{\alpha}$ in $|L|$, and $V^{d,\underline{\alpha}}(S,L)$ to be the closure of $V_0^{d,\underline{\alpha}}(S,L)$. Then $V_0^{d,\underline{\alpha}}(S,L)$ is smooth, $V^{d,\underline{\alpha}}(S,L)$ is irreducible, and their codimensions in $|L|$ are both $\operatorname{codim}(\underline{\alpha}) = \sum_{i=1}^{l(\underline{\alpha})} \tau(\alpha_i)$.*

Proof. Let $\mathcal{C}$ be the universal family of curves in $|L|$ with projection $\mathcal{C} \to |L|$, and let $\mathcal{C}^{[n]}$ be the relative Hilbert scheme of n points of the family. There are natural projections from $\mathcal{C}^{[n]}$ to $S^{[n]}$ and from $\mathcal{C}^{[n]}$ to $|L|$.

Consider the following commutative diagram.

$$\begin{tikzcd} |L| & \mathcal{C}^{[N(\underline{\alpha})]} \arrow[l] \arrow[r] & S^{[N(\underline{\alpha})]} \\ V^{d,\underline{\alpha}}(S,L) \arrow[u] & \Sigma_0 \arrow[u] \arrow[r] \arrow[ul] & S^0(\underline{\alpha}) \arrow[u] \end{tikzcd}$$

The right part of the diagram is Cartesian; i.e. Σ_0 is defined to be $\mathcal{C}^{[N(\underline{\alpha})]} \underset{S^{[N(\underline{\alpha})]}}{\times} S^0(\underline{\alpha})$, and a closed point of Σ_0 is a subscheme $\xi \in S^0(\underline{\alpha})$ satisfying $\xi \subset C$ for some curve $[C] \in |L|$.

If L is $(N(\underline{\alpha})-1)$-very ample, fibers of $\mathcal{C}^{[N(\underline{\alpha})]} \to S^{[N(\underline{\alpha})]}$ are all projective spaces of constant dimension $\dim |L| - N(\underline{\alpha})$, so $\Sigma_0 \to S^0(\underline{\alpha})$ is a projective bundle. Because $S^0(\underline{\alpha})$ is smooth and connected [GLS07], Σ_0 is also smooth connected and thus irreducible. Moreover,

$$\dim \Sigma_0 = \dim S^0(\underline{\alpha}) + \dim |L| - N(\underline{\alpha}) = \dim |L| - \operatorname{codim}(\underline{\alpha}).$$

The image $\operatorname{Im}(\Sigma_0) \subset |L|$ is the collection of curves which contain at least one closed subscheme in $S^0(\underline{\alpha})$, and thus it contains $V_0^{d,\underline{\alpha}}(S,L)$. Its closure $\overline{\operatorname{Im}(\Sigma_0)}$ contains $V^{d,\underline{\alpha}}(S,L)$ as a closed subset. By Proposition 2.4, $V_0^{d,\underline{\alpha}}(S,L)$ intersects a general linear subspace of $\operatorname{codim}(\underline{\alpha})$ at finite points. Therefore $V_0^{d,\underline{\alpha}}(S,L)$ is of codimension $\operatorname{codim}(\underline{\alpha})$ in $|L|$ and so is its closure $V^{d,\underline{\alpha}}(S,L)$.

Since $\operatorname{Im}(\Sigma_0)$ is irreducible and

$$\dim \operatorname{Im}(\Sigma_0) \leqslant \dim |L| - \operatorname{codim}(\underline{\alpha}) = \dim V^{d,\underline{\alpha}}(S,L),$$

$\overline{\operatorname{Im}(\Sigma_0)}$ is an irreducible variety that contains a closed subset $V^{d,\underline{\alpha}}(S,L)$ of the same dimension, which implies $V^{d,\underline{\alpha}}(S,L) = \overline{\operatorname{Im}(\Sigma_0)}$ is irreducible.

Next, we prove $V_0^{d,\underline{\alpha}}(S,L)$ is smooth. Let $s \in H^0(L)$ define a curve C with singularity type $\underline{\alpha}$. The germ of $|L|$ at s maps to the miniversal deformation space Def of the singularities of C. The corresponding map of tangent spaces $T_s|L| = H^0(L)/\langle s\rangle \to H^0(L \otimes \mathcal{O}_C/J(s))$ is onto if $h^0(L \otimes \mathcal{O}_C/J(s)) = \sum_{i=1}^{l(\underline{\alpha})} \tau(\alpha_i) \leqslant N(\underline{\alpha}) + 3$, because L is $(N(\underline{\alpha})+2)$-very ample. For every singularity type $\alpha_i \in \underline{\alpha}$, let f_{α_i} be a germ that defines α_i at the origin. Since f_{α_i} is (analytically) $k(\alpha_i)$-determined, $\mathfrak{m}^{k(\alpha_i)+1} \subseteq \mathfrak{m}J(f_{\alpha_i}) + \langle f_{\alpha_i}\rangle$ (Theorem 2.1). We check

$$\begin{aligned}\tau(\alpha_i) &= \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle f_{\alpha_i}, J(f_{\alpha_i})\rangle \leqslant \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle f_{\alpha_i}, \mathfrak{m}J(f_{\alpha_i})\rangle \\ &\leqslant \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\langle f_{\alpha_i}, \mathfrak{m}^{k(\alpha_i)+1}\rangle = N(\alpha_i).\end{aligned}$$

So $\sum_{i=1}^{l(\underline{\alpha})} \tau(\alpha_i) \leqslant \sum_{i=1}^{l(\underline{\alpha})} N(\alpha_i) = N(\underline{\alpha})$. It follows that the map to miniversal deformation space is a surjective map and thus $V_0^{d,\underline{\alpha}}(S,L)$ is smooth and of codimension $\operatorname{codim}(\underline{\alpha}) = \sum_{i=1}^{l(\underline{\alpha})} \tau(\alpha_i)$ in $|L|$. □

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