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A-TOPOLOGY FOR MINKOWSKI SPACE

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Abstract

We consider in this paper a topology (which we call the A-topology) on Minkowski space, the four-dimensional space–time continuum of special relativity and derive its group of homeomorphisms. We define the A-topology to be the finest topology on Minkowski space with respect to which the induced topology on time-like and light-like lines is one-dimensional Euclidean and the induced topology on space-like hyperplanes is three-dimensional Euclidean. It is then shown that the group of homeomorphisms of this topology is precisely the one generated by the inhomogeneous Lorentz group and the dilatations.

1. Introduction

Many topologies have now been suggested [1, 2, 3, 5] for the Minkowski space, the four-dimensional space–time continuum of special relativity, all conforming to the requirement that the homeomorphism group in each case is the one generated by the inhomogeneous Lorentz group and dilatations (call this group G). That Minkowski space be given a topology appropriate to the algebraic structure of the space (that is, a topology which "fits" the indefinite fundamental form and the null cones associated with it) was first suggested by Zeeman [5]. The homeomorphism group of Zeeman's fine topology is precisely the group G. Later, it was found, on investigation, that there exists a wide class of topologies on Minkowski space having the same property, that is, the homeomorphism group of each of these topologies is G [1, 2, 3]. However, there exists another nice property of the fine topology: If f is a continuous $<$-preserving map of the unit interval I into Minkowski space (the definition of $<$-preserving follows in the next section), then fI is a connected union of time-like intervals. Since one intuitively thinks of a path of a particle as the

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continuous image of I, this result implies that photons are excluded from the category of particles whose paths are intuitively thought of as continuous image of I. If, however, one wants to include the photons in this category (of course, in this case the order $<$ has to be changed to an order $\ll$), then the fine topology will be unsuitable and we have to put a new topology on Minkowski space which we call the A-topology. It is defined to be the finest topology with respect to which the induced topology on time-like and light-like lines is one-dimensional Euclidean and the induced topology on space-like hyperplanes is three-dimensional Euclidean. It is then shown that (i) the homeomorphism group of the A-topology is G and (ii) if f is a $\ll$-preserving map of I into the Minkowski space, then fI is a connected union of a finite number of time-like and (or) light-like intervals.

2. Notation and terminology

Let M denote Minkowski space with characteristic quadratic form Q:

$$M = \{(x_0, x_1, x_2, x_3)\colon x_i \text{ are reals}\}$$

$$Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Let

$$K = \{x \in M\colon Q(x) > 0 \text{ and } x_0 > 0\}$$

and

$$L = \{x \in M\colon Q(x) \geqslant 0 \text{ and } x_0 > 0\}.$$

It is easy to prove that K satisfies the following three conditions.

(i) $K+K = \{x+y\colon x \in K,\ y \in K\} \subset K$;

(ii) $aK = \{ax\colon x \in K\} \subset K$ for every positive real a;

(iii) $K \cap (-K) = \varnothing$ where $-K = \{-x\colon x \in K\}$.

L also satisfies (i), (ii) and (iii). K and L are so-called positive cones and each of them generates a partial order on M as follows:

$$x < y \Leftrightarrow y - x \in K,$$

$$x \ll y \Leftrightarrow y - x \in L.$$

There is also another relation $<.$ on M defined by $x <. y \Leftrightarrow Q(y-x) = 0$ and $y_0 - x_0 > 0$, that is, $y - x$ is a future-pointing light-like (null) vector; this is not a partial order due to lack of transitivity. With this notation, it is easy to establish that

$$x \ll y \Leftrightarrow \begin{cases} \text{either } & x < y \\ \text{or } & x <. y \end{cases}$$

A mapping f of $(M, <)$ into itself is said to be order-preserving if $x < y \Rightarrow f(x) < f(y)$.

If f is one–one, then it is said to be inverse-order-preserving if f^{-1} is order-preserving. A one–one mapping of M onto itself which is both order-preserving and inverse-order-preserving is called an automorphism of M. Since we have two partial orders $<$ and $\ll$, we shall write $<$-automorphism or $\ll$-automorphism depending on which order we use. Similarly, we shall use the words $<$-preserving or $\ll$-preserving *etc.*

We shall denote by G the group of one–one mappings of M onto itself consisting of (i) the Lorentz group, that is, all linear maps which leave the quadratic form Q invariant (ii) translations and (iii) dilatations. G_0 will denote the subgroup of G consisting of the $<$-automorphisms of M. Since every element of G either preserves or reverses the partial order $<$ in M, it follows that G_0 is of index 2 in G. We also have the following theorem.

THEOREM 1. *The group of $\ll$-automorphisms of M is G_0* [3].

We have the following cones at x:

$$\begin{aligned} \text{Light cone at } x: &\quad C^L(x) = \{y\colon Q(y-x) = 0\}, \\ \text{Time cone at } x: &\quad C^T(x) = \{y\colon Q(y-x) > 0\} \cup \{x\}, \\ \text{Space cone at } x: &\quad C^S(x) = \{y\colon Q(y-x) < 0\} \cup \{x\}, \\ &\quad C^{LT}(x) = C^L(x) \cup C^T(x). \end{aligned}$$

It may be easily seen that if $K^*(x) = (K+x) \cup \{x\}$ and $L^*(x) = (L+x) \cup \{x\}$ then $C^T(x) = K^*(x) \cup (-K^*(x))$ and $C^{LT}(x) = L^*(x) \cup (-L^*(x))$. Since every element of G leaves the sign of Q fixed, it is clear that all these cones are invariant under G. With the above notation, it is also easy to see that f is a $<$-automorphism if and only if $f(K^*(x)) = K^*(fx)$ for every x and similarly f is a $\ll$-automorphism if and only if $f(L^*(x)) = L^*(fx)$ for every x.

In what follows d will denote the usual Euclidean distance function on M, that is, $d(x,y) = \{\sum_{i=0}^{3}(x_i - y_i)^2\}^{\frac{1}{2}}$ and $N_\varepsilon^E(x)$ will denote a Euclidean neighbourhood of radius ε about x, that is, $N_\varepsilon^E(x) = \{y \in M\colon d(x,y) < \varepsilon\}$.

3. Definition and properties of the A-topology

DEFINITION. *The A-topology on M is defined to be the finest topology on M with respect to which the induced topology on every time-like line and light-like line is one-dimensional Euclidean and the induced topology on every space-like hyperplane is three-dimensional Euclidean.*

Denote by M^A and M^E the set M equipped with the A-topology and the Euclidean topology, respectively.

We say that a topology on M has the property (P) if the induced topology on every time-like or light-like line is one-dimensional Euclidean and the induced topology on every space-like hyperplane is three-dimensional Euclidean. Let $F = \{T_i\}_{i \in I}$ be the set of all topologies on M having the property (P). It is clear that the set F is non-empty; for, the usual Euclidean topology on M has the property (P) and therefore belongs to F. Introduce a partial order on F as follows: $T_i \leqslant T_j$ in F if and only if $T_i \subset T_j$. Let T be the topology generated by $\bigcup_{i \in I} T_i$, that is, the elements of $\bigcup T_i$ form an open sub-base for the topology T. It is plain that $T_i \leqslant T$ for each $i \in I$. Let U be a basic open set in the topology T, so that

$$U = A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_n},$$

where $A_{i_k} \in T_{i_k}$. If L denotes a time-like or light-like line (or a space-like hyperplane) then

$$U \cap L = \bigcap_{k=1}^{n} (U \cap A_{i_k})$$

which is clearly a one-dimensional Euclidean (or a three-dimensional Euclidean) open set in L. Thus, the topology T, is the finest topology on M having the property (P). Moreover, it is also unique: Suppose that $T' \in F$ is such that for each i, $T_i \leqslant T'$, then $T \leqslant T' \leqslant T$ showing that $T = T'$. We have thus shown the existence and uniqueness of the A-topology. By definition, the A-topology is finer than the Euclidean topology and hence the Hausdorff topology. The open sets of M^A are peculiar and it is not possible to describe the "general" open sets (that is, the basic open sets of the topology). To see that the A-topology is strictly finer than the Euclidean topology, consider a one–one C^1 map h: $[0, 1] = I \to M^E$ satisfying the following conditions:

(i) $h(0) = x$ and $h(1) = y$;

(ii) $t_1 < t_2 \Rightarrow h(t_1) < h(t_2)$ for all $t_1, t_2 \in I$ (where $<$ means the usual partial ordering of the reals);

(iii) any space-like hyperplane, time-like line or light-like line through x meets hI at most at a point.

Let $P = hI - \{x, y\}$. It is clear from the definition of A-topology that P is closed in M^A. We assert that P is not closed in M^E for, if it were, then, P^c (c denotes the complement) would be an open set in M^E about the point x. But from the construction of P, it is easy to see that any Euclidean neighbourhood of x will meet P thus giving a contradiction.

4. Zeno sequences

The notion of a Zeno sequence is due to Zeeman [5]. The definition given here in the case of A-topology can easily be modified to apply whenever there is a topology on M which is finer than the Euclidean topology. It will be evident in the following sections that the technique of using Zeno sequences is quite powerful in determining the homeomorphisms of the finer topology.

Definition. *A Zeno sequence $Z = \{z_n\}$ is a sequence of distinct points of M not containing z such that $z_n \to z$ in M^E by $z_n \not\to z$ in M^A.*

Proposition 1. *A Zeno sequence Z is closed in M^A but not closed in M^E. Conversely, a sequence $Z = \{z_n\}$ of distinct points of M not containing z, which converges to z in M^E is a Zeno sequence if it is closed in M^A.*

We omit the proof of this proposition since it is identical to that of Proposition 1 [2]. We now give a few examples.

Example 1. Let $\{t_n\}$ be a sequence of distinct time-like or light-like lines (or both) passing through a point z. Let $z_n \in t_n$ be such that $d(z_n, z) \to 0$. Let $\{z_n\} = Z$. To prove that Z is a Zeno sequence, we have only to show that Z is closed in M^A and in view of the definition of the A-topology, it is enough to prove that $Z \cap L$ is finite and hence closed for every space-like hyperplane, time-like line or light-like line L. Suppose to the contrary that $Z \cap L$ contains an infinite number of points. Since the induced topology on L is Euclidean, L is complete and Z is a Cauchy sequence, it follows that $Z \cap L$ converges to a point of L; but since the space is Hausdorff, it must be z and, therefore, $z \in L$. If L is a space-like hyperplane, then by construction $Z \cap L = \emptyset$, and if L is a time-like or a light-like line then $Z \cap L$ is at most a singleton; thus in either case we get a contradiction and our assertion that Z is a Zeno sequence is proved.

Example 2. Let $\{s_n\}$ be a sequence of space-like hyperplanes passing through a point z. Choose $z_n \in s_n$ such that $d(z_n, z) \to 0$ and not more than a finite number of z_n's lie on any space-like hyperplane. Following the same argument as above, it is easy to prove that $Z = \{z_n\}$ is a Zeno sequence.

The most important property of Zeno sequences that will be used very frequently is the following:

Lemma 1. *A compact set of M^A cannot contain a Zeno sequence.*

PROOF. Suppose to the contrary that D is a compact set of M^A containing a Zeno sequence Z. Now Z being a closed subset of D is compact. It is easy to see that the topology induced by M^A on Z is discrete because one can always choose a Euclidean ε-neighbourhood about any point $z_n \in Z$ which contains no other point of Z. Since the points of Z are distinct, it follows that Z is an infinite discrete set and therefore cannot be compact and we have a contradiction. This proves the lemma.

5. Homeomorphisms of M^A

In this section, we shall prove that the group of homeomorphisms of M^A is G. Before going into the proof, it will be worthwhile at this point to observe that there are essentially two types of compact sets in M^A, namely three-dimensional ones contained in space-like hyperplanes and the one-dimensional ones contained in time-like lines or light-like lines. Moreover, it is easy to distinguish the two types of compact sets by their connectedness properties. A closed linear interval on a time-like or a light-like line, which is compact in the induced topology, will be disconnected if one point other than an end point is removed; on the other hand, a closed three-dimensional ball in a space-like hyperplane, which is compact in the induced topology, will remain connected even after a countable set of points is removed from it. This distinction is essential and is used frequently in the course of the proof. We now start by proving a sequence of lemmas and finally make use of Theorem 1 to derive the result.

LEMMA 2. *Let B be a closed ball of arbitrary radius with centre x and contained in a space-like hyperplane H which passes through the point x; let h be a homeomorphism of M^A; then $hB \subset C^{LT}(hx)$* is false.

REMARK. We first give an intuitive sketch and then proceed to the proof. Observe that hB is compact and Hausdorff in M^A, since B is, and h is a homeomorphism. Now, a compact Hausdorff space cannot be compact in a finer topology and cannot be Hausdorff in a coarser topology. But the A-topology is strictly finer than the underlying Euclidean topology; therefore, hB will fail to be Hausdorff in the topology induced by the Euclidean topology unless the topologies induced on hB by the A-topology and the Euclidean topology are same. Now hB cannot contain any Zeno sequence so that if one assumes that hB is contained in $C^{LT}(hx)$, then one concludes that hB is a connected union of a finite number of time-like and (or) light-like intervals; but, then, removal of a finite set of points will make this set disconnected, whereas the pre-image will still remain connected.

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PROOF. Suppose to the contrary that $hB \subset C^{LT}(hx)$, then hB meets either (i) only a finite number of time-like and (or) light-like lines passing through hx or (ii) an infinite number of them. In the first case, removal of a single point hx from hB will make the set disconnected; on the other hand, B remains connected even after a countable set is removed from it; thus we arrive at a contradiction.

In the second case, observe that every Euclidean ε-neighbourhood of hx will meet hB such that the points of intersection are on an infinite number of time-like or light-like lines emerging from hx; for, otherwise, we can arrive at a contradiction in the same manner as above. Thus we have enough points in hB in any arbitrary Euclidean ε-neighbourhood of hx to construct a Zeno sequence. Choose a point $z_n \in t_n$, where $\{t_n\}$ is a sequence of distinct time-like and (or) light-like lines passing through hx and intersecting hB, such that $z_n \to hx$ in M^E where each $z_n \in hB$. This is a Zeno sequence contained in hB and by Lemma 1 we have a contradiction. Hence the lemma is proved.

LEMMA 3. *Let B and h be as in Lemma* 2; *then there exists a ball B_r or radius r such that* (*i*) $B_r \subset B$ *and* (*ii*) $hB_r \subset C^S(hx)$.

PROOF. If B is such that $hB \subset C^S(hx)$, then choose $B_r = B$. If not, then from Lemma 2 it follows that hB is partly in $C^{LT}(hx)$ and partly in $C^S(hx)$. Let $A = hB \cap C^{LT}(hx)$. If A is finite, say $\{x_1, x_2, \ldots, x_n\}$, then the pre-image $\{h^{-1}x_1, h^{-1}x_2, \ldots, h^{-1}x_n\}$ can be removed from B and a suitable $B_r \subset B - h^{-1}A$ can be chosen such that $hB_r \subset C^S(hx)$.

If A is infinite, then there are two cases to be considered: (i) hx is a limit point of A in M^E and (ii) hx is not a limit point of A in M^E. In the second case, there exists a neighbourhood 0 of hx which does not meet A. Then choose a suitable $B_r \subset (h^{-1}0) \cap B$, so that $hB_r \subset C^S(hx)$. In the first case when every Euclidean neighbourhood of hx meets A, three cases may arise: (1) A meets an infinite number of time-like or light-like lines through hx; (2) A meets only a finite number of such lines with at least one point on a time-like line; (3) A lies on only a finite number of light-like lines through hx. In case (1), a Zeno sequence can be constructed as in Lemma 2; in case (2), removal of the point hx will make $hB - hx$ disconnected thus giving contradictions. We are now left with case (*i*) (3). We shall show that this case cannot occur.

Now write $D = hB \cap C^S(hx)$. Since B is compact, so is hB; therefore hB cannot contain a Zeno sequence by virtue of Lemma 1. Hence one can choose $\varepsilon > 0$ such that $N^E_\varepsilon(hx) \cap D = F_\varepsilon$ is contained in the union of a finite number of space-like hyperplanes passing through hx. For, if every F_ε required infinitely many space-like hyperplanes, we could construct a Zeno sequence. Observe that F_ε is non-empty; for otherwise we can arrive at a contradiction as follows. Suppose that F_ε is empty, then $N^E_\varepsilon(hx) \cap hB$ is contained in a finite number of light-like lines through hx.

Since $h^{-1}(N_\varepsilon^E(hx))$ is an open set about x, $U = h^{-1}(N_\varepsilon^E(hx)) \cap B$ is an open set about x in the induced topology of B. Choose a closed ball $V \subset U$ with centre x. It is then clear that hV is contained in a finite number of light-like lines through hx. This is a contradiction in view of Lemma 2.

Now choose a closed ball B_r of radius r in $B \cap h^{-1}(N_\varepsilon^E(hx))$ so that hB_r is a union of sets contained in a finite number of space-like hyperplanes and a finite number of light-like lines through hx. This is impossible since $B_r - x$ is connected, whereas $hB_r - hx$ is disconnected. This completes the proof of the lemma.

For convenience, we now make the following definition.

Definition. *A set is said to be a s-set if it is contained in a space-like hyperplane.*

The next lemma helps us to determine the nature of the image hB_r completely.

Lemma 4. *If B_r and h are as in Lemma 3, then there exists a ball $B_s \subset B_r$ such that hB_s is a union of a finite number of s-sets.*

Proof. The argument is essentially a repetition of the one given above. Suppose that hB_r is not contained in any finite union of space-like hyperplanes through hx. We already know from Lemma 1 that we cannot have a Zeno sequence in hB_r, that is, it is impossible to choose a Cauchy sequence in hB_r with one point each on each space-like hyperplane; hence it is possible to choose $\varepsilon > 0$ such that $N_\varepsilon^E(hx) \cap hB_r$ lies in some finite union of space-like hyperplanes through hx. If we choose $B_s \subset h^{-1}(N_\varepsilon^E(hx)) \cap B_r$, then clearly hB_s is a union of a finite number of s-sets. This proves the assertion.

Moreover, since B is compact, the topology induced on hB_s by M^Δ is Euclidean; therefore

$$h|B_s\colon B_s \xrightarrow{h} M^\Delta \xrightarrow{id} M^E$$

is an imbedding, where id denotes the identity map of M; so that it becomes easy to form an idea about the nature of the image hB_s.

Lemma 5. *Let H be a space-like hyperplane through x and h a homeomorphism of M^Δ; then (i) hH is a countable union of s-sets and (ii) $M^E - hH$ has two components.*

Proof. At each rational point x of H (a point whose coordinates are rationals with respect to some frame of reference) choose a ball $B_s(x)$ as in Lemma 4 above; $hB_s(x)$ is then a union of a finite number of s-sets. Since the rational points of H are countable and dense, the first assertion follows.

To prove the second assertion, note that the induced topology on H is Euclidean, so that $hH \cong R^3$. Moreover, $M^A - hH$ has two components; therefore, $M^E - hH$ has at most two components, since the A-topology is finer than the Euclidean topology. We claim that $M^E - hH$ has exactly two components; for, if not, then $M^E - hH$ is connected and therefore any two points p and q of $M^E - hH$ can be arc-connected by a polygonal path; since the topology induced by the A-topology on the polygonal path is Euclidean, it follows that $M^A - hH$ is arc-connected, which is a contradiction.

LEMMA 6. *Let $x \ll y$ and h a homeomorphism of M^A, then there exists a point $z \in [x, y]$, the line segment joining the points x and y, such that $h[x, z] = [hx, hz]$ with $hx \ll hz$ or $hz \ll hx$.*

PROOF. Suppose to the contrary that there exists no such z with the above property, that is, for no $z \in [x, y], h[x, z]$ is contained in a time-like line or a light-like line emerging from hx. Choose a sequence of points $Z = \{z_n\}$ in $[x, y]$ such that $z_n \to x$ and the lines $\{t_n\}$ joining hz_n and hx are all distinct. Passing to a subsequence if necessary, assume that all the t_n's are either (i) time-like and (or) light-like lines or (ii) space-like lines. In the first case, $\{hz_n\}$ is a Zeno sequence in the compact set $h[x, y]$ giving a contradiction. In the second case we claim that no more than a finite number of points of $\{hz_n\}$ lie in any space-like hyperplane through hx. For, otherwise, one can choose a subsequence $\{hz_{n_k}\}$ lying on a space-like hyperplane at hx which still converges to hx. Choose a ball B_δ (as in Lemma 4) with centre hx. Since the subsequence converges to hx, B_δ contains all but a finite number of points of the subsequence. On the other hand, $h^{-1} B_\delta \subset C^S(x)$ does not meet $(x, y]$ at all. Thus in either case we arrive at a contradiction. Hence our assumption that such a z does not exist is false.

DEFINITION. *Let $x \ll y$ and h a homeomorphism of M^A, then h is said to preserve or reverse the orientation of $[x, y]$ according as the sign of the expression*

$$(hz)_0 - (hx)_0 / (z_0 - x_0)$$

is positive or negative, where z is determined as in Lemma 6.

LEMMA 7. *Let $x \ll y, y'$; then a homeomorphism h of M^A either preserves both or reverses both orientations.*

PROOF. Suppose to the contrary that h preserves the orientation of one and reverses that of the other, that is, assume that $x \ll z$, $x \ll z'$, $hx \ll hz$ and $hz' \ll hx$. Take a space-like hyperplane H at z. Clearly $(x, z]$ and $(x, z']$ belong to the same component of $M^{\Delta}-H$. On the other hand, by Lemma 5 $(hx, hz]$ and $(hx, hz']$ are in different components of $h(M^{\Delta}-H) = M^{\Delta}-hH$. Since components are preserved by any homeomorphism we have a contradiction.

In case $[x, y]$ and $[x, y']$ are oriented oppositely and h preserves one and reverses the other orientation, the same argument applies since h^{-1} is also a homeomorphism. This completes the proof of the lemma.

LEMMA 8. *Any homeomorphism h of M^{Δ} either preserves or reverses the partial order $\leqslant$.*

PROOF. Let $x \leqslant y$ and $hx \leqslant hz$, that is, h preserves the orientation of $[x, y]$. By applying Lemma 6, we have for each $z \in [x, y]$, a neighbourhood N_z such that hN_z consists of one or two time-like and (or) light-like intervals. As $[x, y]$ is compact, we can obtain a finite covering N_{z_i} $(i = 1, 2, \ldots, n)$ and hence a finite set of points $x = z_0 \leqslant z_1 \leqslant z_2 \leqslant \ldots$ such that for each i, $h[z_{i-1}, z_i]$ is an interval on a time-like and (or) a light-like line. Applying Lemma 7, we obtain $hx \leqslant hy$ in a finite number of steps. Similarly, if h reverses the orientation of $[x, y]$, then $hy \leqslant hx$.

To prove that h either preserves or reverses the partial order $\leqslant$, we proceed as follows: Let $x \leqslant y$ and $p \leqslant q$ where x, y, p, q are arbitrary points of M. Choose a point u such that $x \leqslant u$ and $p \leqslant u$. From the first paragraph and Lemma 7 it follows that if $hx \leqslant hy$ then $hx \leqslant hu$ and since $x \leqslant u$, and $p \leqslant u$, $hx \leqslant hu$ implies $hp \leqslant hu$. Similarly $hp \leqslant hq$. Thus all partial orders are preserved if one is preserved. In the same manner, all orders are reversed if only one is reversed.

We have therefore the following:

THEOREM 2. *The group of homeomorphisms of M^{Δ} is G.*

PROOF. Let h be a homeomorphism of M^{Δ}; then by Lemma 8, h either preserves or reverses the partial order $\leqslant$. In the latter case, compose it with the time reflection g defined by

$$g(x_0, x_1, x_2, x_3) = (-x_0, x_1, x_2, x_3);$$

so that $h \circ g$ becomes $\leqslant$-preserving. By Theorem 1, either h or $h \circ g$ belongs to G_0. In any case $h \in G_0 \cup G_0 g^{-1} = G$. This completes the proof.

The A-topology has a very interesting property. Intuitively, one thinks about the path of a particle as the image of a continuous map $f\colon I = [0, 1] \to M$ such that at each point the path enters into the null cone L; in other words, if $t_1, t_2 \in I$, then

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$t_1 < t_2$ implies that $f(t_1) \ll f(t_2)$. (Here $<$ denotes the usual ordering of the real numbers.) In case of Zeeman's fine topology, the image of a continuous map $f: I \to M^F$ satisfying $t_1 < t_2 \Rightarrow f(t_1) < f(t_2)$ is a union of a finite number of time-like intervals (M^F denotes the set M equipped with the fine topology). In particular, this excludes photons since photons travel along null lines. In case of A-topology, however, one can follow exactly the same argument as that of Zeeman to prove the following:

PROPOSITION 2. *Let $f: I \to M^A$ be a continuous map such that $t_1, t_2 \in I$ with $t_1 < t_2$ implies that $f(t_1) \ll f(t_2)$; then fI is a connected union of finite number of time-like and (or) light-like intervals.*

6. Final remarks

Williams in his paper [4] has suggested a topology on Minkowski space which is characterized by the property that the induced topology on time-like and space-like lines is Euclidean and that it is the finest such topology on M having this property. This definition, which uses only time-like and space-like lines, offers a few advantages. For example, it lends itself readily for possible generalization to curved space–times where curves are important, whereas space-like hypersurfaces are of little physical significance.

However, this topology differs significantly from the A-topology (or from Zeeman's fine topology) in its group of homeomorphisms. Williams has proved that the C^1-subgroup of homeomorphisms of this topology is G [4, Theorem 3]. Without the C^1-condition, it is doubtful whether the result will still be valid. Another notable feature of Williams' topology is that if $f: I \to M^w$ is a continuous map, where M^w denotes Minkowski space with Williams' topology, then $f(I)$ is a connected union of time-like and (or) space-like intervals. If, however, f is assumed to be order-preserving, then it follows that $f(I)$ is a connected union of time-like intervals representing the path of an inertial particle under a finite number of collisions. This, as pointed out in the introduction, excludes the path of photons. The A-topology is significantly different from William's topology in this respect.

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References

[1] S. Nanda, Ph.D. Thesis, Queen's University at Kingston, Canada (1969).
[2] S. Nanda, "Topology for Minkowski space", *J. Math. Phys.* 12 (1971), 394–400.
[3] S. Nanda, "Weaker versions of Zeeman's conjectures on topologies for Minkowski space", *J. Math. Phys.* 13 (1972), 12–15.
[4] G. Williams, "Fine topologies for Minkowski space", *Proc. Camb. Phil. Soc.* 76 (1974), 503–509.
[5] E. C. Zeeman, "Topology for Minkowski space", *Topology* 6 (1967), 161–171.

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