

Modelling rollers for shallow water flows

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Hydraulic jumps, roll waves or bores in open channel flows are often treated as singularities by hydraulicians while slowly varying shallow water flows are described by continuous solutions of the Saint-Venant equations. Richard & Gavrilyuk (*J. Fluid Mech.*, vol. 725, 2013, pp. 492–521) have enriched this model by introducing an equation for roller vorticity in a very elegant manner. This new model matches several experimental results that have resisted theoretical approaches for decades. This is the case of the roller of a stationary hydraulic jump as well as the oscillatory instability that the jump encounters when the Froude number is increased. The universality of their approach as well as its convincing comparisons with experimental results open the way for significant progress in the modelling of open channel flows.

Key words: channel flow, shallow water flows, shear layer turbulence

1. Introduction

Hydraulicians like to deal with slowly varying open flows in shallow waters since the Saint-Venant equations (Saint-Venant 1871), often referred to as the ‘shallow water equations’ yield realistic solutions in most situations. But, as written in their most famous reference book by Chow (1959), they ‘have long ago come to regard the various phenomena of rapidly varied flows as a number of isolated cases each requiring its own specific empirical treatment’. This is the case for stationary hydraulic jumps or travelling ones such as roll waves or bores, such as the one shown in the figure by the title (copyright Arnold Price; licensed for reuse under the Creative Commons Attribution-ShareAlike 2.0 license).

The shallow water equations deal with vertically averaged quantities, which removes one spatial dimension in the flow representation. Owing to their hyperbolic mathematical nature, these equations develop shocks whose discontinuities can be

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modelled through the Rankine–Hugoniot relations arising from the conservation laws for mass and momentum. The energy loss through the jump is a consequence of these two conservation laws since there is no internal energy in this model. But experimental results show that this shock model fails to represent the real phenomena in a transition zone whose extent can be one or two orders of magnitude larger than the layer depth.

The lack of a generic model to describe correctly hydraulic jumps has motivated the development of experimentally derived empirical laws. For instance, Hager, Bremen & Kawagoshi (1990) state that the roller length L is related to the depth h_- at the toe of a stationary hydraulic jump by the relation $L/h_- = 8F - 12$ when the upstream Froude number F is in the range $[2.5, 8]$. In the same range, Chanson (2011) describes the depth profile $h(x)$ of the roller by $(h(x) - h_-)/(h_+ - h_-) = (x/L)^{0.441}$ where h_+ is the downstream depth.

Recently, Richard & Gavriluk (2012, 2013) opened the way for a robust theoretical framework that matches a lot of experimental results for stationary hydraulic jumps, as well as for roll waves such as the results of Brock (1969, 1970). In a clever way, they enrich the Saint-Venant equations by taking into account the roller vorticity dynamics. One bright and convincing result of their model lies in its capacity to describe numerically the oscillatory instability encountered by the stationary hydraulic jump when the Froude number F is increased beyond the threshold $F_c = 1.5$, in agreement with the experimental results of Mok (2004).

2. Overview

There is an analogy between the dynamics of an isentropic compressible gas in a tube and an incompressible open shallow water flow, where the mass density ρ of the former corresponds to the layer depth h of the latter. In the one-dimensional horizontal case, the fluxes of the section-averaged momentum ρU and hU are respectively the pressure $P = B\rho^\gamma$ and $P = gh^2/2$, where B and γ are constants describing the perfect gas and g is gravity applied to the incompressible fluid. Conservation equations of these models form a hyperbolic system with characteristic velocities $U \pm c$ such that $c^2 = \gamma P/\rho$ and $c^2 = gh$ respectively.

These models neglect the variance $R = \langle u'^2 \rangle$ of the velocity perturbation u' to the section-averaged U , which can be justified by an asymptotic expansion when the flow is slowly varying. Although they violate this hypothesis, singularities such as shock waves and hydraulic jumps are nevertheless classically modelled with the Rankine–Hugoniot relations derived from mass conservation and the conservation law for momentum.

Richard & Gavriluk (2012, 2013) overcome this drawback in the open channel case by restoring the non-vanishing variance in the momentum flux $P = gh^2/2 + hR$. Neglecting $\langle u'^3 \rangle$ with an hypothesis of ‘weakly sheared flow’, they establish that PU is the flux of the energy hE with $E = (gh + U^2 + R)/2$. The continuous part of this model is still hyperbolic with $c^2 = gh + 3R$. A new Riemann invariant $\Omega = R/h^2$, propagating with the velocity U , can be seen as the ‘enstrophy’ of the flow. The model also provides a new Rankine–Hugoniot relation stating that the hydraulic head $H = h + (1/2g)(U^2 + 3R)$ is conserved through the jump.

The quantity R can be seen as the internal energy and the hydraulic head corresponds to the total enthalpy for the compressible gas analogy. But the analogy stops here since the energy R of the incompressible fluid can be transformed into heat. The parameterization of this transformation is made in Richard & Gavriluk (2013) by dimensional analysis leading to a dissipation term for the enstrophy

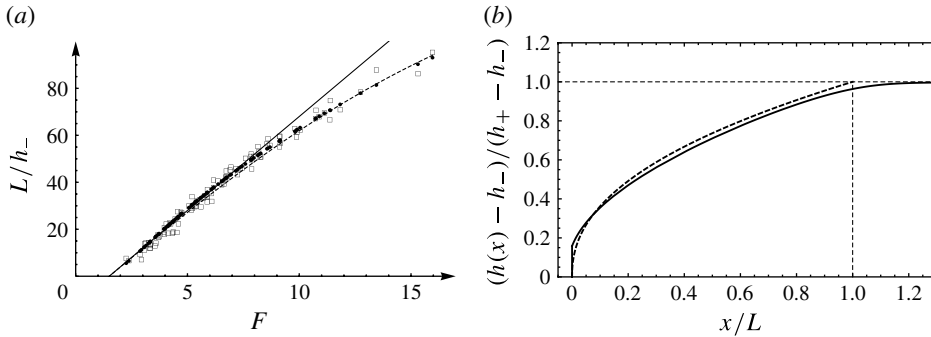


FIGURE 1. (a) Roller length L as function of the upstream Froude number F . Model values (\bullet) compared with experimental measures (\square) of Hager *et al.* (1990). (b) Free surface profile $h(x)$ for $F = 5.56$, linking depth h_- at the toe of the hydraulic jump up to h_+ at its end. Model values (thick curve) compared with experimental law (dashed curve) of Chanson (2011).

Ω of the form $-\Lambda(\Omega)|U|^3/h^3$, with $\Lambda \geq 0$. This monotonic decrease of the vorticity is consistent with the observation by Svendsen *et al.* (2000). Assuming the existence of a constant-ensrophy background φ attributed to the vorticity generated by friction on the channel bottom, the ensrophy dissipation coefficient is set to $\Lambda(\Omega) = 2C_r(1 - \varphi/\Omega)$ where C_r is adjusted to fit the experimental law of Hager *et al.* (1990). The momentum dissipation obeys the classical parameterization $-C_f(Re)|U|U$ where the friction coefficient C_f depends here on the Reynolds number Re through the Colebrook–White formula for smooth bottoms.

The predictions of this model are compared with experimental results in figure 1. The roller length increases with the upstream Froude number $F = U/\sqrt{gh}$ in figure 1(a) and the theoretical curve fits the experimental values remarkably well. In figure 1(b), the model exhibits a discontinuity for the depth at the toe of the hydraulic jump, followed by a smooth backwater curve $h(x)$ converging to its asymptotic value. Again, these values match surprisingly well the experimental results, in view of the fact that C_r and φ are the only adjustable parameters of the model.

But the highlight of Richard & Gavrilyuk (2013) lies in the oscillating regime that the model exhibits through a numerical simulation of its equations. It is well known that the usual Saint-Venant equations fail to model such a regime. In agreement with the experimental results of Mok (2004), this enriched model exhibits the critical Froude number $F_c = 1.5$ for the Hopf bifurcation encountered by the stationary hydraulic jump and predicts accurate oscillation frequencies. Richard & Gavrilyuk (2013) associate this critical value with the fact that, for a given flow discharge, the depth discontinuity at the toe of the stationary hydraulic jump is maximum for $F_c = 1.5$. Thanks to a physical reasoning involving the shape of the backwater curves and the downward propagation of information, the authors are able to provide a convincing explanation of the unstable feedback loop that leads to the oscillating regime.

3. Future

The ‘Richard–Gavrilyuk (RG) equations’ (Richard & Gavrilyuk 2012, 2013) are destined to become famous due to their ability to describe rapidly varying open channel flows with few adjustable parameters. So far, only stationary hydraulic jumps

and roll waves have been studied for a comparison between theory and experimental results. Other hydraulic singularities such as abrupt shrinking, both in open channels and pipes, could be considered. Further validation of the RG equations can be looked for by measuring backwater curves or wave velocities in the vicinity of hydraulic singularities. A greater challenge would lie in the explanation of some wavy backwater curves that are missed by the Saint-Venant equations and do not come from wave dispersion.

The RG approach will certainly be relevant to the field of developed turbulence. For instance, it would be interesting to relate the streamwise dissipation of the energy or enstrophy generated in a shock with theories of decaying turbulence through a Taylor hypothesis. The equation for the enstrophy Ω in the RG equation can be compared to turbulent closures such as TKE (turbulent kinetic energy k), k - ϵ or k - ω . Finding an oscillatory regime coming out of a feedback loop with the turbulent variables would resonate with the concept of elastic turbulence. Since the Saint-Venant equations have been rigorously derived from the Navier–Stokes equations for viscous flows in the framework of slowly varying flows (see for instance Boutounet *et al.* 2008), mathematicians are likely to extend their asymptotic expansions to rapidly varying flow to recover the RG equations. Finally, the RG theory might also apply to non-Newtonian flows and gives an interesting framework to parameterize, for different rheologies, the dissipation of the enstrophy generated through various singularities.

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