

A positive Liapunov exponent for the critical value of an S-unimodal mapping implies uniform hyperbolicity

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Abstract. A positive Liapunov exponent for the critical value of an S-unimodal mapping implies a positive Liapunov exponent of the backward orbit of the critical point, uniform hyperbolic structure on the set of periodic points and an exponential diminution of the length of the intervals of monotonicity. This is the proof of the Collet-Eckmann conjecture from 1981 in the general case.

1. Introduction

The existence of an invariant measure absolutely continuous with respect to Lebesgue measure for maps on the interval is often connected with the existence of some hyperbolic structure i.e. the exponential growth of derivatives of the iterations on some subset of the interval. (But, according to [B, C], sometimes there is only subexponential growth.)

By virtue of examples we recall the works of Lasota and Yorke [L, Y], Ruelle [R], Misiurewicz [M] and Szlenk [Sz] where the assumed conditions, sufficient for the existence of the absolutely continuous invariant measure, imply the uniform hyperbolic structure over the whole interval, the image of the critical point or some Cantor set.

In 1981 Collet and Eckmann [CE1] proved the existence of such a measure for S-unimodal mappings which have the uniform hyperbolic structure on the image of critical point and on the set of the preimages of the critical point i.e. satisfying following conditions: there exist $K_E > 0$ and $\lambda_E > 1$ such that for all n :

$$C1 \quad \left| \frac{d}{dx} (f^n)(f(c)) \right| > K_E \lambda_E^n \quad \text{and}$$

$$C2 \quad \left| \frac{d}{dx} (f^n)(z) \right| > K_E \lambda_E^n \quad \text{if } f^n(z) = c,$$

where c is the (unique) critical point of f and $f^n = f \circ f \circ \dots \circ f$ n -times. Collet and Eckmann conjectured that C1 implies C2. This has been proven [N1] for symmetric mappings (that means such that $f(x) = f(1-x)$ for $x \in [0, 1]$). In the present paper we prove that the conjecture is also true for nonsymmetric S-unimodal mappings.

The condition C1 is fulfilled quite often [CE3], but it seems that it is not necessary [B, C].

2. Notation and technical lemmas

From now on we shall assume that the mapping $f: [0; 1] \rightarrow [0; 1]$ satisfies the following conditions:

A0: $f \in C^3[0; 1]; f(0) = f(1) = 0$.

A1: f is S -unimodal; that means that there exists a unique $c \in (0; 1)$ such that f is increasing on $(0; c)$ and decreasing on $(c; 1)$ and $Sf < 0$ where $Sf = f'''/f' - 3(f''/f')^2/2$.

A2: $f''(c) \neq 0$.

We shall use the following notation: $f^1 = f$ and for $n \geq 1$ $f^{n+1} = f^n \circ f; x_n = f^n(x); Df^n = df^n/dx$.

Now we quote without proof some lemmas:

LEMMA 1. [CE2, II.4]. *If $Sf \leq 0$ then $Sf^n \leq 0$. $Sf \leq 0$ implies that $|f'|$ has no positive local minima.*

LEMMA 2. [CE2, II.4]. *If f satisfies A1 and C1 then $f|_{(c_2, c_1)}$ has no sinks and no attractive periodic orbits.*

Remark 3. By Lemma 2 we may assume later on that $f'(0) > 1$ and that there is no other fixed point of f on $(0; c)$.

LEMMA 4. [CE1, 2.2]. *There are two constants $M > 0, m > 0$, such that for $x \in [0; 1]$*

$$m|x - c| \leq |f'(x)| \leq M|x - c|$$

and

$$m|x - c|^2/2 \leq |f(x) - f(c)| \leq M|x - c|^2/2.$$

We define $\hat{x}(x)$ by: $\hat{c} = c$, and for $x \neq c, f(\hat{x}) = f(x)$ and $\hat{x} \neq x$.

LEMMA 5. [N2, 3.4]. *For $x \neq c, |\hat{x} - c|/|x - c| \leq (M/m)^{1/2}$ and $|f'(\hat{x})|/|f'(x)| \leq (M/m)^{3/2}$.*

LEMMA 6. [CE1, 2.6]. *If $Sf < 0$ and $Df^n|_{(x,y)} \neq 0$ then $|x_n - y_n| \geq (Df^n(x) \cdot Df^n(y))^{1/2}|x - y|$.*

LEMMA 7. [N1; 7]. *Suppose $g \in C^3[u, v]; Sg \leq 0; g'(u) = g'(v) = 0; g'|_{(u,v)} \neq 0$. For a fixed $x \in (u, v)$ define a function $h(t), t \in [u, v]$ by $h(t) = (g(t) - g(x))/(t - x)$ if $t \neq x$ and $h(x) = g'(x)$. Then $h(t)$ has only one local extremum in (u, v) and it is a maximum. In particular for $a \leq t \leq b$ we have $|h(t)| \geq \min(|h(a)|; |h(b)|)$.*

LEMMA 8. [CE2, II.4]. *The set $C_{-\infty} = \bigcup_{j \geq 0} f^{-j}(c)$ is dense in $[0; 1]$.*

3. Preliminary estimations

Let us denote by Δ_n the family of the intervals of monotonicity of f^n i.e. $(\alpha, \gamma) \in \Delta_n$ if and only if $\alpha, \gamma \in \bigcup_{0 \leq j < n} f^{-j}(c) \cup \{0; 1\}$ and $Df^n|_{(\alpha,\gamma)} \neq 0$. Let $|A|$ denote the Lebesgue measure of the set $A \subset [0; 1]$.

LEMMA 9. *Let α^j be a sequence of preimages of c such that $(0, \alpha^j) \in \Delta_j$. Then there exist two constants $K_0 > 0$ and $\lambda_0 > 1$ such that:*

$$P(j) = \min (|f^j(0, \alpha^{j+1})|/|(0, \alpha^j)|; |f^j(\alpha^{j+1}, \alpha^j)|/|(\alpha^{j+1}, \alpha^j)|)$$

Proof. First observe that $\alpha^0 = c$ and $\alpha^{j+1} = f^{-1}(\alpha^j) \cap (0, \alpha^j)$; so such a sequence exists. We have moreover $f^j(\alpha^{j+1}) = c$. By density of \mathbb{C}_∞ , there is a j_0 such that for $j > j_0$: $\min(c_1; c_1 - c) < \lambda_0 \alpha^j$ where $\lambda_0 = (f'(0))^{1/2} > 1$. Hence for $j \geq j_0$

$$P(j) \geq \min\left(\frac{c}{\alpha^{j+1}}, \frac{c_1 - c}{\alpha^j}\right) > 1.$$

By Lemma 7. $Df^j(\alpha^{j+1}) \geq P(j) > 1$, and by Lemma 6

$$\frac{c}{\alpha^{j+1}} = \frac{f^j(\alpha^{j+1}) - f^j(0)}{\alpha^{j+1} - 0} \geq (Df^j(0) \cdot Df^j(\alpha^{j+1}))^{1/2} > \lambda_0^j.$$

Therefore $\alpha^j < (c/\lambda_0)\lambda_0^{-j}$ for all $j > j_0$ and

$$P(j) > \frac{\min(c; c_1 - c)}{\lambda_0} \cdot \lambda_0^j \quad \text{for } j > j_0.$$

We set $K_0 = \lambda_0^{-j_0} \min_{j \geq j_0} (P(j); c; c_1 - c)$ and obtain the assertion. □

COROLLARY 10. *There is a constant K_1 and a sequence β^j such that for all $j(\beta^j; 1) \in \Delta_j$ and*

$$\min(|f^j(\beta^{j+1}, 1)|/|(\beta^{j+1}; 1)|; |f^j(\beta^j; \beta^{j+1})|/|(\beta^j; \beta^{j+1})|) > K_1 \lambda_0^j.$$

Proof. We set $\beta^j = \hat{\alpha}^j$, use Lemma 9 and Lemma 5. $K_1 = K_0 \cdot (m/M)^{1/2}$. □

We denote by (a, b) the interval with endpoints a, b independent on their mutual order (i.e. not necessary $a < b$).

LEMMA 11 [N2, 10]. *Let $(a, c) \in \Delta_{n+1} \setminus \Delta_n$, then there are two periodic points of period n, p and q such that:*

$$(q, \hat{q}) \subset (a, \hat{a}) \subset (\hat{p}, p) \quad \text{and} \quad Df^n|_{(c,p)} \neq 0.$$

Moreover $a_n = \hat{a}_n = c$.

PROPOSITION 12. *Suppose that f satisfies A0–A2 and C1. Then there are two constants $K_A > 0$ and $\lambda_A > 1$ (independent on n) such that if $(a, c) \in \Delta_{n+1} \setminus \Delta_n$ then*

$$|f^n(a, c)|/|(a, c)| > K_A \lambda_A^n \quad n = 1, 2, \dots$$

Proof. Assume first that f^n is decreasing on (c, a) . By Lemma 11 $(c, q) \subset (c, a) \subset (c, \hat{p})$ for some periodic points $q = q_n$ and $p = p_n$. We study f^n on the interval (a_1, c_1) . We have $q_1 \in (a_1; c_1)$ and $f^n(q_1) = q_1$, hence by Lemma 2, $|Df^n(q_1)| > 1$. By Lemma 6, C1 and Lemma 4 we have:

$$\begin{aligned} (*) \quad |q_{n+1} - c_{n+1}| &\geq (Df^n(q_1) Df^n(c_1))^{1/2} |q_1 - c_1| \geq (K_E \lambda_E^n)^{1/2} |q_1 - c_1| \\ &\geq (K_E \lambda_E^n)^{1/2} \frac{m}{2} |q - c|^2. \end{aligned}$$

Now we come back to the interval (c, a) and as f has no attractive periodic orbit we have by Lemma 5 and Lemma 1

$$|Df^n(x)| \geq \min(|Df^n(q)|; |Df^n(p)|) \geq (m/M)^{3/2}$$

for all $x \in (q, p)$. We used

$$\begin{aligned} Df^n(\hat{p}) &= Df^{n-1}(p_1) \cdot f'(\hat{p}) = Df^{n-1}(p_1) \cdot f'(p) \cdot \frac{f'(\hat{p})}{f'(p)} \\ &\geq Df^n(p) \left(\frac{m}{M}\right)^{3/2}. \end{aligned}$$

Thus

$$|c - q|/|a - q| = |a_n - q_n|/|a - q| \geq (m/M)^{3/2}$$

and

$$|a - c| = |c - q| + |a - q| \leq |c - q|(1 + (M/m)^{3/2}).$$

By (*), Lemma 4 and the last inequality

$$\begin{aligned} M|c_n - a_n|^2/2 &\geq |c_{n+1} - a_{n+1}| > |c_{n+1} - q_{n+1}| > (K_E \lambda_E^n)^{1/2} m|q - c|^2/2 \\ &\geq (K_E \lambda_E^n)^{1/2} m(m^{3/2}/(M^{3/2} + m^{3/2}))^2 |a - c|^2/2 \end{aligned}$$

Thus $|c_n - a_n|/|c - a| > K_B \lambda_A^n$ where $\lambda_A = \lambda_E^{1/4}$ and

$$K_B = K_E^{1/2} m^2 / (M^2 + mM\sqrt{m}).$$

Now consider that f^n is increasing on (a, c) . Then it is decreasing on (c, \hat{a}) and by previous estimations and Lemma 5 we have:

$$\frac{|f^n(c) - f^n(a)|}{|c - a|} = \frac{|f^n(c) - f^n(\hat{a})|}{|c - \hat{a}|} \cdot \frac{|c - \hat{a}|}{|c - a|} \geq K_B \lambda_A^n (m/M)^{1/2}.$$

The assertion of Proposition 12 follows for $K_A = K_B(m/M)^{1/2}$.

Remark 13. By Proposition 12 there is an N_0 such that for $n > N_0$ if $(c, a) \in \Delta_{n+1} \setminus \Delta_n$ then $|c_n - a_n|/|c - a| > \lambda_A^{n/2}$.

4. The intervals of monotonicity

The intention of the following estimations is as follows: we try to generalise Proposition 12 on all intervals $(\alpha, \beta) \in \Delta_{n+1} \setminus \Delta_n$. For this we assume without loss of generality that for $(\alpha, \beta) \in \Delta_{n+1} \setminus \Delta_n$ there is a $k < n$ such that $\alpha_k = \beta_n = c$. We write

$$\frac{|\alpha_n - \beta_n|}{|\alpha - \beta|} = \frac{|\alpha_n - \beta_n|}{|\alpha_k - \beta_k|} \frac{|\alpha_k - \beta_k|}{|\alpha - \beta|},$$

$(\alpha_k, \beta_k) = (c, \beta_k) \in \Delta_{n-k+1} \setminus \Delta_{n-k}$ and the first quotient may be estimated by Proposition 12. As for the second we observe that for some $(\gamma, \delta) \in \Delta_k$, $(\alpha, \beta) \subset (\gamma, \delta)$ and both (γ, α) and (α, δ) belong to $\Delta_{k+1} \setminus \Delta_k$. We can use Lemma 7 with $g = f^k$ and

$$x = \frac{|\alpha_k - \beta_k|}{|\alpha - \beta|} \geq \min \left(\frac{|\alpha_k - \gamma_k|}{|\alpha - \gamma|}, \frac{|\alpha_k - \delta_k|}{|\alpha - \delta|} \right)$$

and we reduced the estimation for an interval from $\Delta_{n+1} \setminus \Delta_n$ to the estimation for an interval from $\Delta_{k+1} \setminus \Delta_k$, $k < n$. So we could try the inductive proof. That is how the proof goes in the symmetric case, where we can set $K_A = 1$ in homolog of Proposition 12. But in general $K_A < 1$ and the constant may reappear and spoil the exponential growth every time we use Proposition 12. So we are obliged to study different cases as for example $n - k \geq N_0$, when we can use Remark 13, change the constant λ_A or find some other way to estimate $|\alpha_n - \beta_n|/|\alpha - \beta|$. In fact although we shall use the above idea we shall prove that $|\alpha - \beta| < K\lambda^{-n}$ ($K > 0, \lambda > 1$), and then conclude (Proposition 19) that f has uniform hyperbolic structure on the set of periodic points (i.e. there is a $\lambda > 1$ such that for all s , if $p_s = p$ then $|Df^s(p)| > \lambda^s$). This will enable us to use Proposition 3.9 [N2] which states that f has uniform hyperbolic structure on the set of periodic points if and only if it satisfies C2.

Before we generalise Proposition 12 we introduce some new object, which will describe precisely the intervals $(\alpha, \beta) \in \Delta_{n+1} \setminus \Delta_n$.

For an $x \in (0; 1) \setminus \mathbb{C}_{-\infty}$ we define the family $\xi(x)$ of the intervals by:

$$\xi(x) = \left\{ \xi \in \bigcup_{n=0}^{\infty} \Delta_n : x \in \xi \right\}.$$

Since for $k < n$ the critical points of f^k are also the critical points for f^n (this follows from $Df^n(y) = Df^{n-k}(y_k) \cdot Df^k(y)$) we have a natural order in $\xi(x)$ by inclusion. We numerate the intervals $\xi \in \xi(x)$ accordingly $\xi_0 = (0; 1)$ and $\xi_{i+1} \subsetneq \xi_i$, $i = 0, 1, \dots$

By density of $\mathbb{C}_{-\infty}$ we have two easy lemmas.

LEMMA 14.

(A) $\bigcap_{\xi_i \in \xi(x)} \text{cl}(\xi_i) = x$.

(b) There is a $i_0 = i_0(x)$ such that $\{0; 1\} \cap \xi_{i_0} = \emptyset$.

LEMMA 15. For every i there is a point $\beta^i \in \xi_i$ such that $f^{n_i}(\beta^i) = c$, where n_i is the maximal n with $\xi_i \in \Delta_n$. Moreover β^i is the endpoint of ξ_{i+1} .

LEMMA 16. Suppose $i \geq i_0$ and let $\xi_i = (\alpha, \gamma)$. Then there are $k, m < n_i$ such that $\alpha_k = \gamma_m = c$, $k \neq m$. If $m < k$ then $k = n_{i-1}$ and $m = n_{i-1}$, for some $l \geq 2$.

Proof. As $i \geq i_0$ and $\xi_i \in \bigcup_{n=0}^{\infty} \Delta_n$ the endpoints of ξ_i belong to $\mathbb{C}_{-\infty}$. As $\xi_i \in \Delta_{n_i}$ we have $k, m < n_i$, $k \neq m$ by Rolle's theorem. We have $\xi_i \subset \xi_{i-1} \subset \xi_{i-2}$ and it is enough to remark that $\alpha = \beta^{i-1}$, (β^i defined in Lemma 15). □

Let $\xi_i = (\alpha, \gamma)$ and $\beta = \beta^i \in \xi_i$. We shall denote (α, β) by $\bar{\xi}_i$ and (β, γ) by $\bar{\bar{\xi}}_i$, ($\alpha_k = \gamma_m = c$; $k > m$).

PROPOSITION 17. Suppose f satisfies A0-A2 and that the assertion of Proposition 12 is true. Then there exist positive constants K_T, d, w, λ_T (with $\lambda_T > 1$) (independent on n and x) such that for every $x \in (0; 1) \setminus \mathbb{C}_{-\infty}$ there is a nondecreasing sequence $\{s_j(x)\}$ such that for all i and all $\xi_i \in \xi(x)$ we have

$$A(i): \quad |f^{n_i}(\bar{\xi}_i)|/|\bar{\xi}_i| \geq K_T d^s \lambda_T^{n_i} \text{ and } |f^{n_i}(\bar{\bar{\xi}}_i)|/|\bar{\bar{\xi}}_i| > K_T d^s \lambda_T^{n_i}$$

and

$$B(i): \quad |\bar{\xi}_i| \leq (w/(1+w))^{s_i} \text{ and } |\bar{\bar{\xi}}_i| \leq (w/(1+w))^{s_i}.$$

The idea of the proof. The proof will consist in estimating $A(i)$ with the aid of the decomposition described at the beginning of this section. We shall fix x and define s_i in such a way that $B(i)$ holds. We shall investigate several cases, in most of them $s_{i+1} = s_i$, and rarely $s_{i+1} = s_i + 1$. The sequence s_i counts how many times we are obliged to reintroduce the constant d in $A(i)$. But every time we spoil $A(i)$ we diminish the length of ξ_i by a constant factor. In fact, as we shall see in Proposition 18 $A(i)$ and $B(i)$ together prove that $|\xi_i|$ diminish exponentially.

Proof. First we define the constant mentioned in Proposition 17. We recall that λ_A, K_A come from Proposition 12, N_0 from Remark 13, K_1, λ_0 from Corollary 10.

We may assume $K_A < 1$. We fix $N_1 > N_0$ such that $\lambda_A^{N_1/2} > 2\lambda_A^{N_1/4}/K_A$. Let $d = \min_{x \neq y} |x - y|$, where

$$x, y \in \bigcup_{s=0}^{2N_1} f^{-s}(c) \cup \{c_s\}_{s=0}^{N_1},$$

$d > 0$ otherwise f would have a sink. Obviously $d < 1$.

$$w = (K_A d^3)^{-1} > 1,$$

$$\lambda_T = \min \left(\lambda_A^{1/4}; \lambda_0; \left(\frac{1+w}{w} \right)^{1/2N_1} \right) > 1,$$

$$K_T = \min(K_A; K_1) > 0.$$

Now we fix $x \in (0, 1) \setminus \mathbb{C}_{-\infty}$ and prove $A(i)$ and $B(i)$ by induction. For $j = 1, n_j = 1$. We set $s_1 = 0$ and $A(1)$ is true by Lemma 9 and Proposition 12. $B(1)$ is obvious.

Suppose that $A(j)$ and $B(j)$ are true for all $j \leq i$. We want to prove them for $j = i + 1$. If $\xi_{i+1} \cap \{0, 1\} \neq \emptyset$ then $i + 1 < i_0(x)$ (i_0 defined in Lemma 14) and we use directly Corollary 10. Otherwise both ends of ξ_{i+1} are critical points of $f^{n_{i+1}}$ and we have by Lemma 16: $\beta = \beta^{i+1} \in \xi_{i+1} = (\alpha, \gamma)$ with $\alpha_{n_i} = \gamma_{n_{i-1+1}} = \beta_{n_{i+1}} = c$. We denote $n_{i+1} = n; n_i = k; n_{i-1+1} = m; n > k > m$. By convention $\bar{\xi}_{i+1} = (\alpha, \beta); \bar{\xi}_{i+1} = (\beta, \gamma)$. We have also

$$(\alpha_k; \beta_k) = (c, \beta_k) \in \Delta_{n-k+1} \setminus \Delta_{n-k},$$

$$(\gamma_m, \beta_m) = (c, \beta_m) \in \Delta_{n-m+1} \setminus \Delta_{n-m},$$

$$(\gamma_m, \alpha_m) = (c, \alpha_m) \in \Delta_{k-m+1} \setminus \Delta_{k-m},$$

and for these intervals we can use Proposition 12 and Remark 13. We shall consider three general cases

(1) $n - k > N_1,$

(2.1) $n - k \leq N_1$ and $k - m > N_1,$

(2.2) $n - k \leq N_1$ and $k - m \leq N_1.$

We shall divide (2.1) into four subcases and (2.2) into three subcases.

(1) $n - k > N_1$. We set $s_{i+1} = s_i$. We estimate $A(i + 1)$ for $\bar{\xi}_{i+1}$.

$$\begin{aligned} \frac{|f^{n_{i+1}}(\bar{\xi}_{i+1})|}{|\bar{\xi}_{i+1}|} &= \frac{|\alpha_n - \beta_n|}{|\alpha - \beta|} = \frac{|\alpha_n - \beta_n|}{\alpha_k - \beta_k} \cdot \frac{|\alpha_k - \beta_k|}{|\alpha - \beta|} \\ &\geq \lambda_A^{(n-k)/4} \cdot K_T d^s \lambda_T^k > K_T d^{s+1} \lambda_T^n \end{aligned}$$

For the first quotient we used Remark 13 and (1) ($N_1 > N_0$) then the definition of $\lambda_T, (\lambda_T \leq \lambda_A^{1/4})$. For the second we used $A(i)$ and Lemma 7, as announced at the beginning of this section. The estimation of $A(i + 1)$ for $\bar{\xi}_{i+1}$ is similar as $n - m > n - k > N_1$ and $S_i \geq S_{i-1}$. $B(i + 1)$ follows from the fact that

$$|\alpha - \gamma| \geq \max(|\alpha - \beta|; |\beta - \gamma|) = \max(|\bar{\xi}_{i+1}|; |\bar{\xi}_{i+1}|)$$

and $(\alpha; \gamma)$ is either $\bar{\xi}_i$ or $\bar{\xi}_i$ by Lemma 16.

So we can consider the following case:

(2.1) We assume $n - k \leq N_1$ and $k - m > N_1$.

First we remark that as in (1) since $n - m > k - m > N_1$ we have

$$|f^{n_{i+1}}(\bar{\xi}_{i+1})|/|\bar{\xi}_{i+1}| \geq K_T d^s \lambda_T^n$$

so we have to estimate $A(i+1)$ only for $\bar{\xi}_{i+1}$. But now we cannot use Remark 13 and we shall get rid of the constant K_A from Proposition 12 with the “help” of the constant w . We distinguish four subcases:

$$(2.1.1) \quad |\alpha - \beta| > w|\gamma - \beta| \text{ and } |\gamma_k - \beta_k| > d|\alpha_k - \beta_k|,$$

$$(2.1.2) \quad |\alpha - \beta| > w|\gamma - \beta| \text{ and } |\gamma_k - \beta_k| \leq d|\alpha_k - \beta_k|,$$

$$(2.1.3) \quad |\alpha - \beta| \leq w|\gamma - \beta| \text{ and } |\gamma - \beta| > w|\alpha - \beta|,$$

$$(2.1.4) \quad |\alpha - \beta| \leq w|\alpha - \beta| \text{ and } |\gamma - \beta| \leq w|\alpha - \beta|,$$

(2.1.1): we set $s_{i+1} = s_i$. We define $y \in (\beta; \gamma)$ by $|Df^k(y)| = |\gamma_k - \beta_k|/|\gamma - \beta|$. As $n - k \leq N_1$ we have $|\alpha_k - \beta_k| \geq d$, and

$$\begin{aligned} \frac{|\alpha_k - \beta_k|}{|\alpha - \beta|} &= \frac{|\alpha_k - y_k|}{|\alpha - y|} \cdot \frac{|\alpha - y|}{|\alpha - \beta|} \cdot \frac{|\alpha_k - \beta_k|}{|\alpha_k - \beta_k|} \\ &\geq (Df^k(y) \cdot Df^k(\alpha))^{1/2} \cdot \frac{d}{1} \cdot 1 \\ &= \left(\frac{|\gamma_k - \beta_k|}{|\gamma - \beta|} \cdot |Df^k(\alpha)| \right)^{1/2} d \\ &= \left(\frac{|\gamma_k - \beta_k|}{|\alpha_k - \beta_k|} \cdot \frac{|\alpha_k - \beta_k|}{|\alpha - \beta|} \cdot \frac{|\alpha - \beta|}{|\gamma - \beta|} |Df^k(\alpha)| \right)^{1/2} d \\ &\geq \left(d \cdot \frac{|\alpha_k - \beta_k|}{|\alpha - \beta|} \cdot w \cdot |Df^k(\alpha)| \right)^{1/2} \cdot d. \end{aligned}$$

We used Lemma 6, the assumptions of (2.1.1) and the definition of d . This gives by definition of w , Lemma 7 and $A(i)$

$$\frac{|\alpha_k - \beta_k|}{|\alpha - \beta|} \geq d^3 w |Df^k(\alpha)| \geq d^3 (K_A d^3)^{-1} K_T d^{s_i} \lambda_T^k = K_T d^{s_i} \lambda_T^k / K_A.$$

In order to end the estimation of $A(i+1)$ for $\bar{\xi}_{i+1}$ we observe that $|\alpha_n - \beta_n|/|\alpha_k - \beta_k| \geq K_A \lambda_T^{n-k}$ by Proposition 12 and that K_A will be simplified in the product of two quotients. $B(i+1)$ is obvious as in (1).

(2.1.2): we assume $n - k \leq N_1$, $k - m > N_1$, $|\alpha - \beta| > w|\alpha - \beta|$ and $|\gamma_k - \beta_k| \leq d|\alpha_k - \beta_k|$. We set $S_{i+1} = S_i$. $B(i+1)$ is obvious by $B(i)$ and $A(i+1)$ for $\bar{\xi}_{i+1}$ is already done. We estimate $A(i+1)$ for $\bar{\xi}_{i+1}$. Again we estimate only $|\alpha_k - \beta_k|/|\alpha - \beta|$ and then we use Proposition 12.

$$\begin{aligned} \frac{|\alpha_k - \beta_k|}{|\alpha - \beta|} &= \frac{|\gamma_k - \alpha_k|}{|\gamma - \alpha|} \cdot \frac{|\alpha_k - \beta_k|}{|\gamma_k - \alpha_k|} \cdot \frac{|\gamma - \alpha|}{|\alpha - \beta|} = \frac{|\gamma_k - \alpha_k|}{|\gamma_m - \alpha_m|} \cdot \frac{|\gamma_m - \alpha_m|}{|\gamma - \alpha|}, \\ \frac{|\alpha_k - \beta_k|}{|\alpha_k - \beta_k| + |\gamma_k - \beta_k|} \cdot \frac{|\gamma - \alpha|}{|\alpha - \beta|} &\geq (\lambda_A^{1/2})^{k-m} \cdot (K_T d^{s_{i-1}} \lambda_T^m) \cdot \frac{1}{1+d} \cdot 1 \\ &\geq \frac{2}{K_A} (\lambda_A^{1/4})^{k-m} \cdot K_T d^{s_i} \lambda_T^m \cdot \frac{1}{2} \geq K_T d^{s_i} \lambda_T^k / K_A. \end{aligned}$$

We used the fact that $k - m > N_1 > N_0$, Remark 13, the definition of N_1 , $A(i-1)$ with Lemma 7 for the interval $|\gamma - \alpha|$ and $S_{i-1} \leq S_i$. We conclude $A(i+1)$ for $\bar{\xi}_{i+1}$ as in (2.1.1).

(2.1.3): we assume $n - k \leq N_1, k - m > N_1, |\alpha - \beta| \leq w|\gamma - \beta|$ and $|\gamma - \beta| > w|\alpha - \beta|$.

We set $s_{i+1} = s_i$.

$$\frac{|\alpha_k - \beta_k|}{|\alpha - \beta|} = \frac{|\alpha_k - \gamma_k|}{|\alpha - \gamma|} \cdot \frac{|\alpha - \gamma|}{|\alpha - \beta|} \cdot \frac{|\alpha_k - \beta_k|}{|\alpha_k - \gamma_k|} \geq K_T \lambda_T^k d^{s_i} \cdot \frac{|\alpha - \beta| + |\beta - \gamma|}{|\alpha - \beta|} \cdot \frac{d}{1} \\ \geq K_T d^{s_i} \lambda_T^k \cdot (1 + w) \cdot d \geq K_T d^{s_i} \lambda_T^k / K_A.$$

We used $A(i)$ for the interval $|\alpha - \gamma|$, (2.1.3) and the definition of w and the definition of d . Now we proceed as in (2.1.1) and obtain $A(i+1)$ for $\bar{\xi}_{i+1}$.

(2.1.4): we assume $n - k \leq N_1, k - m > N_1, |\alpha - \beta| \leq w|\gamma - \beta|$ and $|\gamma - \beta| \leq w|\alpha - \beta|$.

For the first time we are obliged to set $s_{i+1} = s_i + 1$. $A(i+1)$ for $\bar{\xi}_{i+1}$ is obvious as now $d^{s_i} > d^{s_{i+1}}$ and we can use (1). We have to prove $A(i+1)$ for $\bar{\xi}_{i+1}$ and $B(i+1)$. We begin with $B(i+1)$. We have $|\alpha - \gamma| = |\alpha - \beta| + |\gamma - \beta|$ and by (2.1.4)

$$(**) \quad |\alpha - \gamma| \geq \left(1 + \frac{1}{w}\right) \max(|\alpha - \beta|; |\beta - \gamma|).$$

Hence

$$\max(|\alpha - \beta|; |\beta - \gamma|) \leq \frac{w}{1 + w} |\alpha - \gamma| \leq \frac{w}{1 + w} \left(\frac{w}{1 + w}\right)^{s_i} = \left(\frac{w}{1 + w}\right)^{s_{i+1}},$$

as we applied $B(i)$ to the interval (α, γ) . This proves $B(i+1)$. Now for $A(i+1)$ for $\bar{\xi}_{i+1}$ we estimate directly using (**).

$$\frac{|\alpha_n - \beta_n|}{|\alpha - \beta|} = \frac{|\alpha - \gamma|}{|\alpha - \beta|} \cdot \frac{|\alpha_k - \gamma_k|}{|\alpha - \gamma|} \cdot \frac{|\alpha_n - \beta_n|}{|\alpha_k - \gamma_k|} \geq \frac{w + 1}{w} \cdot \frac{|\alpha_k - \gamma_k|}{|\alpha - \gamma|} \cdot \frac{d}{1} \\ \geq \lambda_T^{2N_1} K_T d^{s_i} \lambda_T^k d \geq K_T d^{s_{i+1}} \lambda_T^n.$$

We used the definition of $\lambda_T, n - k \leq N_1, A(i)$ applied to the interval (α, γ) , the definition of d and the definition of s_{i+1} . This proves $A(i+1)$ for $\bar{\xi}_{i+1}$ and ends the case (2.1). Now we consider the case (2.2) $n - k \leq N_1$ and $k - m \leq N_1$.

(2.2.1) $n - k \leq N_1, k - m \leq N_1$ and $|\alpha - \beta| > w|\gamma - \beta|$. We set $S_{i+1} = S_i$. First we remark that $\beta_k \in f^{-(n-k)}(c)$, thus by definition of $d: |\gamma_k - \beta_k| \geq d > d|\alpha_k - \beta_k|$. Now we can prove $A(i+1)$ for $\bar{\xi}_{i+1}$ as in (2.1.1). As for $\bar{\xi}_{i+1}$ we have

$$\frac{|\gamma_n - \beta_n|}{|\gamma - \beta|} = \frac{|\gamma_n - \beta_n|}{|\gamma_m - \beta_m|} \cdot \frac{|\gamma_m - \beta_m|}{|\gamma_m - \beta_m|} \cdot \frac{|\gamma_k - \beta_k|}{|\gamma - \beta|} \geq K_A \lambda_T^{n-m} \cdot \frac{d}{1} \cdot \left(\frac{|\alpha_k - \beta_k|}{|\alpha - \beta|} \cdot d \cdot w\right) \\ \leq K_A \lambda_T^{n-m} d^2 K_T d^{s_i} \lambda_T^k (K_A d^3)^{-1} > K_T d^{s_{i+1}} \lambda_T^n.$$

We used Proposition 12, the definition of d , again (2.1.1) to estimate $|\alpha_k - \beta_k|/|\alpha - \beta|$ the inequalities of (2.2.1) then the definition of λ_T and w . $B(i+1)$ is obvious.

(2.2.2): we assume $n - k \leq N_1, k - m \leq N_1$ and $|\alpha - \beta| \leq |\gamma - \beta|w$ and $|\gamma - \beta| > w|\alpha - \beta|$. Similarly as in (2.2.1) we have $|\alpha_m - \beta_m| > d|\beta_m - \gamma_m|$. We set $s_{i+1} = s_i$ and prove $A(i+1)$ for $\bar{\xi}_{i+1}$ as we proved $A(i+1)$ for $\bar{\xi}_{i+1}$ in (2.1.1). The proof of $A(i+1)$ for $\bar{\xi}_{i+1}$ goes as in (2.1.3).

(2.2.3): we assume $n - k \leq N_1, k - m \leq N_1$ and $|\alpha - \beta| \leq |\gamma - \beta|w$ and $|\gamma - \beta| \leq |\alpha - \beta|w$. We set for the second time $s_{i+1} = s_i + 1$ and for both $\bar{\xi}_{i+1}$ and $\bar{\xi}_{i+1}$ we repeat the proof of (2.1.4) for $\bar{\xi}_{i+1}$. This ends the proof in the case (2.2) and hence the proof of Proposition 17. □

5. Consequences

PROPOSITION 18. Under the assumptions of Proposition 17 there exists a constant $\lambda_N > 1$ (independent of x and i) such that for all $x \in (0, 1) \setminus \mathbb{C}_{-\infty}$ and every i

$$\text{if } \xi_i \in \xi(x) \text{ then } |\xi_i| < K\lambda_N^{-n_i},$$

where $K > 0$ is a constant independent of i .†

Consider

$$\frac{1}{n_i} \log \frac{|f^{n_i}(\xi_i)|}{|\xi_i|}$$

by Proposition 17, $A(i)$ we have

$$\frac{1}{n_i} \log \frac{|f^{n_i}(\xi_i)|}{|\xi_i|} \geq \frac{1}{n_i} \log (K_T d^{s_i} \lambda_T^{n_i}).$$

We have two possibilities: either

$$\frac{1}{n_i} \log (K_T d^{s_i} \lambda_T^{n_i}) \geq (\log \lambda_T)/2$$

and in this case

$$|\xi_i| \leq |f^{n_i}(\xi_i)| (\lambda_T^{1/2})^{-n_i} < (\lambda_T^{1/2})^{-n_i}$$

or

$$\frac{1}{n_i} \log (K_T d^{s_i} \lambda_T^{n_i}) < (\log \lambda_T)/2,$$

and then

$$\frac{\log K_T}{n_i} + \frac{s_i}{n_i} \log d + \frac{1}{2} \log \lambda_T < 0.$$

We take i_1 such that

$$\frac{\log K_T}{n_i} + \frac{1}{4} \log \lambda_T \geq 0$$

for $i \geq i_1$ and hence for $i \geq i_1$ we have

$$\frac{s_i}{n_i} \log d + \frac{1}{4} \log \lambda_T < 0 \quad \text{and} \quad \frac{s_i}{n_i} > \kappa = -\frac{\log \lambda_T}{4 \log d} > 0.$$

But in this case we have by $B(i)$

$$|\xi_i| = |\bar{\xi}_i| + |\xi_i| \leq 2 \left(\frac{w}{1+w} \right)^{s_i} < 2 \left(\frac{w}{1+w} \right)^{\kappa n_i}.$$

In order to finish the proof we set

$$\lambda_N = \min \left(\lambda_T^{1/2}, \left(\frac{1}{1+w} \right)^{-\kappa} \right) \geq 1.$$

The constant K is adjusted according to i_1 . □

† E. Vizinescu pointed out that in fact K does not depend on x .

LEMMA 19. Suppose that f satisfies A0–A3 and there is $\lambda_N > 1$ such that for all x there is a constant $K(x)$ such that for all i :

$$\text{if } \xi_i \in \xi(x) \text{ then } |\xi_i| < K(x)\lambda_N^{-n}.$$

Then f has uniform hyperbolic structure on the set of periodic points: i.e. if $p = p_s$ then $|Df^s(p)| \geq \lambda_N^s$.

Proof. Suppose that Lemma 19 is not true. Then there exists a periodic point p of period S , $p_s = p$, such that $|Df^s(p)| = \lambda^s < \lambda_N^s$. we may assume $Df^s(p) > 0$, otherwise we take p and $2s$. Take $\alpha, \beta \in \bigcup_{0 \leq r \leq s} f^{-r}(c)$ such that $p \in (\alpha, \beta)$ and there is no smaller interval of such a property. We construct the intervals $(\alpha^j; \beta^j)$ by $\alpha = \alpha, \beta^1 = \beta$ and $\alpha^{j+1} = f^{-s}(\alpha^j) \cap (\alpha^j; p)$ and $\beta^{j+1} = f^{-s}(\beta^j) \cap (p, \beta^j)$. By virtue of Lemmas 1, 2 and $Df^s(p) > 0$ such interval exists for every j . By Lemma 1 $Df^s(p) > Df^s(x)$ either for all $x \in (\alpha^1, p)$ or for all $x \in (p, \beta^1)$. Suppose the first possibility holds (the other case is similar). Then for every j $Df^{js}(p) > Df^{js}(y)$ for all $y \in (\alpha^j, p)$. In order to prove it we decompose

$$Df^{js}(y) = \prod_{r=0}^{j-1} Df^s(y_{rs})$$

and as by construction of α^j for every $0 < r < j$

$$y_{rs} \in (\alpha^{j-r}, p) \subset (\alpha^1, p)$$

we have $Df^{js}(y) < (Df^s(p))^j = Df^{sj}(p)$. Hence

$$\lambda^{sj} = Df^{sj}(p) > \frac{|f^{sj}(\alpha^j, p)|}{|(\alpha^j; p)|} = \frac{|f^s(\alpha) - p|}{|\alpha^j - p|}.$$

Now consider $\xi(p)$. We have for all $j(\alpha^j, \beta^j) \in \xi(p)$ with $(\alpha^j; \beta^j) = \xi_{i(j)}$ and obviously $|s \cdot j - \eta_{i(j)}| \leq s$. Hence by the assumption

$$\lambda^{sj} > \frac{|f^s(\alpha) - p|}{K(p) \cdot L^s} \lambda_N^{sj}, \text{ where } L = \sup_{x \in (0,1)} |f'(x)|.$$

The factor $|f^s(\alpha) - p|/K(p)L^s$ does not depend on j so the last inequality contradicts $\lambda < \lambda_N$. This proves Lemma 19. □

COROLLARY 20. f satisfies C2.

Proof. By Proposition 3.9, [N2]. For S -unimodal functions C2 is equivalent to uniform hyperbolic structure of the set of periodic points. □

COROLLARY 21. There exist a constant $\lambda > 1$ such that for every n and every $\xi \in \Delta_n$, $|\xi| < \lambda^{-n}$.

Proof. This follows from Proposition 5.2 [N2] which states that C1 and C2 imply uniform exponential diminution of length of the intervals of monotonicity. □

COROLLARY 22. There are constants $K > 0$ and $\lambda > 1$ such that for every $x \in (0; 1)C_{-\infty}$ and every i if $\xi_i \in \xi(x)$ then $|f^{n_i}(\xi_i)|/|\xi_i| > K\lambda^{n_i}$.

Proof. One has to remake the proof of Proposition 17 and everytime when $A(i)$ has to be spoiled (i.e. $s_{i+1} = s_i + 1$) by adding a factor d one apply directly Corollary 21 as then $|f^{n_i}(\xi_i)| \geq d$. □

6. Final remarks

The last three corollaries show that there are no differences between hyperbolic properties of S -unimodal functions satisfying $C1$ depending on symmetry. One would like to show this through a diffeomorphic conjugation of a unimodal function f with symmetrical unimodal function g (with $g = h \circ f \circ h^{-1}$, where $h(\hat{x}) = (1 + x - x)/2$), but the S property may not be conserved.

Proposition 17 does not need $C1$ but only the assertion of Proposition 12, which is true for example for f with uniform hyperbolic structure on periodic points and a weak assumption: $|Df^k(c_1)| > K\lambda_N^{-n/2} n = 1, 2, \dots$ [N2, 3]. It is tempting to prove Proposition 12 only with $C2$.

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