ON THE BOUNDARY SPECTRUM IN BANACH ALGEBRAS

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We investigate some properties of the set $S_\delta(a) = \{ \lambda \in \mathbb{C} : \lambda - a \in \partial S\}$ (which we call the boundary spectrum of $a$) where $\partial S$ denotes the topological boundary of the set $S$ of all non-invertible elements of a Banach algebra $A$, and where $a$ is an element of $A$.

1. Introduction and Preliminaries

Let $A$ be a complex Banach algebra with unit 1. We shall denote the spectrum of an element $a$ in $A$ by $\sigma(a)$ and the spectral radius of $a$ in $A$ by $r(a)$ (or by $\sigma(a, A)$ and $r(a, A)$ respectively, if the particular Banach algebra needs to be emphasized). The distance from an element $a \in \mathbb{C}$ to a subset $E$ of $\mathbb{C}$ will be denoted by $d(a, E)$, and $\delta(a)$ (or $\delta(a, A)$, if necessary) will indicate the distance $d(0, \sigma(a))$ from 0 to the spectrum of $a$. If $\lambda \in \mathbb{C}$, then we shall write $\lambda$ for the element $\lambda 1$ in $A$. We recall that if $\alpha \notin \sigma(a)$, then $d(\alpha, \sigma(a)) = 1/(r((\alpha - a)^{-1}))$ ([1, Theorem 3.3.5]).

If $E$ is a subset of a metric space $\mathcal{X}$, then $\partial_X E$ denotes the topological boundary of $E$ and $\text{int}_X E$ the topological interior of $E$ relative to $\mathcal{X}$. For an $r > 0$ and an element $x$ in $\mathcal{X}$, the notation $B_X(x, \epsilon)$ will be used to denote the open ball relative to $\mathcal{X}$ with centre $x$ and radius $\epsilon$. (If the choice of a metric space $\mathcal{X}$ is clear, the subscript $\mathcal{X}$ will be dropped.)

In this paper we consider the set $S_\delta(a) = \{ \lambda \in \mathbb{C} : \lambda - a \in \partial S\}$ (or $S_\delta(a, A)$, if the particular Banach algebra needs to be emphasized) where $S$ (or $S_A$, if necessary) denotes the set of all non-invertible elements of $A$. Some properties of this set are investigated: in particular the relationship between $S_\delta(a, A)$ and $S_\delta(a, B)$ where $B$ is a closed subalgebra of a Banach algebra $A$ such that $B$ contains the unit of $A$, and the relationship between $S_\delta(a, A)$ and $S_\delta(Ta, B)$ where $B$ is another Banach algebra and $T : A \rightarrow B$ a homomorphism. Finally, some results involving the boundary spectrum $S_\delta(a)$ of a positive element $a$ in an ordered Banach algebra are obtained.

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2. Boundary Spectrum

Let \( A \) be a complex Banach algebra with unit 1 and let \( S \) be the set of all non-invertible elements of \( A \). Then \( S \) is a closed subset of \( A \). Define, for \( a \in A \), the set \( S_\alpha(a) \) in the complex plane as follows:

\[
S_\alpha(a) = \{ \lambda \in \mathbb{C} : \lambda - a \in \partial S \}
\]

We shall call this set the boundary spectrum of \( a \) in \( A \). Also define, for \( a \in A \),

\[
r_1(a) = \sup\{|\lambda| : \lambda \in \partial \sigma(a)\}
\]

and

\[
r_2(a) = \sup\{|\lambda| : \lambda \in S_\alpha(a)\}.
\]

**Proposition 2.1.** Let \( A \) be a Banach algebra and \( a \in A \). Then \( \partial \sigma(a) \subseteq S_\alpha(a) \subseteq \sigma(a) \); and therefore \( r_1(a) = r_2(a) = r(a) \), and if \( \alpha \notin \sigma(a) \), then \( d(\alpha, \partial \sigma(a)) = d(\alpha, S_\alpha(a)) = d(\alpha, \sigma(a)) \).

**Proof:** To prove that \( \partial \sigma(a) \subseteq S_\alpha(a) \), let \( \lambda \in \partial \sigma(a) \) and \( \varepsilon > 0 \). Then there exist a \( \lambda_1 \in B(\lambda, \varepsilon) \cap \sigma(a) \) and a \( \lambda_2 \in B(\lambda, \varepsilon) \cap (\mathbb{C} \setminus \sigma(a)) \). If \( b_1 = \lambda_1 - a \) and \( b_2 = \lambda_2 - a \), then \( b_1 \in S \), \( b_2 \notin S \) and \( b_1, b_2 \in B(\lambda - a, \varepsilon) \). Therefore \( \lambda - a \in \partial S \), so that \( \lambda \in S_\alpha(a) \). This proves that \( \partial \sigma(a) \subseteq S_\alpha(a) \), and since \( S \) is closed, \( \partial S \subseteq S \), so that \( S_\alpha(a) \subseteq \sigma(a) \).

It follows from Proposition 2.1 that, for every \( a \in A \), the set \( S_\alpha(a) \) is non-empty. Since \( \partial S \) is closed, \( S_\alpha(a) \) is closed, and since \( S_\alpha(a) \) is contained in the spectrum of \( a \), it is bounded as well; in fact, \( S_\alpha(a) \subseteq \{ \lambda \in \mathbb{C} : |\lambda| \leq r(a) \} \). Therefore \( S_\alpha(a) \) is a compact set.

In general, \( \partial \sigma(a) \neq S_\alpha(a) \). We proceed to illustrate this with an example.

**Example 2.2.** ([1, Remark 1, p.56]) Let \( l^2(\mathbb{Z}) \) be the Hilbert space of all bilateral square-summable sequences and \( (e_n : n \in \mathbb{Z}) \) the orthonormal basis where, for each integer \( n \), the vector \( e_n \) is \((\ldots, \xi_{-1}, (\xi_0), \xi_1, \ldots)\), where \( \xi_n = 1 \) and \( \xi_i = 0 \) for all integers \( i \) different from \( n \). (In this case, the term in round brackets indicates the one corresponding to index zero.) Let \( T, R : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \) be the weighted shifts

\[
T e_n = \begin{cases}
0 & \text{if } n = -1 \\
e_{n+1} & \text{if } n \neq -1
\end{cases}
\]

and

\[
R e_n = \begin{cases}
e_0 & \text{if } n = -1 \\
0 & \text{if } n \neq -1.
\end{cases}
\]

Then \( 0 \in \sigma(T) \) and \( \sigma(T + \lambda R) \) is contained in the unit circle for all \( \lambda \neq 0 \).
Moreover, if $0 < |\lambda| < 1$, then $\lambda$ is an eigenvalue of $T$. Indeed, if $\xi_j = 0$ for all $j \in \mathbb{N} \cup \{0\}$ and $\xi_{-j} = \lambda^{-j}$ for all $j \in \mathbb{N}$, then the (non-zero) element $(\ldots, \xi_{-1}, (\xi_0), \xi_1, \ldots)$ of $l^2(\mathbb{Z})$ is in the kernel of $T - \lambda I$.

(The above facts about the operators $T$ and $R$ also follow from ([3, Problem 84]).)

**Example 2.3.** Let $l^2(\mathbb{Z})$ be the Hilbert space of all bilateral square-summable sequences, $A$ the Banach algebra $\mathcal{L}(l^2(\mathbb{Z}))$ of all bounded linear operators on $l^2(\mathbb{Z})$ and $T$ the element of $A$ defined in Example 2.2. Then $\partial \sigma(T)$ is properly contained in $S_\theta(T)$.

**Proof:** Let $(\lambda_n)$ be a sequence different from zero converging to zero, $R$ the operator defined in Example 2.2 and $T_n = T + \lambda_n R$ ($n \in \mathbb{N}$). Then $\|T_n - T\| \to 0$. Moreover, by Example 2.2, each $T_n$ is invertible and $T$ is not invertible. Hence $T \notin \partial S$. Therefore $-T \in \partial S$, and so $0 \in S_\theta(T)$. However, Example 2.2 together with the remark thereafter imply that 0 is an interior point of $\sigma(T)$, and so $0 \notin \partial \sigma(T)$. \qed

We recall the following well-known property of boundary points of the set of invertible (or non-invertible) elements:

**Theorem 2.4.** ([10, Theorem 2.5, p. 397]) Let $A$ be a Banach algebra and $a \in A$. If $a \in \partial S$, then $a$ is a topological divisor of zero.

From the above theorem we immediately obtain the following property of the boundary spectrum of $a$:

**Corollary 2.5.** Let $A$ be a Banach algebra and $a \in A$. If $\lambda \in S_\theta(a)$, then $\lambda - a$ is a topological divisor of zero.

**Lemma 2.6.** Let $A$ be a Banach algebra, $a \in \partial S$ and $d$ an invertible element. Then $ad \in \partial S$ and $da \in \partial S$.

**Proof:** If $a \in \partial S$ and $d$ is invertible, then for each $\varepsilon > 0$ there exist elements $c_1 \in S \cap B(a, (\varepsilon/\|d\|))$ and $c_2 \in (A \setminus S) \cap B(a, (\varepsilon/\|d\|))$. It follows that $c_1 d \in S \cap B(ad, \varepsilon)$ and $c_2 d \in (A \setminus S) \cap B(ad, \varepsilon)$. Hence $ad \in \partial S$, and similarly $da \in \partial S$. \qed

It follows from Lemma 2.6 that $a \in \partial S$ if and only if $\lambda a \in \partial S$, for all $\lambda \neq 0$.

**Proposition 2.7.** Let $a$ be an invertible element of a Banach algebra $A$. Then $S_\theta(a^{-1}) = (S_\theta(a))^{-1}$.

**Proof:** For any $\lambda \neq 0$ and any invertible element $a \in A$ we have $\lambda - a^{-1} = (a - (1/\lambda))^{-1}$. So if $\lambda \in S_\theta(a^{-1})$, then $\lambda (a - (1/\lambda))a^{-1} \in \partial S$. It follows from Lemma 2.6 that $a - (1/\lambda) \in \partial S$, so that $1/\lambda \in S_\theta(a)$. We have proved that $S_\theta(a^{-1}) \subseteq (S_\theta(a))^{-1}$ for all invertible elements $a$, and therefore also $(S_\theta(a))^{-1} \subseteq S_\theta(a^{-1})$ for all invertible $a$. \qed

Further mapping properties of $S_\theta$ will be investigated in a future paper.

Let $B$ be a closed subalgebra of a Banach algebra $A$ such that $B$ contains the unit element 1 of $A$. It is well known that if $a \in B$, then $\partial \sigma(a, B) \subseteq \partial \sigma(a, A)$ ([1, Theorem
We shall show that $S_\theta(a, B) \subseteq S_\theta(a, A)$ holds as well. In order to do this, we need the following results, some of which are interesting in their own right.

**Theorem 2.8.** ([1, Theorem 3.2.13 (i)]) Let $B$ be a closed subalgebra of a Banach algebra $A$ such that $B$ contains the unit element 1 of $A$. Then $B \setminus S_B$ is the union of all components of $B \cap (A \setminus S_A)$ containing points of $B \setminus S_B$.

**Lemma 2.9.** Let $B$ be a closed subalgebra of a Banach algebra $A$ such that $B$ contains the unit element 1 of $A$. If $E$ is a subset of $A$, then $\partial_B E \subseteq \partial_A E$.

**Theorem 2.10.** Let $B$ be a closed subalgebra of a Banach algebra $A$ such that $B$ contains the unit element 1 of $A$. Then $S_B$ is the union of $S_A$ and all the components of $B \cap (A \setminus S_A)$ containing points of $S_B$.

**Proof:** Clearly $S_A \subseteq S_B$. If $x \in S_B$ and $x \notin S_A$, then $x \in B \cap (A \setminus S_A)$, so that $x$ is contained in a component of $B \cap (A \setminus S_A)$ which contains points of $S_B$. Hence $S_B$ is contained in the union of $S_A$ and all the components of $B \cap (A \setminus S_A)$ containing points of $S_B$.

Conversely, let $\Omega$ be a component of $B \cap (A \setminus S_A)$ which contains points of $S_B$. If $\Omega \not\subseteq S_B$, then $\Omega$ is a component of $B \cap (A \setminus S_A)$ which contains a point of $B \setminus S_B$. Theorem 2.8 implies that $\Omega \subseteq B \setminus S_B$, which contradicts the fact that $\Omega$ contains points of $S_B$. Hence $\Omega \subseteq S_B$.

The following result was proved in [2], using the fact that boundary points of the set of invertible elements of a Banach algebra are topological divisors of zero (see Theorem 2.4) and therefore permanently singular. We provide an alternative proof.

**Theorem 2.11.** ([2, Corollary 18, p. 14]) Let $B$ be a closed subalgebra of a Banach algebra $A$ such that $B$ contains the unit element 1 of $A$. Then $\partial_B S_B \subseteq \partial_A S_A$.

**Proof:** To prove that $\partial_B S_B \subseteq \partial_B S_A$, suppose that $x \notin \partial_B S_A$. If $x \notin B$, then $x \notin \partial_B S_B$, so suppose that $x \in B$. Then there exists an $\varepsilon > 0$ such that either (i) $B_B(x, \varepsilon) \subseteq S_A$ or (ii) $B_B(x, \varepsilon) \subseteq B \setminus S_A$. Since $S_A \subseteq S_B$, case (i) implies that $B_B(x, \varepsilon) \subseteq S_B$, so that $x \notin \partial_B S_B$, so suppose that $B_B(x, \varepsilon)$ is contained in a component $\Omega$ of $B \cap (A \setminus S_A)$. If $\Omega$ contains points of $S_B$, then by Theorem 2.10, $\Omega$ is contained in $S_B$, so that $x \notin \partial_B S_B$. If $\Omega$ contains no points of $S_B$, then $\Omega \subseteq B \setminus S_B$, so that once again, $x \notin \partial_B S_B$.

We have proved that $\partial_B S_B \subseteq \partial_B S_A$. Together with Lemma 2.9 the result follows.

**Corollary 2.12.** Let $B$ be a closed subalgebra of a Banach algebra $A$ such that $B$ contains the unit element 1 of $A$. If $a \in B$, then $S_\theta(a, B) \subseteq S_\theta(a, A)$.

**Proof:** If $\lambda \in S_\theta(a, B)$, then $\lambda - a \in \partial_B S_B$. It follows from Theorem 2.11 that $\lambda - a \in \partial_A S_A$, so that $\lambda \in S_\theta(a, A)$.

Now we consider the situation where $A$ and $B$ are Banach algebras (with $B$ not necessarily a subalgebra of $A$) and $T : A \to B$ a homomorphism, and investigate the
relationship between \( S_0(a, A) \) and \( S_0(Ta, B) \), where \( a \in A \). We first establish some properties involving \( TS_A \) and \( S_B \), and \( T(\partial_A S_A) \) and \( \partial_B S_B \). The proof of the next lemma is trivial:

**Lemma 2.13.** Let \( A \) and \( B \) be Banach algebras and \( T : A \rightarrow B \) a homomorphism. Then the following hold:
1. \( T^{-1} S_B \subseteq S_A \).
2. If \( T \) is surjective, then \( S_B \subseteq TS_A \).
3. If \( T \) is bijective, then \( T^{-1} S_B = S_A \) and \( TS_A = S_B \).

**Theorem 2.14.** Let \( A \) and \( B \) be Banach algebras and \( T : A \rightarrow B \) a continuous isomorphism. Then \( T(\partial_A S_A) = \partial_B S_B \).

**Proof:** If \( x \in \partial_A S_A \), then there exist sequences \((x_n)\) in \( S_A \) and \((y_n)\) in \( A \setminus S_A \) such that \( x_n \rightarrow x \) and \( y_n \rightarrow x \). It follows from Lemma 2.13 (3) that \( Ty_n \in B \setminus S_B \) and \( Tx_n \in S_B \). Since \( T \) is continuous, \( Tx_n \rightarrow Tx \) and \( Ty_n \rightarrow Tx \). Hence \( Tx \in \partial_B S_B \).

Conversely, if \( y \in \partial_B S_B \), say \( y = Tx \) with \( x \in A \), then there exist sequences \((z_n)\) in \( S_B \) and \((w_n)\) in \( B \setminus S_B \) such that \( z_n \rightarrow y \) and \( w_n \rightarrow y \). It follows from Lemma 2.13 (3) that \( z_n = Tx_n \) with \( x_n \in S_A \) and that \( w_n \in B \setminus TS_A \), so that \( w_n = Tu_n \) with \( u_n \in A \setminus S_A \). Since \( T \) is bijective, linear and bounded, \( T^{-1} \) exists and is linear and bounded (by the Bounded Inverse Theorem), which implies that \( x_n \rightarrow x \) and \( u_n \rightarrow x \). Since \( (x_n) \) is in \( S_A \) and \( (u_n) \) is in \( A \setminus S_A \), it follows that \( x \in T^{-1} S_B \).

In the following result \( \ker T \) will denote the kernel of \( T \).

**Theorem 2.15.** Let \( A \) and \( B \) be Banach algebras, \( T : A \rightarrow B \) a continuous isomorphism and \( a \in A \). Then

\[
S_0(a, A) = S_0(Ta, B) = \bigcup_{b \in \ker T} S_0(a + b, A).
\]

**Proof:** If \( \lambda \in S_0(a, A) \), then \( \lambda - a \in \partial_A S_A \), so that Theorem 2.14 implies that \( \lambda - Ta = T(\lambda - a) \in \partial_B S_B \), and so \( \lambda \in S_0(Ta, B) \).

If \( \lambda \in S_0(a + b, A) \) for some \( b \in \ker T \), then \( \lambda - Ta = T(\lambda - a - b) \in \partial_B S_B \), by Theorem 2.14, so that \( \lambda \in S_0(Ta, B) \).

We have proved that

\[
S_0(a, A) \subseteq S_0(Ta, B) \quad \text{and} \quad \bigcup_{b \in \ker T} S_0(a + b, A) \subseteq S_0(Ta, B).
\]

If \( \lambda \in S_0(Ta, B) \), then \( T(\lambda - a) = \lambda - Ta \in \partial_B S_B \), so that Theorem 2.14 implies that \( T(\lambda - a) \in T(\partial_A S_A) \). The injectivity of \( T \) implies that \( \lambda - a \in \partial_A S_A \), so that \( \lambda \in S_0(a, A) \). Since \( 0 \in \ker T \), we obtain the following inclusions:

\[
S_0(Ta, B) \subseteq S_0(a, A) \subseteq \bigcup_{b \in \ker T} S_0(a + b, A).
\]

Hence the results follow.
3. APPLICATIONS IN ORDERED BANACH ALGEBRAS

In this section we investigate certain results in ordered Banach algebras involving the boundary spectrum. From ([9, Section 3]) we recall that an algebra cone \( C \) of a complex Banach algebra \( A \) with unit 1 is a subset of \( A \) containing 1 which is closed under the following operations: addition, positive scalar multiplication, and multiplication. If \( A \) has an algebra cone \( C \), then \( A \), or more specifically \((A, C)\), is called an ordered Banach algebra (OBA). If, in addition, \( C \cap -C = \{0\} \), then \( C \) is called proper.

An algebra cone \( C \) of \( A \) induces an ordering "\( \leq \)" on \( A \) in the following way:

\[ a \leq b \text{ if and only if } b - a \in C \]

(where \( a, b \in A \)). This ordering is reflexive and transitive. Furthermore, \( C \) is proper if and only if the ordering has the additional property of being antisymmetric. Considering the ordering that \( C \) induces we find that \( C = \{ a \in A : a \geq 0 \} \) and therefore we call the elements of \( C \) positive.

An algebra cone \( C \) of \( A \) is called closed if it is a closed subset of \( A \). Furthermore, \( C \) is said to be normal if there exists a constant \( \alpha > 0 \) such that it follows from \( 0 \leq a \leq b \) in \( A \) that \( \|a\| \leq \alpha \|b\| \). It is well known that if \( C \) is normal, then \( C \) is proper. If \( C \) has the property that if \( a \in C \) and \( a \) is invertible, then \( a^{-1} \in C \), then \( C \) is said to be inverse-closed. If \( B \) is a Banach algebra such that \( 1 \in B \subseteq A \), then \( C \cap B \) is an algebra cone of \( B \), and hence \((B, C \cap B)\) is an OBA.

In [9, 8], and later in [4, 5, 6, 7], some spectral theory of positive elements in ordered Banach algebras was developed. In particular, we recall the following results:

**Theorem 3.1.** ([9, Theorem 4.1(1)]) Let \((A, C)\) be an OBA with \( C \) normal. If \( a, b \in A \) such that \( 0 \leq a \leq b \), then \( r(a) \leq r(b) \).

We refer to the above property by saying that the spectral radius in \((A, C)\) is monotone.

**Theorem 3.2.** ([9, Theorem 5.2]) Let \((A, C)\) be an OBA with \( C \) closed and such that the spectral radius in \((A, C)\) is monotone. If \( a \in C \), then \( r(a) \in \sigma(a) \).

Using the boundary spectrum we obtain the following (slightly stronger) analogues of Theorem 3.2 and ([6, Theorem 3.3]):

**Proposition 3.3.** Let \((A, C)\) be an OBA with \( C \) closed and such that the spectral radius in \((A, C)\) is monotone. If \( a \in C \), then \( r(a) \in \partial \sigma(a) \).

**Proof:** If \( a \in C \), then by Theorem 3.2 \( r(a) \in \sigma(a) \). Hence \( r(a) \in \partial \sigma(a) \) and so \( r(a) \in \partial \sigma(a) \).

**Proposition 3.4.** Let \((A, C)\) be an OBA with \( C \) closed and inverse-closed, and such that the spectral radius in \((A, C)\) is monotone. If \( a \) is an invertible element of \( C \), then \( \delta(a) \in S_\theta(a) \).
**PROOF:** If \( a \in C \) and \( a \) is invertible, then \( a^{-1} \in C \), since \( C \) is inverse-closed. Proposition 3.3 implies that \( r(a^{-1}) \in S_{\delta}(a^{-1}) \). Hence \( r(a^{-1}) = 1/\lambda_0 \) for some \( \lambda_0 \in S_{\delta}(a) \), by Proposition 2.7. Since \( r(a^{-1}) = 1/(\delta(a)) \), the result follows. \( \square \)

In the following result \( B \) is a subalgebra of \( A \) but not necessarily closed in \( A \).

**Theorem 3.5.** Let \((A, C)\) be an OBA and \( B \) a Banach algebra with \( 1 \in B \subseteq A \).

1. Suppose that the spectral radius in \((A, C)\) is monotone. If \( 0 \leq a \leq b \) with \( a, b \in B \) and either \( \partial \sigma(a, B) = \partial \sigma(a, A) \) or \( S_{\delta}(a, B) = S_{\delta}(a, A) \), then \( r(a, B) \leq r(b, B) \).
2. Suppose that the spectral radius in \((B, C \cap B)\) is monotone. If \( 0 \leq a \leq b \) with \( a, b \in B \) and either \( \partial \sigma(b, B) = \partial \sigma(b, A) \) or \( S_{\delta}(b, B) = S_{\delta}(b, A) \), then \( r(a, A) \leq r(b, A) \).

**Proof:**

1. Since \( B \) is a subalgebra of \( A \), we have that \( \sigma(b, A) \subseteq \sigma(b, B) \), so that \( r(b, A) \leq r(b, B) \). The monotonicity of the spectral radius in \((A, C)\) implies that \( r(a, A) \leq r(b, A) \). Finally, the assumption that either \( \partial \sigma(a, B) = \partial \sigma(a, A) \) or \( S_{\delta}(a, B) = S_{\delta}(a, A) \) yields \( r(a, B) = r(a, A) \), by Proposition 2.1. Combining the results, it follows that \( r(a, B) \leq r(b, B) \).

2. Similarly as in (1), the fact that \( B \) is a subalgebra of \( A \), the monotonicity of the spectral radius in \((B, C \cap B)\) and the additional assumption imply, respectively, that \( r(a, A) \leq r(a, B) \), \( r(a, B) \leq r(b, B) \) and \( r(b, B) = r(b, A) \), which yield the result. \( \square \)

We note that Theorem 3.5 (2) is a stronger version of ([9, Proposition 4.5]).

For our next result we need the following lemma and theorem:

**Lemma 3.6.** ([7, Lemma 4.1]) Let \( A \) be a Banach algebra, \( x, y \in A \) and \( \alpha \in \mathbb{C} \). If \( \alpha - x \) is invertible and \( r((\alpha - x)^{-1}(x - y)) < 1 \), then \( \alpha - y \) is invertible.

**Theorem 3.7.** ([7, Proof of Theorem 4.2]) Let \((A, C)\) be an OBA with \( C \) closed and normal, and let \( x \in C \). If \( y \in C \) such that \( x \leq y \) and either \( xy \leq yx \) or \( yx \leq xy \), and \( \alpha \) is a positive real number such that \( \alpha > r(x) \), then

\[
r((\alpha - x)^{-1}(y - x)) \leq r((\alpha - x)^{-1})r(y - x).
\]

Now let \((A, C)\) be an OBA. Define, for each \( x \in C \), an analogue \( A'(x) \) of the set \( A(x) \) (defined in ([7, Section 4])) as follows:

\[
A'(x) = \{ y \in A : x \leq y, \ xy \leq yx \ or \ yx \leq xy \ and \ d(r(y), S_{\delta}(x)) \geq d(\alpha, S_{\delta}(x)) \ for \ all \ \alpha \in S_{\delta}(y) \}\n\]

Then \( x \in A'(x), A'(x) \subseteq C \) and \( A'(0) = C \). Finally, the following theorem is a complementary result to ([7, Theorem 4.2]):
**Theorem 3.8.** Let \((A, C)\) be an OBA with \(C\) closed and normal, and let \(x \in C\). Then \(S_\theta(y) \subseteq S_\theta(x) + r(x - y)\) for all \(y \in A'(x)\).

**Proof:** Let \(y \in A'(x)\). Then \(0 \leq x \leq y\), so that \(r(x) \leq r(y)\), by Theorem 3.1. If \(r(x) = r(y)\), then \(d(r(y), S_\theta(x)) = 0\), by Proposition 3.3, so that, by the assumption, \(d(\alpha, S_\theta(x)) = 0\) for all \(\alpha \in S_\theta(y)\). This implies that \(d(\alpha, S_\theta(x)) \leq r(x - y)\) for all \(\alpha \in S_\theta(y)\), so that \(S_\theta(y) \subseteq S_\theta(x) + r(x - y)\).

So suppose that \(r(x) < r(y)\), and suppose there exists an \(\alpha \in S_\theta(y)\) such that \(d(\alpha, S_\theta(x)) > r(x - y)\). Proposition 3.3 implies that \(r(y) \in S_\theta(y)\) and hence, by the assumption, we may take \(\alpha \in \mathbb{R}^+\) with \(\alpha > r(x)\). Since \(\alpha \notin \sigma(x)\), it follows from Proposition 2.1 that \(d(\alpha, S_\theta(x)) = d(\alpha, \sigma(x))\), so that \(d(\alpha, S_\theta(x)) = 1/\left(r((\alpha - x)^{-1})\right)\). Therefore \(r((\alpha - x)^{-1})r(x - y) < 1\) with \(\alpha \in \mathbb{R}^+\) and \(\alpha > r(x)\).

It follows from Theorem 3.7 that \(r((\alpha - x)^{-1}(y - x)) < 1\), so that \(\alpha \notin \sigma(y)\), by Lemma 3.6. Hence \(\alpha \notin S_\theta(y)\) — a contradiction. Therefore \(d(\alpha, S_\theta(x)) \leq r(x - y)\) for all \(\alpha \in S_\theta(y)\), so that \(S_\theta(y) \subseteq S_\theta(x) + r(x - y)\). ☐

**References**