

THE EXPONENTIAL REPRESENTATION OF HOLOMORPHIC FUNCTIONS OF UNIFORMLY BOUNDED TYPE

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Abstract

It is shown that if E, F are Fréchet spaces, $E \in (H_{ub}), F \in (DN)$ then $H(E, F) = H_{ub}(E, F)$ holds. Using this result we prove that a Fréchet space E is nuclear and has the property (H_{ub}) if and only if every entire function on E with values in a Fréchet space $F \in (DN)$ can be represented in the exponential form. Moreover, it is also shown that if $H(F^*)$ has a LAERS and $E \in (H_{ub})$ then $H(E \times F^*)$ has a LAERS, where E, F are nuclear Fréchet spaces, F^* has an absolute basis, and conversely, if $H(E \times F^*)$ has a LAERS and $F \in (DN)$ then $E \in (H_{ub})$.

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1. Introduction

Let E and F be locally convex spaces. A holomorphic function f from E to F is said to be of *uniformly bounded type* if there exists a neighbourhood U of $0 \in E$ such that $f(rU)$ is bounded for all $r > 0$. By $H_{ub}(E, F)$ we denote the linear subspace of the space of holomorphic functions from E to F , consisting of all functions of uniformly bounded type. We write $H_{ub}(E)$ rather than $H_{ub}(E, \mathbb{C})$. We say that a locally convex space E has the *property* (H_{ub}) (and write $E \in (H_{ub})$) if the identity $H(E) = H_{ub}(E)$ holds.

Recently Le Mau Hai and Thai Thuan Quang [3] have shown that $H_b(E, F) = H_{ub}(E, F)$ for Fréchet spaces E, F and $E \in (H_{ub}), F \in (\overline{DN})$.

In Section 3, we extend the above result to the more general case. Namely, the property (\overline{DN}) of the space F is replaced by the property (DN) (Theorem 3.1).

By using Theorem 3.1, in Section 4 we prove that a Fréchet space E is nuclear and $E \in (H_{ub})$ if and only if every entire function on E with values in a Fréchet space $F \in (DN)$ can be represented in the exponential form (Theorem 4.1). The final portion of this section will deal with the exponential representation of holomorphic functions on $E \times F^*$, where E is Fréchet, $E \in (H_{ub})$ and F is nuclear Fréchet such that F^* has an absolute basis and $H(F^*)$ has a linearly absolutely exponential representation system (Theorem 4.2). The proof is based on Theorems 3.1, 4.1 and Proposition 4.3 which say that if F is a Fréchet space such that F^* has an absolute basis then $F \in (DN)$ if and only if $H(F^*) \in (DN)$.

2. Preliminaries

We may frequently use the standard notation of the theory of locally convex spaces as presented in the book of Pietsch [9]. A locally convex space always is a complex vector space with a locally convex Hausdorff topology.

For a Fréchet space E , we always assume that its locally convex structure is generated by an increasing system $\{\|\cdot\|_k\}$ of semi-norms. Then we denote by E_k the completion of the canonically normed space $E/\ker\|\cdot\|_k$ and $\omega_k : E \rightarrow E_k$ denotes the canonical map and U_k denotes the set $\{x \in E : \|x\|_k < 1\}$.

2.1. Holomorphic function Let E and F be locally convex spaces and let $D \subset E$ be open, $D \neq \emptyset$. A function $f : D \rightarrow F$ is called *holomorphic* if f is continuous and Gâteaux-analytic. By $H(D, F)$ we denote the vector space of all holomorphic functions on D with values in F . We use $H_b(E, F)$ to denote the space of holomorphic functions from E to F which are bounded on every bounded set in E . The space $H_b(E, F)$ is equipped with the topology τ_b of uniform convergence on all bounded sets.

2.2. The property (DN) Let E be a Fréchet space with a fundamental system of semi-norms $\{\|\cdot\|_k\}$. We say that E has the *property (DN)* if

$$\exists p \exists d \forall q \exists k, C > 0 \text{ such that } \|\cdot\|_q^{1+d} \leq C \|\cdot\|_k \|\cdot\|_p^d.$$

2.3. Separately holomorphic function Let E, F be locally convex spaces. For a function $f : E \times F \rightarrow \mathbb{C}$, we put

$$\begin{aligned} f_x(y) &= f(x, y) \quad \text{for } y \in F; \\ f_y(x) &= f(x, y) \quad \text{for } x \in E. \end{aligned}$$

The function f is called *separately holomorphic* if $f_x : F \rightarrow \mathbb{C}$ and $f_y : E \rightarrow \mathbb{C}$ are holomorphic for all $x \in E$ and $y \in F$ respectively.

2.4. Linearly absolutely exponential representation system Let E be a locally convex space and $\{x_k\}$ be a sequence in E . We say that $\{x_k\}$ is a *linearly absolute representation system* (abbreviated LARS) if every element $x \in E$ can be written in the form $x = \sum_{k \geq 1} \xi_k(x)x_k$, where the series is absolutely convergent.

A LARS in $H(D)$ of the form $\{\exp u_k\}$, where u_k are continuous linear functionals on E and D is an open set in E , is said to be a linearly absolutely exponential representation system of $H(D)$. It is denoted by LAERS.

3. Holomorphic functions of uniformly bounded type

In this section we prove the following theorem which was proved in [3] by Le Mau Hai and Thai Thuan Quang in the case when $F \in (\overline{DN})$.

THEOREM 3.1. *Let E be a Fréchet space. Then $E \in (H_{ub})$ if and only if*

$$H(E, F) = H_{ub}(E, F)$$

for all Fréchet spaces $F \in (DN)$.

PROOF. 1. *Necessary.* Since $F \in (DN)$, by Vogt [13] F can be considered as a subspace of the space $B\widehat{\otimes}_\pi s$ for some Banach space, where s denotes the space of rapidly decreasing sequences.

On the other hand, $B\widehat{\otimes}_\pi s$ is a subspace of $H_b((B\widehat{\otimes}_\pi s)'_b)$ and $H_b((B\widehat{\otimes}_\pi s)'_b) \cong H_b(B'\widehat{\otimes}_\pi s')$.

Given $f \in H(E, F) \subset H(E, H_b(B'\widehat{\otimes}_\pi s'))$, we write the Taylor expansion of $f(z)$ at $0 \in B'\widehat{\otimes}_\pi s'$ for $z \in E$

$$f(z)(t) = \sum_{n \geq 0} \sum_{\substack{j_1, \dots, j_n \geq 1 \\ k_1, \dots, k_n \geq 1}} \widehat{P_n f}(z)(u_{k_1} \otimes e_{j_1}^*, \dots, u_{k_n} \otimes e_{j_n}^*),$$

where

$$t = \sum_{k \geq 1} u_k \otimes v_k \in B'\widehat{\otimes}_\pi s', \quad P_n f(z)(t) = \frac{1}{2\pi i} \int_{|q|=r>0} \frac{f(z)(qt)}{q^{n+1}} dQ,$$

and $\{e_j\}$ is the canonical basis of s with the dual basis $\{e_j^*\}$, $\widehat{P_n f}$ is the symmetric n -linear form associated to $P_n f$.

Since $\{e_j\}_{j \geq 1}$ is an absolute basis, for $p \geq 1$, choose $q \geq p$ such that

$$\sum_{j \geq 1} \|e_j^*\|_q^* \|e_j\|_p < \frac{1}{(p+1)e^2},$$

where $\|e_j^*\|_q^* = \sup\{|e_j^*(x)|, \|x\|_q \leq 1\} = 1/j^q$.

For each $p \geq 1$, consider a family $\mathcal{F}_p = \{f_{p,n,u_1,\dots,u_n}\}_{n \geq 0} \subset H_b(E)$ given by

$$f_{p,n,u_1,\dots,u_n}(z) = \sum_{j_1,\dots,j_n \geq 1} p^n \widehat{P_n f}(z)(u_1 \otimes e_{j_1}^*, \dots, u_n \otimes e_{j_n}^*) \|e_{j_1}\|_p \cdots \|e_{j_n}\|_p$$

where $u_1, \dots, u_n \in W$, the unit ball of B' .

Then for each $p \geq 1$, the family \mathcal{F}_p is bounded in $H_b(E)$. Indeed, for every bounded set K in E , we have

$$\begin{aligned} & \sup_{z \in K} \sup_{\substack{u_1, \dots, u_n \in W \\ n \geq 0}} \left\{ p^n \sum_{j_1, \dots, j_n \geq 1} \left| \widehat{P_n f}(z)(u_1 \otimes e_{j_1}^*, \dots, u_n \otimes e_{j_n}^*) \right| \|e_{j_1}\|_p \cdots \|e_{j_n}\|_p \right\} \\ &= \sup_{z \in K} \sup_{\substack{n \geq 0 \\ u_1, \dots, u_n \in W}} \left\{ p^n \sum_{j_1, \dots, j_n \geq 1} \left| \widehat{P_n f}(z) \left(u_1 \otimes \frac{e_{j_1}^*}{\|e_{j_1}^*\|_q^*}, \dots, u_n \otimes \frac{e_{j_n}^*}{\|e_{j_n}^*\|_q^*} \right) \right| \right. \\ & \quad \left. \times \|e_{j_1}\|_p \|e_{j_1}^*\|_q^* \cdots \|e_{j_n}\|_p \|e_{j_n}^*\|_q^* \right\} \\ &\leq \sup_{z \in K} \sup_{t \in \text{conv}(W \otimes U_q^o)} |f(z)(t)| \left\{ \sup_{n \geq 0} \frac{n^n p^n}{n!} \sum_{j_1, \dots, j_n \geq 1} \|e_{j_1}\|_p \|e_{j_1}^*\|_q^* \cdots \|e_{j_n}\|_p \|e_{j_n}^*\|_q^* \right\} \\ &\leq \|f\|_{K \times \text{conv}(W \otimes U_q^o)} \sup_{n \geq 0} \left\{ \left(\frac{np}{(p+1)e^2} \right)^n \frac{1}{n!} \right\} \\ &\leq C_p \|f\|_{K \times \text{conv}(W \otimes U_q^o)}, \end{aligned}$$

where

$$U_q = \{x \in s : \|x\|_q < 1\} \quad \text{with the polar } U_q^o,$$

$$C_p = \sup_{n \geq 0} \left\{ \left(\frac{np}{(p+1)e^2} \right)^n \frac{1}{n!} \right\}.$$

Since $E \in (H_{ub})$, by Meise and Vogt [5], there exists $\alpha \geq 1$ such that the family \mathcal{F}_p is bounded in $H_b(E_\alpha)$.

However, for every bounded set $K \subset E_\alpha, p \geq 1$, we have the following estimate

$$\begin{aligned} (3.1) \quad & \sup_{z \in K} \sup_{t \in \text{conv}(W \otimes U_q^o)} \sum_{n \geq 0} |P_n f(z)(t)| \\ & \leq \sup_{z \in K} \sup_{\substack{u_{k_1}, \dots, u_{k_n} \in W \\ \sum_{k \geq 1} |\lambda_k| \leq 1}} \left\{ \sum_{n \geq 0} \frac{1}{p^n} \sum_{k_1, \dots, k_n \geq 1} |\lambda_{k_1}| \cdots |\lambda_{k_n}| \right. \\ & \quad \left. \times \sum_{j_1, \dots, j_n \geq 1} p^n \left| \widehat{P_n f}(z)(u_{k_1} \otimes e_{j_1}^*, \dots, u_{k_n} \otimes e_{j_n}^*) \right| \|e_{j_1}\|_p \cdots \|e_{j_n}\|_p \right\} \end{aligned}$$

$$\begin{aligned} &\leq \sup_{z \in K} \sup_{\substack{u_1, \dots, u_n \in W \\ n \geq 0}} \left\{ p^n \sum_{j_1, \dots, j_n \geq 1} \left| \widehat{P_n f}(z)(u_1 \otimes e_{j_1}^*, \dots, u_n \otimes e_{j_n}^*) \right| \right. \\ &\quad \left. \times \|e_{j_1}\|_p \cdots \|e_{j_n}\|_p \right\} \sup_{\sum_{k \geq 1} |\lambda_k| \leq 1} \left\{ \sum_{n \geq 0} \frac{1}{p^n} \sum_{k_1, \dots, k_n \geq 1} |\lambda_{k_1}| \cdots |\lambda_{k_n}| \right\} \\ &\leq C_K \sum_{n \geq 0} \frac{1}{p^n} \end{aligned}$$

where

$$\begin{aligned} C_K = \sup_{z \in K} \sup_{\substack{u_1, \dots, u_n \in W \\ n \geq 0}} \left\{ p^n \sum_{j_1, \dots, j_n \geq 1} \left| \widehat{P_n f}(z)(u_{k_1} \otimes e_{j_1}^*, \dots, u_{k_n} \otimes e_{j_n}^*) \right| \right. \\ \left. \times \|e_{j_1}\|_p \cdots \|e_{j_n}\|_p \right\} < \infty. \end{aligned}$$

Put

$$(3.2) \quad g(z, t) = \sum_{n \geq 0} P_n f(z)(t)$$

with $z \in E_\alpha, t \in (B \widehat{\otimes}_\pi s)'$.

From (3.1), it follows that the right-hand side of (3.2) converges and defines a separately holomorphic function on $E_\alpha \times (B \widehat{\otimes}_\pi s)'$.

It is easy to see that g is bounded on every bounded set on $(B \widehat{\otimes}_\pi s)'$. By Galindo, Garcia, Maestre [1] the holomorphic function of bounded type

$$\widehat{g} : (B \widehat{\otimes}_\pi s)' \rightarrow H_b(E_\alpha)$$

which is induced by g , can be factorized through a Banach space by an entire function of bounded type. Because every ball in a Banach space is bounded we infer $f \in H_{ub}(E, F)$.

2. *Sufficient.* In the case $F = \mathbb{C}$, by hypothesis we obtain $E \in (H_{ub})$.

The theorem is proved. □

4. The exponential representation

First we recall that a locally convex space E has the property (H_u) and write $E \in (H_u)$ if every holomorphic function f on E is of uniform type. This means that there exists a continuous semi-norm ϱ on E such that f can be factorized holomorphically through the canonical map $\omega_\varrho : E \rightarrow E_\varrho$, where E_ϱ denote the space associated to ϱ .

In [7] Nguyen Minh Ha and Nguyen Van Khue proved that a Fréchet space E is nuclear and $E \in (H_u)$ if and only if every holomorphic function on E with values in a Banach space B can be written in the form $f(x) = \sum_{k \geq 1} \xi_k \exp u_k(x)$, where the series is absolutely convergent in the space $H(E, B)$ of holomorphic functions on E with values in B equipped with the compact-open topology.

In this section we shall consider the above result in another situation with the note that if $E \in (H_{ub})$ then $E \in (H_u)$ and if F is Banach then $F \in (DN)$. Namely, we are going to prove the following:

THEOREM 4.1. *Let E be a Fréchet space. Then E is nuclear and $E \in (H_{ub})$ if and only if every holomorphic function on E with values in a Fréchet space $F \in (DN)$ can be written in the form $f(x) = \sum_{k \geq 1} \xi_k \exp u_k(x)$, where the sequences $(\xi_k) \subset F$, $(u_k) \subset E^*$, the dual space of E , and the series is absolutely convergent in the space $H_b(E, F)$.*

PROOF. First we prove sufficiency of the theorem.

Let $\{p_\alpha\}$ be a fundamental system of semi-norms on E . To prove the nuclearity of E , for every continuous semi-norm ϱ on E write the canonical map $\omega_\varrho : E \rightarrow E_\varrho$ in the form $\omega_\varrho(x) = \sum_{k \geq 1} \xi_k \exp u_k(x)$ in which $\sum_{k \geq 1} \|\xi_k\| \exp \|u_k\|_B^* < \infty$ for every bounded set B in E . This follows from the hypothesis and the property (DN) of the space Banach E_ϱ . Then

$$\omega_\varrho(x) = d\omega_\varrho(0)(x) = \sum_{k \geq 1} \xi_k u_k(x)$$

for $x \in E$ and $\sum_{k \geq 1} \|\xi_k\| \|u_k\|_B^* < \infty$ for every bounded set B in E .

Now we prove that there exists a continuous semi-norm $\beta > \varrho$ in E such that

$$\sum_{k \geq 1} \|\xi_k\| \exp \|u_k\|_\beta^* < \infty.$$

Indeed, if this does not hold, for every α we have $\sum_{k \geq 1} \|\xi_k\| \exp \|u_k\|_\alpha^* = \infty$. Hence for every α there exists k_α such that $\sum_{k \leq k_\alpha} \|\xi_k\| \exp \|u_k\|_\alpha^* > \alpha$. This inequality implies that for each $k \leq k_\alpha$ there exists x_k^α with $\|x_k^\alpha\|_\alpha \leq 1$ such that

$$\sum_{k \leq k_\alpha} \|\xi_k\| \exp |u_k(x_k^\alpha)| > \alpha.$$

Put $B = \{x_1^1, \dots, x_{k_1}^1, \dots, x_1^\alpha, \dots, x_{k_\alpha}^\alpha, \dots\} \cup \{0\}$. Then B is bounded in E and

$$\sum_{k \geq 1} \|\xi_k\| \exp \|u_k\|_B^* > \alpha \quad \text{for every } \alpha \geq 1.$$

This is impossible, because $\sum_{k \geq 1} \|\xi_k\| \exp \|u_k\|_{\beta}^* < \infty$.

By the same argument as above, there exists a continuous semi-norm $\beta > \rho$ in E such that $\sum_{k \geq 1} \|\xi_k\| \|u_k\|_{\beta}^* < \infty$. This means that the canonical map $\omega_{\beta\rho} : E_{\beta} \rightarrow E_{\rho}$ is nuclear. Hence E is nuclear.

Now, since E is nuclear, to prove $E \in (H_{ub})$ by [4] it suffices to show that if E is a topological subspace of a locally convex space G with a fundamental system of continuous semi-norm induced by semi-inner products then every $f \in H(E)$ has an extension $g \in H(G)$.

Given $f \in H(E, \mathbb{C}) = H(E)$, by the hypothesis, we can write

$$f(x) = \sum_{k \geq 1} \xi_k \exp u_k(x)$$

such that $\sum_{k \geq 1} |\xi_k| \exp |u_k(x)| < \infty$. Applying the Hahn-Banach theorem to $u_k \in E^*$, $k \geq 1$, there exist $\hat{u}_k \in G^*$ such that $\hat{u}_k|_E = u_k$ and $\|\hat{u}_k\|^* = \|u_k\|^*$, for all $k \geq 1$. Then the function $g(x) = \sum_{k \geq 1} \xi_k \exp \hat{u}_k(x)$ defines a holomorphic function on G and $g|_E = f$.

Now, assume that E is nuclear and $E \in (H_{ub})$. By Theorem 3.1, we have $H(E, F) = H_{ub}(E, F)$. Then every $f \in H(E, F)$ is of uniform type. It implies that there exists a continuous semi-norm ρ on E and a holomorphic function g on E_{ρ} such that $f = g\omega_{\rho}$. Take a continuous semi-norm $\beta > \rho$ on E such that $T = \omega_{\beta\rho}$ is nuclear. Write

$$T(x) = \sum_{j \geq 1} t_j u_j(x) e_j$$

with $a = \sum_{j \geq 1} |t_j| < \infty$ and $\|u_j^*\| < 1$, $\|e_j\| < 1$, $e_j \in E_{\rho}$, $u_j \in E_{\beta}^*$ for $j \geq 1$. Consider the Taylor expansion of g at $0 \in E$

$$g(x) = \sum_{n \geq 0} P_n g(x),$$

where

$$P_n g(x) = \frac{1}{2\pi i} \int_{|t|=r} \frac{g(tx)}{t^{n+1}} dt.$$

Choose two sequences $\{\xi_k\}$ and $\{\alpha_k\}$ in \mathbb{C} such that

$$(4.1) \quad z = \sum_{k \geq 1} \xi_k \exp(\alpha_k z)$$

for $z \in \mathbb{C}$ and

$$(4.2) \quad C_r = \sum_{k \geq 1} |\xi_k| \exp(r|\alpha_k|) < \infty$$

for all $r > 0$. Such sequences exist by [2]. Formally, we have

$$\begin{aligned}
 (gT)(x) &= g(Tx) = \sum_{ng \in 0} P_n g(Tx) = \sum_{n \geq 0} P_n g \left(\sum_{j \geq 1} t_j u_j(x) e_j \right) \\
 &= \sum_{n \geq 0} \sum_{j_1, \dots, j_n \geq 1} t_{j_1} \cdots t_{j_n} u_{j_1}(x) \cdots u_{j_n}(x) \widehat{P}_n g(e_{j_1}, \dots, e_{j_n}) \\
 &= \sum_{n \geq 0} \sum_{j_1, \dots, j_n \geq 1} t_{j_1} \cdots t_{j_n} \widehat{P}_n g(e_{j_1}, \dots, e_{j_n}) \\
 &\quad \times \left(\sum_{k \geq 1} \xi_k \exp \alpha_k u_{j_1}(x) \right) \cdots \left(\sum_{k \geq 1} \xi_k \exp \alpha_k u_{j_n}(x) \right) \\
 &= \sum_{n \geq 0} \sum_{\substack{j_1, \dots, j_n \geq 1 \\ k_1, \dots, k_n \geq 1}} t_{j_1} \cdots t_{j_n} \xi_{k_1} \cdots \xi_{k_n} \widehat{P}_n g(e_{j_1}, \dots, e_{j_n}) \\
 &\quad \times \exp[\alpha_{k_1} u_{j_1}(x) + \cdots + \alpha_{k_n} u_{j_n}(x)]
 \end{aligned}$$

where $\widehat{P}_n g$ is the symmetric n -linear form associated to $P_n g$.

It remains to check that the right-hand side is absolutely convergent in $H(E, F)$.

For each $r > 0$, take $s > C_r a e$. Since

$$\|P_n g(e_{j_1}, \dots, e_{j_n})\|_q \leq \left(\frac{n^n}{n! s^n} \right) \|g\|_{s,q}$$

where

$$\|g\|_{s,q} = \sup\{\|g(x)\|_q : \|x\| < s\}$$

and without loss generality by the nuclearity of E , we may assume that g is bounded on every bounded set in E_ρ . We have

$$\begin{aligned}
 \|g(Tx)\|_q &\leq \sum_{n \geq 0} \sum_{\substack{j_1, \dots, j_n \geq 1 \\ k_1, \dots, k_n \geq 1}} |t_{j_1}| \cdots |t_{j_n}| |\xi_{k_1}| \cdots |\xi_{k_n}| \\
 &\quad \times \|\widehat{P}_n g(e_{j_1}, \dots, e_{j_n})\|_q \exp[r(|\alpha_{k_1}| + \cdots + |\alpha_{k_n}|)] \\
 &\leq \|g\|_{s,q} \sum_{n \geq 0} \frac{C_r^n a^n n^n}{n! s^n} < \infty
 \end{aligned}$$

for $\|x\| < r$.

The theorem is proved. □

In order to complete this section we will prove the following:

THEOREM 4.2. *Let E and F be nuclear Fréchet spaces such that F^* has an absolute basis. Then*

- (i) $H(E \times F^*)$ has a LAERS if $H(F^*)$ has a LAERS and $E \in (H_{ub})$;
- (ii) conversely, if $H(E \times F^*)$ has a LAERS and $F \in (DN)$ then $E \in (H_{ub})$.

The proof is based on Theorem 3.1 and the following:

PROPOSITION 4.3. *Let F be a Fréchet space such that F^* has an absolute basis. Then $F \in (DN)$ if and only if $H(F^*) \in (DN)$.*

PROOF. Sufficiency is obvious because F can be considered as a subspace of $H(F^*)$.

The proof of necessary condition is based on the results of Ryan [10] which introduces a convenient system of semi-norms defining the topology of $H(F^*)$.

Assume that $F \in (DN)$ such that F^* has an absolute basis $\{e_j^*\}$. By the open mapping theorem, the topology of F can be defined by the system of semi-norms

$$\|x\|_k = \sum_{j \geq 1} |e_j^*(x)| \|e_j\|_k,$$

where $\{e_j\}_j$ is the sequence of coefficient functionals associated to $\{e_j^*\}$. Choose $p \geq 1$ such that (DN) holds. Since $\|\cdot\|_p$ is a norm, we have $\|e_j\|_p \neq 0$, for $j \geq 1$. Hence

$$\|e_j^*\|_q^* \leq \|e_j^*\|_p^* \leq \frac{1}{\|e_j\|_p} < \infty, \quad \forall j \geq 1, \forall q \geq p.$$

Moreover, by the definition of (DN) and by the equality

$$\|e_j^*\|_q^* = \frac{1}{\|e_j\|_q}, \quad \forall j \geq 1,$$

there exists d such that for every q there exist $k, C > 0$ such that

$$(4.3) \quad \|e_j^*\|_q^{*1+d} \geq C \|e_j^*\|_k^* \|e_j^*\|_p^{*d} \quad \forall j \geq 1.$$

For each $k \geq 1$, put

$$F^*(k) = \left\{ x^* \in F^* : \|x^*\|_k^* = \sum_{j \geq 1} |e_j^*(x^*)| \|e_j^*\|_k^* < \infty \right\}.$$

It is easy to check that $F^*(k)$ is a Banach space and $\{e_j^*\}$ is also an absolute basis for $F^*(k)$. On the other hand, since $F^* = \bigcup_{k=1}^\infty F^*(k)$ and every bounded set in F^* is contained and bounded in some $F^*(k)$, we conclude that the topology of $H(F^*)$ can be defined by the system of semi-norms $\{\|\cdot\|_{k,r} : k \geq 1, r > 0\}$, where

$$\|f\|_{k,r} = \sup\{\|f(x^*)\| : \|x^*\|_k \leq r\}.$$

By applying results of Ryan [10], we obtain, for every $f \in H(F^*)$, the representation

$$f(x^*) = f\left(\sum_{j \geq 1} t_j e_j^*\right) = \sum_M b_m(f) r^m$$

which converges absolutely and uniformly on every bounded set in F^* , where

$$M = \{(m_1, m_2, \dots, m_n, 0, \dots), m_i \in \mathbb{N}, i = 1, 2, \dots\},$$

$$b_m(f) = \left(\frac{1}{2\pi i}\right)^n \int_{|t_1|=\varrho_1} \dots \int_{|t_n|=\varrho_n} \frac{f(t_1 e_1^* + \dots + t_n e_n^*)}{t_1^{m_1+1} \dots t_n^{m_n+1}} dt$$

with $dt = dt_1 \dots dt_n$.

Since for every $k \geq p$ the map

$$\varphi_k : \ell^1 \rightarrow F^*(k)$$

$$(\xi_j) \mapsto \sum_{j \geq 1} \xi_j \frac{e_j^*}{\|e_j^*\|_k^*}$$

is an isomorphism, again by Ryan [10], the system $\{\|\cdot\|_{k,r} : k \geq p, r > 0\}$ is equivalent to the system of semi-norms $\{\|\cdot\|_{k,r} : k \geq p, r > 0\}$ where

$$\|f\|_{k,r} = \sup \left\{ \frac{r^{|m|} |b_m(f)| m^m}{a_{\cdot,k}^m |m|^{|m|}}, m \in M \right\}$$

and $|m| = m_1 + \dots + m_n$; $a_{\cdot,k}^m = \|e_1^*\|_k^{*m_1} \dots \|e_n^*\|_k^{*m_n}$, $m^m = m_1^{m_1} \dots m_n^{m_n}$.

From (4.3) we get

$$\begin{aligned} \|\cdot\|_{q,r}^{1+d} &\leq \sup \left\{ \left[\frac{r^{|m|} |b_m(\cdot)| m^m}{a_{\cdot,q}^m |m|^{|m|}} \right]^{1+d}, m \in M \right\} \\ &\leq \sup \left\{ \left(\frac{r^{1+d}}{C} \right)^{|m|} \frac{|b_m(\cdot)| m^m}{a_{\cdot,k}^m |m|^{|m|}}, m \in M \right\} \sup \left\{ \left[\frac{|b_m(\cdot)| m^m}{a_{\cdot,p}^m |m|^{|m|}} \right]^d, m \in M \right\} \\ &= \|\cdot\|_{k,r^{1+d}/C} \cdot \|\cdot\|_{p,1}^d \end{aligned}$$

Hence $H(F^*) \in (DN)$. The proposition is proved. □

Now we prove Theorem 4.2.

(i) First note that F is reflexive, that is, $F^{**} = F$, because it is a Fréchet nuclear space. Since F is nuclear and $H(F^*)$ has a LAERS, by [7], $F \in (DN)$. According to Proposition 4.3, we have $H(F^*) \in (DN)$. Because E is nuclear and $H(E \times F^*) \cong H(E, H(F^*))$, it follows from Theorem 4.1 that every $f \in H(E, H(F^*))$ can be

written in the form $f(x) = \sum_{k \geq 1} \xi_k \exp u_k(x)$, where $(\xi_k) \subset H(F^*)$ and the series is absolutely convergent in the $H(E, H(F^*))$.

Moreover, since $H(F^*)$ has a LAERS, every $\xi_k \in H(F^*)$ can be also written in the form

$$\xi_k(y^*) = \sum_{j \geq 1} \eta_{j,k} \exp v_j(y^*), \quad \forall k \geq 1,$$

where $(\eta_{j,k}) \subset \mathbb{C}$ and $v_j \in F^{**} = F, j \geq 1$ and the series is absolutely convergent in the $H(F^*)$. Thus

$$\begin{aligned} f(x, y^*) &= \sum_{k,j \geq 1} \eta_{j,k} \exp v_j(y^*) \exp u_k(x) \\ &= \sum_{k,j \geq 1} \eta_{j,k} \exp [\langle u_k, x \rangle + \langle v_j, y^* \rangle] \end{aligned}$$

and $\sum_{k,j \geq 1} |\eta_{j,k}| \exp [\|u_k\|_K^* + \|v_j\|_L^*] < \infty$ for every compact set $K \subset E, L \subset F^*$.

(ii) It is an immediate consequence of Theorem 4.1.

This completes the proof of the theorem. \square

REMARK 4.4. Recently, Phan Thien Danh and Duong Luong Son [8] have proved that every separately holomorphic function on an open subset $U \times V$ of $E \times F^*$, where $E \in (\tilde{\Omega})$ is a Fréchet nuclear space having a basis, has a local Dirichlet representation if and only if $F \in (DN)$ for every Fréchet nuclear space F .

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