A JOURNEY ROUND THE TRIANGLE 217

A journey round the triangle
Lester's circle, Kiepert's hyperbola and a configuration from Morley

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In a recent article [1], Ron Shail has given a Cartesian proof of an interesting theorem due to J. A. Lester. This states that, for any triangle, the circumcentre \( O \), the nine-point centre \( O_9 \) and the two Fermat points \( F \) and \( F' \), (which are the points of concurrence of the joins of its vertices to the vertices of equilateral triangles drawn outwards/inwards on the opposite sides), are concyclic. He refers to Lester's own treatment as needing complex coordinates with computer-assisted algebra; his own proof uses an unpromising method, and results in similar problems. Contemplation of the configuration would suggest that the location of the point of intersection of \( FF' \) with the Euler line \( OO_9 \) might lead to a simple proof. The theorem is in fact a corollary from the properties of a remarkable configuration originating with Morley [2, p. 209], and shown in Figure 1. He did not deduce Lester's result, nor label the crucial point \( J \) in the diagram, which was drawn without that particular intersection. Also involved in this figure is a rectangular hyperbola known by the name of its describer Kiepert [3]. It is helpful to discuss all three together. What follows is a journey through country nowadays rather unfamiliar, avoiding the computerised motorway and using older tracks via complex numbers, trilinear coordinates and Euclidean methods which reveal much more than is apparent from a Cartesian treatment.

![Figure 1](https://www.cambridge.org/core/)

**Notation**

Take the circumcentre of \( \triangle ABC \) as origin in a complex plane, and the circumradius as unity. Let the affixes of the vertices \( A, B, C \) be \( a, b, c \), a notation I shall follow throughout. The complex conjugate of \( a, a^* \) will then be \( 1/a \), and similarly for \( b, c \) and all points on the circumcircle, which are called *turns*, since multiplying by one of them is equivalent to rotation. Write \( a + b + c = s_1 \), which is the affix of the orthocentre \( H \), because
\[ s_1 - a = b + c \] tells us that \( AH \) is equal and parallel to \( 2OA' \), where \( A' \) is the midpoint of \( BC \), and similarly for \( BH \) and \( CH \). Then \( G \), the centroid, has affix \( s_1 / 3 \), and the nine-point centre \( O_9 \) is \( s_1 / 2 \). Write \( bc + ca + ab = s_2 \), and \( abc = s_3 \). Then \( s_1^2 = s_2 / s_3 \). I shall also make much use of the directed angle \( \angle PQR \), which is the measure, mod \( \pi \), of the rotation that carries the line \( PQ \) (not ray) to the line \( QR \). (The positive sense is usually, but not necessarily, taken to be anti-clockwise, but must of course be consistently maintained.)

**Isogonal conjugates** [4, p. 214]

Two points \( P \) and \( Q \), not on the sides of a triangle \( ABC \), are isogonal conjugates with respect to that triangle if \( \angle BAP = \angle QAC \), and similarly for vertices \( B \) and \( C \). Since the directed angles are equal,

\[
\frac{(p - a)(q - a)}{(b - a)(c - a)} = \frac{(p - a)(q - a)}{(b - a)(c - a)}
\]

must be real. Thus

\[
\frac{(p - a)(q - a)}{(b - a)(c - a)} = \frac{(p - a^*)(q - a^*)}{(b - a^*)(c - a^*)} = \frac{(p - a^{-1})(q - a^{-1})}{(b^{-1} - a^{-1})(c^{-1} - a^{-1})}
\]

Therefore \( (p - a)(q - a) = bc(1 - ap^*)(1 - aq^*) \) which can be written as \( T(p, q) = bcT^*(p, q) \), if \( T(p, q) = p + q + abc* q^* - s_1 \). Then we must also have \( T(p, q) = caT^*(p, q) \) at \( B \), which is impossible unless \( T(p, q) = 0 \), when also at \( C \), \( T(p, q) = abT^*(p, q) \) will follow. So we have

**Lemma.** \( P \) and \( Q \) are isogonal conjugates if and only if \( p + q + s_3 p^* q^* = s_1 \).

**Theorem 1.** If \( ABC \) is any triangle, and \( \mathfrak{A} \) is a fixed diameter of its circumcircle \( \mathcal{U} \), the locus of the isogonal conjugates of the points of \( \mathfrak{A} \) is a rectangular hyperbola \( \mathfrak{R} \) with the following properties:

(a) \( \mathfrak{R} \) contains \( ABC \) and the orthocentre \( H \).

(b) The centre of \( \mathfrak{R} \) is on the nine-point circle.

(c) The asymptotes of \( \mathfrak{R} \) are the Simson lines of the points in which \( \mathfrak{A} \) meets \( \mathcal{U} \).

(d) The fourth intersection of \( \mathfrak{R} \) with \( \mathcal{U} \) is diametrically opposite to the orthocentre \( H \) on \( \mathfrak{R} \), and diametrically opposite the point \( R \) on \( \mathcal{U} \) whose Simson line is parallel to \( \mathfrak{A} \) (Figure 2).

**Proof.** Take \( \mathcal{U} \) as \( z^* = 1 \). If \( \mathfrak{A} \) is \( z = \lambda t \) for fixed \( t \), then \( \mathfrak{R} \) is the locus given by \( z + \lambda t + \lambda s_3 z^* / t = s_1 \Leftrightarrow t(z - s_1) = -\lambda(s_3 z^* + t^2) \). Taking the conjugate, \( t(s_3 z^* - s_2) = -\lambda(t^2 z + s_3) \), and eliminating \( \lambda \), we find the equation of \( \mathfrak{R} \) to be

\[
(z - s_1)(z + r) = rs_3 \left(z^* - s_1^*\right) \left(z^* + r^*\right),
\]

where \( r = s_3 / t^2 \), which is a turn.
Alternatively, we can eliminate \( z^* \) to get the parametric equation

\[ z(1 - \lambda^2) = s_1 - \lambda(t + s_2/t) + \lambda^2 r. \]  

(2)

It is immediate (from (1)) that \( \mathcal{R} \) is a conic, and (from (2)) that it contains \( s_1 \) \( (\lambda = 0) \), \( a \) when \( \lambda = t/a \) etc., and that when \( \lambda \to \pm 1 \), \( z \to \infty \).

As \( \lambda \to +1 \), \( z/z^* \to -s_3/t = -rt \), and as \( \lambda \to -1 \), \( z/z^* \to +s_3/t = +rt \), so that the asymptotes are perpendicular and \( \mathcal{R} \) is a rectangular hyperbola.

Writing (1) in the form

\[ [z - (s_1 - r)/2]^2 - rs_3 [z^* - (s_1^* - r^*)/2]^2 = \text{constant}, \]

we find the centre of \( \mathcal{R} \) to be the point \( \frac{1}{2} (s_1 - r) \), whose distance from \( \frac{1}{2} s_1 \), the nine-point centre, is \( \frac{1}{2} \), since \( r \) is a turn, thus proving (b). It is obvious from (1) that \( \mathcal{R} \) contains \( -r \), which is diametrically opposite to \( r \) on \( \mathcal{U} \) and to \( H \) on \( \mathcal{R} \); it now only remains to identify the Simson lines of \( t \), \( -t \), and \( r \).

To do this I shall use the line of images in the sides of the triangle of a point on the circumcircle. This is parallel to the Simson line and really more fundamental, though historically it was conceived later. First we need the equation of \( BC \). This must be such that when \( z \) is a turn \( u \) it is equivalent to the quadratic \( u^2 - u(b + c) + bc = 0 \) or \( u - (b + c) + bcu^* = 0 \). It is therefore \( z + bcz^* = b + c \) (whose truth is obvious once it is written down). The self-conjugate equation of a line states that a point on the line is coincident with its reflection in the line, so that the image of \( z \) in \( BC \) is \( b + c - bcz^* = s_1 - a - s_2z^*/a \). When \( z \) is a fixed turn \( u \), the three images in the sides will all belong to the locus given by \( z - s_1 = -(\theta + s_3/u\theta) \) for a variable turn \( \theta \). But this is part of a straight line through the orthocentre \( s_1 \) since \( (z - s_1)/(z - s_1^*) = s_3/u \), a fixed turn, which gives the direction of the line of images and the Simson line of \( u \). When \( u = t \), this is \( rt \) and the line is parallel to the asymptote at \( \lambda = -1 \). Similarly the asymptote at \( \lambda = 1 \) is the Simson line of \( -t \).
Finally, the line of reflections of $r$ is parallel to $z/z^* = s_3/r = r^2$, which is $\mathbb{D}$, so (d) is proved and the proof of the theorem is complete.

**Isogonal conjugates in trilinear coordinates**

If $P(\alpha, \beta, \gamma)$ is inside the base triangle $ABC$, ($\alpha\beta\gamma \neq 0$), then so also is $Q$ its isogonal conjugate, which I shall label $(\alpha', \beta', \gamma')$. Then

$$\frac{\beta}{\gamma} = \frac{AP \sin \angle PAC}{AP\sin \angle BAP} = \frac{\sin \angle BAQ}{\sin \angle QAC} = \frac{\gamma'}{\beta'}$$

so that $\beta\beta' = \gamma\gamma'$. Similarly, from the situation at $B$, $\gamma\gamma' = \alpha\alpha'$. Now, if the ray $AP$ is rotated to cross $AB$ so that $P$ is outside, $\gamma$ will change sign. But $AQ$ will then rotate to bring $Q$ across $AC$ and $\beta'$ will change sign so that $\beta\beta'$ will still be equal to $\gamma\gamma'$. Any other position of $P$ in the plane can be reached by repetition of such changes, so that for any $P$ in the plane other than $A$, $B$, $C$, $P$ and $Q$ are related by the equations $\alpha\alpha' = \beta\beta' = \gamma\gamma'$. Now suppose $P$ is on $BC$, so that $\alpha = 0$. Unless $P$ is at $B$ or $C$, $\beta$ and $\gamma$ are non-zero and $\beta' = \gamma' = 0$, so that $Q$ is at $A$. We may therefore say that a vertex of the triangle is isogonally conjugate to any point on the opposite side, and consideration of the isogonal rays at the other two vertices confirms this.

**Theorem 2.** If $P, Q$ are isogonal conjugates, $\angle BPC + \angle BQC = \angle BAC$.

**Proof.**

$$\angle BPC = \angle PBA + \angle BAC + \angle ACP = \angle CBQ + \angle BAC + \angle QCB$$

$$= \angle BAC + \angle CQB,$$ and the result follows.

**Theorem 3.** If $P, Q$ are isogonal conjugates: (a) their pedal triangles $P_aP_bP_c$ and $Q_aQ_bQ_c$ have the same circumcircle, whose centre is the midpoint $M$ of $PQ$; (b) $P_bP_c$ is perpendicular to $AQ$, and five similar statements (Figure 3).
A JOURNEY ROUND THE TRIANGLE

Proof

(a) If \( \angle BAP = \theta \),

\[
\frac{AP_b}{AP_c} = \frac{\cos(A - \theta)}{\cos \theta} = \frac{AQ_c}{AQ_b}
\]

\[\Rightarrow AP_b.AQ_b = AP_c.AQ_c \Rightarrow P_bQ_bP_cQ_c \text{ are concyclic;}
\]

by considering the mediators of \( P_bQ_b \) and \( P_cQ_c \), we see that this circle has centre \( M \). Likewise \( P_cQ_cP_aQ_a \) lie on a circle with centre \( M \), but, since both circles pass through \( P_c \), they must coincide, so triangles \( P_aP_bP_c \) and \( Q_aQ_bQ_c \) have the same circumcircle, with centre \( M \).

(b) Let \( AQ \) meet \( P_bP_c \) at \( R \). In \( \Delta s \) \( ARP_c, AP_bP, \angle AP_cR = \angle APP_b \) and \( \angle RAP_c = \angle P_bAP \), therefore \( \angle ARP_c = \angle AP_bP = \frac{1}{2}\pi \).

The isodynamic (Hessian) points

The Apollonius circle \( K_a \) is the locus of a point \( X \) when \( BX/XC = BA/AC \). \( K_b \) and \( K_c \) are similarly defined. These last two meet in two points \( S \) and \( S' \) where \( SA.BC = SB.CA = SC.AB \) which also lie on \( K_a \). Historically, these are the points with affixes \( h_1 \) and \( h_2 \) for which the cubic with roots \( a, b \) and \( c \) can be put in the form \( (z - h_1)^3 = \lambda(z - h_2)^3 \) for real \( \lambda \). This method is due to Hesse whose name adheres to the points. The three circles are coaxial; their centres \( L, M, N \) lie on the sides of the triangle and on the line \( bca + ca\beta + ab\gamma = 0 \), the Lemoine line. \( B \) and \( C \) are inverse in \( K_a \), etc., so the circumcircle is orthogonal to all three, \( OSS' \) are collinear, and \( OS.OS' = R^2 \).

![Figure 4](https://www.cambridge.org/core/journals/a-journey-round-the-triangle)

**Figure 4**

**Theorem 4.** The pedal triangle of each Hessian point is equilateral.

**Proof.** Let \( SS_a, SS_b, SS_c \) be the perpendiculars from \( S \) to \( BC, CA \) and \( AB \). Then \( AS_bSS_c \) are concyclic and

\[ S_bS_c = SA \sin A = SA.BC/2R = SB.CA/2R = S_cS_a = S_aS_b \]

similarly.

The same argument holds for \( S' \) (Figure 4).
The isogonic (Fermat) points

These are the points $F, F'$ at which $\angle AFB, \angle AF'B$ etc. are all $\pm \pi/3$. They can be obtained as follows (Figure 5).

**Theorem 5.** Let equilateral triangles $BCP, CAQ, ABR$ be described outwards (inwards) on the sides of a triangle $ABC$. Then (a) the lines $AP, BQ, CR$ concur at $F$ ($F'$), where $\angle BFC = \angle CFA = \angle AFB = \pm(-)2\pi/3$, and (b) the centres $U, V, W$ of triangles $BCP, CAQ, ABR$ form a new equilateral triangle whose centre is $G$, the centroid of $\Delta ABC$.

**Proof.** The proof is the same for $F$ and $F'$. I give the proof for $F$, when the figure is clearer.

(a) $\Delta CAR$, rotated about $A$ through $\pi/3$, coincides with $\Delta QAB$. Therefore $CR = QB$ and the angle between them is $\pi/3$. Similarly for $AP$ and $BQ$. So if $CR, BQ$ meet at $X$, $BXAR$ and $CXAQ$ are concyclic sets, $\angle BXC = 2\pi/3$ and $BXCP$ are concyclic. Then $\angle CXA = 2\pi/3$, $\angle PXC = \pi/3$, $AXP$ is a straight line, so $X = F$, and $BC, CA, AB$ all subtend $2\pi/3$ at $X$.

(b) This is most easily proved by using vectors. To begin with, since the common chord of the circles $BRAF, CQAF$ is $AF$, the join of their centres, $VW$, is perpendicular to $AF$, and similarly $WU$ is perpendicular to $BF$ and $UV$ to $CF$, so that $\Delta UVW$ is equilateral. Now take any origin $E$ and write $EX = x$, and so on. Then we have, since $AP = BQ = CR$, and they make angles of $2\pi/3$ with one another, their vector sum is zero, i.e. $p - a + q - b + r - c = 0$, or $p + q + r = a + b + c = 3g$. Therefore

$$3(u + v + w) = b + c + p + c + a + q + a + b + r$$
$$= 2(a + b + c) + (p + q + r) = 9g,$$

so that $G$ is the centroid of $\Delta UVW$. 

![Figure 5](https://www.cambridge.org/core/core)
Theorem 6. \(S\) and \(F\), \(S'\) and \(F'\) are isogonal conjugate pairs.

Proof. Let \(\hat{S}\) be the isogonal conjugate of \(S\). By theorem 2,

\[
\angle BAC - \angle BSC = \angle SBA + \angle BAC + \angle ACS
\]

\[
= \angle SS_aS_c + \angle S_bS_aS + \angle BAC = \angle S_bS_aS_c + \angle BAC,
\]

therefore

\[
\angle BSC = \angle S_cS_aS_b = \pm \pi/3 = \angle BFC.
\]

Since this same argument holds for each side of the triangle, \(F = \hat{S}\) is indeed a Fermat point. If \(S\) is replaced by \(S'\) the argument remains with \(\pm \pi/3\) for \(-\pi/3\).

Theorem 7. \(FS\) and \(F'S'\) are parallel to the Euler line, \(OGH\).

Proof. Since, by theorems 3 and 5, \(S_bS_c\) and \(VW\) are both perpendicular to \(AF\), and similarly for the other sides, there is a dilatation \(\varphi\) which maps \(\Delta UVW\) to \(\Delta S_aS_bS_c\). By theorems 3(b) and 5(b), \(\varphi(G) = M\), the midpoint of \(SF\). Since \(UO\) and \(SS_a\) are parallel, both being perpendicular to \(BC\), and similarly \(OV\) is parallel to \(SS_b\) and \(OW\) to \(SS_c\), \(\varphi(O) = S\) and therefore \(\varphi(OG) = SM\). So \(SMF\) is parallel to \(OG\) and \(F = \varphi(J)\), where \(J\) is the image of \(O\) in \(G\), i.e., \(OJ = 2OG = \frac{2}{3}OH\). This will be true whichever of \(S, S'\) is taken.

Write the similar dilatation for \(S'\) as \(\varphi'\). The fixed point of \(\varphi\) must be the intersection of \(OS\) and \(JF\). Then the compounded linear transformation \(\varphi'\varphi^{-1}\) maps \(S\) to \(S'\), \(F\) to \(F'\), \(M\) to \(M'\), and must therefore coincide with the perspective mapping with fixed point \(Y\) between the lines \(\infty M\) and \(\infty M'\) in the accompanying projective diagram (Figure 6). \(MY\) is then the polar of \(\infty\) with respect to the quadrangle \(SF'FS'\) and the following can now be deduced:

(a) \(JFF'\) are collinear, as are \(GSF', GS'F\).
(b) \( Y \) is \( \hat{G} = K \), the symmedian point, by Hesse's theorem [5, §2.37] and \( K \) is on \( FF' \).

c) \( GK \) bisects \( FS \) and \( F'S' \).

Further results must await the extension of theorem 5, and the particularisation of theorem 1 to the Kiepert hyperbola.

**The Kiepert hyperbola**

We can now discuss this remarkable curve. This is the case of Theorem 1 when \( \mathcal{D} \) is the Brocard line \( OK \), given by \( t^2 = k/k^* \); then \( r = s_3k^*/k = s \), say, the Steiner point, and \( \mathcal{K} \) becomes \( \mathcal{R} \), given by

\[
(z - s_1)(z + s) = (s_3z^* - s_2)(sz^* + 1).
\]  

It meets the Euler line at \( H \) and \( G \), the isogonal conjugates of \( O \) and \( K \).

**Theorem 8.** \( \mathcal{K} \) contains the tritangent centres of the complementary triangle \( A'B'C' \).

*Proof.* The tritangent centres of \( ABC \), being their own isogonal conjugates, by the lemma, satisfy the equation \( 2i + s_3j^2 = s_1 \), or \( 2s_3i^* + i^2 = s_2 \). The triangle \( A'B'C' \) is obtained by the transformation \( 2z' + z = 3g = s \), giving \( 2i' = (s_1 - i) \).

The symmedian (Lemoine) point \( k \) is the isogonal conjugate of \( G(s_1/3) \), so that \( 3k + s_2k^* = 2s_1 \). Therefore also \( 3s_3k^* + ks_1 = 2s_2 \). From (3), we need to show that \( L = (i' - s_1)(i' + s) = ss_2L^* \), or \( kL = s_2k^*L^* \).

We have

\[
4kL = -(i + s_1)(2s_3k^* + s_1k - ik) = (i + s_1)(ki + s_3k^* - 2s_2)
\]

\[
= k(s_2 - 2s_3i^*) + i(ks_1 + s_3k^* - 2s_2) + si(s_3k^* - 2s_1s_2)
\]

\[
= k(s_2 - 2s_3i^*) - 2s_3k^*i + s_1s_3k^* - 2s_1s_2
\]

\[
= s_3[(s_1k^* + s_1k^* - 2(k^*i + ki^*) - 2s_1s_1^*] = s_3\rho, \text{ where } \rho \text{ is real,}
\]

\[
= s_3^24k^*L^*.
\]

Since this is true for any \( i' \) of the four centres, \( \mathcal{K} \) contains them all.

Because the Fermat points \( F, F' \) are the isogonal conjugates of the isodynamic points \( S, S' \) lying on \( OK, F, F' \) lie on \( \mathcal{K} \).

From Theorem 1, \( \mathcal{K} \) meets the circumcircle \( \mathcal{U} \) again at the point of \( \mathcal{U} \) diametrically opposite the Steiner point \( Z \), which is the point where \( \mathcal{U} \) meets the circumscribing Steiner ellipse \( \Sigma \). This is the Tarry point \( T \), and from the theorem the centre \( X \) of \( \mathcal{K} \) is the midpoint of \( TH \). Therefore \( OH, ZX \) are medians of \( \Delta ZHT \) and meet at \( G \) where \( HG = 2GO \), and \( ZG = 2GX \). So \( X \) is the image of \( Z \) under the dilatation \( [G; -\frac{1}{2}] \) which is \( Z' \), the Steiner point of the complementary triangle \( A'B'C' \), the meet of the images of \( \mathcal{U} \) and \( \Sigma' \)—the nine-point circle and the inner Steiner ellipse, which touches the sides of \( \Delta ABC \) at their midpoints \( A'B'C' \) (Figure 7).
In trilinear coordinates

\( \mathcal{H} \) is the unique conic through \( ABCGH \); its trilinear equation is therefore of the form \( \Sigma p \beta \gamma = 0 \). Since it contains \( G \left( \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right) \), \( \Sigma p a = 0 \), and since it contains \( H \left( \sec A, \sec B, \sec C \right) \), \( \Sigma p \cos A = 0 \). Therefore

\[
p : q : r = (b \cos C - c \cos B) : (c \cos A - a \cos C) : (a \cos B - b \cos A)
\]

\[
= bc(b^2 - c^2) : ca(c^2 - a^2) : ab(a^2 - b^2)
\]

\[
= \sin (B - C) : \sin (C - A) : \sin (A - B)
\]

and the equation for \( \mathcal{H} \) can be put in the algebraic form

\[
\Sigma bc(b^2 - c^2) \beta \gamma = 0,
\]

or the trigonometric form \( \Sigma \beta \gamma \sin (B - C) = 0 \). An important property of \( \mathcal{H} \) is now easily established.

**Theorem 9.** If isosceles triangles \( BCP, CAQ, ABR \) with base angles \( \theta \) are described on the sides of a triangle as bases (reckoning \( \theta \) as positive if \( P, Q, R \) are outside the triangle, and negative if the triangles are drawn inwards from the sides), \( AP, BQ, CR \) concur at a point \( D \) and the locus of \( D \) is the Kiepert hyperbola \( \mathcal{H} \). (I append the proof for completeness, but the same proof in areal coordinates has been given in [3]) (Figure 8).

**Proof.** At the point \( P \), \( \beta = PC \sin (C + \theta) \) and \( \gamma = PB \sin (B + \theta) \), so that the line \( AP \) is given by \( \beta / \gamma = \csc (B + \theta) / \csc (C + \theta) \). Similarly \( BQ \) is given by \( \gamma / \alpha = \csc (C + \theta) / \csc (A + \theta) \), so that \( D \)
FIGURE 8

is the point \([\text{cosec} (A + \theta), \text{cosec} (B + \theta), \text{cosec} (C + \theta)]\) and must also lie on \(CR\). Then

\[
\Sigma \sin (B - C)/\alpha = \Sigma \sin (B - C) \sin (A + \theta) \\
= \cos \theta \Sigma \sin (B - C) \sin (B + C) + \sin \theta \Sigma \sin (B - C) \cos A \\
= \cos \theta \Sigma (\sin^2 B - \sin^2 C) \\
+ \sin \theta [\Sigma \sin B \cos C \cos A - \Sigma \cos B \sin C \cos C] \\
= 0.
\]

So that \(D\) is on \(\mathcal{K}\).

**Special points**

When \(\theta = 0\), \(P\) is at \(A'\) on \(BC\) and \(D\) is the centroid \(G\).

When \(\theta = \frac{1}{2}\pi\), \(P\) is at \(\infty\), and \(D\) is the orthocentre \(H\).

When \(\theta = \pm \pi / 3\), the triangles \(BCP, CAQ, ABR\) are equilateral and \(D\) is at \(F\) and \(F'\).

These four positions of \(P\) form a harmonic range, so that \(GH, FF'\) are a harmonic set on \(\mathcal{K}\). If \(OK\) meets the circumcircle at \(U, U'\), since, by section 2, \(SS'\) are inverse with respect to the circumcircle, \(\{UU', SS'\} = -1\). Since isogonal conjugacy is polar conjugacy with respect to any rectangular hyperbola through the tritangent points, it leaves cross-ratio unchanged, and therefore \(FF'\) are conjugate with respect to the line at infinity; i.e., \(FF'\) is a diameter of \(\mathcal{K}\) and \(GH\) is a conjugate chord; \(FF'\) bisects \(GH\) at \(J\), as we know; and the tangents to \(\mathcal{K}\) at \(F, F'\) are parallel to \(GH\). The polar of \(K(a, b, c)\) is the line \(\Sigma bc (b^2 - c^2)(by + c\beta) = 0\), which reduces to \(\Sigma aa (b^2 - c^2)(b^2 + c^2 - a^2) = 0\), or \(\Sigma a \sin (B - C) \sin A \cos A = 0\) which is \(GH\). Therefore \(K\) is on \(FF'\) and the tangents at \(G, H\) are \(GK, HK\).
Lester's circle and the Morley configuration

We need a simple theorem for the rectangular hyperbola.

Theorem 10. If $LL'$ is a diameter of a rectangular hyperbola, and $MNM'$ a conjugate chord, bisected by $LL'$ at $N$, then $NL.NL' = NM^2$ (Figure 9).

Proof. Let the hyperbola be $x : y : c = t^2 : 1 : t$, and $M, M', L$ the points with parameters $t_1, t_2$ and $t_3$. The tangent at $L$ is $x + t^2y = 2ct_3$ and $MN$ is $x + t_1t_2y = c(t_1 + t_2)$. These are parallel, therefore $t_1t_2 = t_3^2$. ON, NM make equal angles with the x-axis. Let the angle be $\varphi$. $N$ is $\frac{1}{2}[c(t_1 + t_2), c(1/t_1 + 1/t_2)]$. Also $NL = \frac{1}{2}c(t_1 + t_2 - 2t_3) \sec \varphi$, $NL' = \frac{1}{2}c(t_1 + t_2 + 2t_3) \sec \varphi$ and $NM = \frac{1}{2}c(t_1 - t_2) \sec \varphi$.

$$NL.NL' = (\sec^2 \varphi.c^2/4)[(t_1 + t_2)^2 - 4t_3^2] = (\sec^2 \varphi.c^2/4)[t_1 - t_2]^2 = NM^2.$$ 

Then we have for $\mathcal{K}, FF'$ is a diameter conjugate to $GJH$ with $J$ the midpoint of $GH$. Therefore $JF.JF' = JG^2 = JO.JO_9$ and $FF'OO_9$ are concyclic (Figure 1).

![Figure 9](https://www.cambridge.org/core/)

Theorem 11. $FS$ is tangent to $\mathcal{K}$ at $F$ and $F'S'$ is tangent at $F'$.

Proof. $F$ is the point $(\cosec A', \cosec B', \cosec C')$ where $A' = A + \pi/3$, $B' = B + \pi/3$, $C' = C + \pi/3$. $S$ is the point $(\sin A', \sin B', \sin C')$. The tangent at $F$ to $\mathcal{K}$ is given by the equation $\Sigma \sin(B - C)(\beta \cosec C' + \gamma \cosec B') = 0$. Now $B - C = B' - C'$, and $A' + B' + C' = 2\pi$. Therefore, multiplying by $\sin A' \sin B' \sin C'$ and rearranging, we get for the tangent at $F$

$$\Sigma \alpha \sin A' \left[ \sin(C' - A') \sin B' + \sin(A' - B') \sin C' \right] = 0.$$ 

Since $\sin B' = \sin(C' + A')$ and so on, using the identity $\sin(A - B) \sin(A + B) = \sin^2 A - \sin^2 B$, we can reduce this to

$$\Sigma \alpha \sin A' \left[ \sin^2 C' - \sin^2 B' \right] = 0. \quad (6)$$
It is now immediate that $S$, with coordinates $(\sin A', \sin B', \sin C')$, must lie on this line. The argument for $F'S'$ is the same with $-\pi/3$ instead of $\pi/3$. From (6) we now have another proof that $FS$ and $F'S'$ are parallel to $GH$.

Since $J$ is the midpoint of $GH$, which is conjugate to $XJ$ on $\mathcal{C}$, it follows that the circle with centre $J$ and radius $JX$ meets the Euler line at its points of intersection with the asymptotes of $\mathcal{C}$.
The various points, lines, circles, Lester's circle, Steiner's ellipse, Kiepert's hyperbola and their incidences are shown in Figure 10. The following collinearities are noteworthy; proof of those not referred to is left to the interested reader. \( P, Q, R \) are the second intersections of the circumcircle \( \mathcal{U} \) with the Apollonius circles \( K_a, K_b, K_c \). The sign + indicates two similar ones.

\[
\begin{align*}
OGO_gJH & \quad \text{Euler line} & \quad OSKS' & \quad \text{Brocard axis} & \quad LMN & \quad \text{Lemoine line} \\
AGA' + & \quad AHH_a + & \quad BA'CH_aL + & \quad AKPU + & \quad ALVW + & \quad GSF' & \quad GS'F & \quad HXT & \quad ZOT & \quad ZGX & \quad JFKXF'
\end{align*}
\]

Finally, it is noteworthy that the sixteen points we have visited: \( A, B, C; \) \( O, H, \infty \) (on \( OH \) in Figure 6); \( P, Q, R, P', Q', R' \) (in Figure 5); \( F, F', S, S' \); all lie on the Neuberg cubic of the triangle (see [5]). Some achievement for the happy band of pilgrims at our journey's end.

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And the units of probability are ...

Now we measure the probability of error in inches

According to Casper Weinberger, heard on BBC Five Live by Andrew Rogers during the Iraq conflict.