BOUNDEDNESS PROPERTIES IN FUNCTION-LATTICES

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THE continuous real functions on a topological space X are partially ordered in a natural way by putting $f \leqq g$ if and only if $f(x) \leqq g(x)$ for all x in X. With respect to this partial ordering these functions constitute a lattice, the lattice operations $\cup$ and $\cap$ being defined by the relations $(f \cup g)(x) = \max\,(f(x), g(x))$, $(f \cap g)(x) = \min\,(f(x), g(x))$. The lattice character of any partially ordered system merely expresses the existence of least upper and greatest lower bounds for any finite set of elements in the system. Many partially ordered systems enjoy much stronger boundedness properties than these: for example, every subset with an upper bound may have a least upper bound, as in the case of the real number system. It is our purpose in this paper to determine the conditions under which the function-lattice described above exhibits behaviour of this kind. As an illustration of the results established we may take the important case where X is a compact Hausdorff space: here we find that the boundedness property required of the function-lattice is equivalent to the condition that X be the Boolean space associated with a Boolean algebra which has a certain strong additivity property; and many curious and useful relationships among the functions of Baire on the space X appear as a consequence of this condition, which implies that each bounded function of Baire on X differs from a uniquely determined continuous function only on a set of the first category.[1]

1. The spectral analysis of a real function. We shall first lay the foundations for the main investigation by developing what may be called the spectral analysis of a real function. We denote by capital Latin letters the abstract set X and its subsets, by lower case Latin letters real-valued functions on X, by capital Greek letters the real number system Σ and its subsets, and by lower-case Greek letters real numbers. At certain points we shall suppose that X is a topological space and the functions on it continuous. We first make the following observation.

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[1]The results of this paper, for the case where X is compact, were announced in general terms and without proof in the Proceedings of the National Academy of Sciences, vol. 26 (1940), 280-283; and important applications were indicated there. Full discussions for the case where X is a completely regular space satisfying the category condition (C) of Sec. 2 were presented in lectures at Harvard, Brown, Chicago, and the Universidad del Litoral (Rosario, Argentina) in 1942 and 1943. As referee, Professor I. Halperin pointed out that it would be desirable to isolate and minimize the rôle played by the condition (C). His detailed suggestions to this end have been incorporated in Sec. 2.

THEOREM 1. *The set* $E_f(\lambda) = \{x; f(x) < \lambda\} \subset X$ *increases with* λ *and enjoys the properties*

$$\bigcap_{\lambda\epsilon\Sigma} E_f(\lambda) = O, \ \bigcup_{\lambda\epsilon\Sigma} E_f(\lambda) = X, \ \bigcup_{\mu<\lambda} E_f(\mu) = E_f(\lambda),$$

$$\{x; f(x) = \lambda\} = P_f(\lambda) \text{ where } P_f(\lambda) = E_f(\lambda)' \cap \bigcap_{\mu>\lambda} E_f(\mu),$$

(*and even the equivalent properties in which the operations* $\bigcap$, $\bigcup$ *affect only variables* λ, μ *which belong to a set* Λ *everywhere dense in* Σ). *The characteristic function of* $E_f(\lambda)$ *is expressed by the formula* $\phi_\lambda \times f$, *where* $\phi_\lambda(\mu)$ *is* 1 *or* 0 *according as* $\mu < \lambda$ *or* $\mu \geqq \lambda$ *and* $(\phi_\lambda \times f)(x) = \phi_\lambda(f(x))$.

The family of sets $E_f(\lambda)$ may be called the *spectral family* for the function f. The formal analogy between this family and the spectral family for a self-adjoint operator in Hilbert space explains our choice of terminology. Eventually the analogy proves to be much more than formal, though we shall not discuss this circumstance in the present paper.

It is now a simple task to invert the theorem above.

THEOREM 2. *If* $E(\lambda)$, $-\infty < \lambda < +\infty$, *has the properties*

$$\bigcap_{\lambda\epsilon\Sigma} E(\lambda) = O, \ \bigcup_{\lambda\epsilon\Sigma} E(\lambda) = X, \ \bigcup_{\mu<\lambda} E(\mu) = E(\lambda)$$

(*or even the equivalent properties in which the operations* $\bigcap$, $\bigcup$ *affect only variables* λ, μ *which belong to a set* Λ *everywhere dense in* Σ), *then* $E(\lambda)$ *is the spectral family of a function* f *defined by putting* $f(x) = \lambda$ *when and only when* x *is in* $P(\lambda) = E(\lambda)' \cap \bigcap_{\mu>\lambda} E(\mu)$.

Proof. First let us demonstrate that the non-void sets $P(\lambda)$ constitute a partition of X—in other words, that the relations

$$P(\lambda_1) \cap P(\lambda_2) = O \text{ when } \lambda_1 \neq \lambda_2, \ \bigcup_{\lambda\epsilon\Sigma} P(\lambda) = X$$

hold. In the typical case where $\lambda_1 < \lambda_2$, the first relation follows immediately from the relations $P(\lambda_1) \subset E(\lambda_2)$, $P(\lambda_2) \subset E(\lambda_2)'$. The other is not quite so easy to prove. If x is a fixed element in X, the relations $x \epsilon E(\mu)$ and x non $\epsilon E(\mu)$ define respectively the upper and lower sections of a Dedekind cut in Σ by virtue of the fact that $\mu_1 < \mu_2$ implies $E(\mu_1) \subset E(\mu_2)$. Let $\lambda(x)$ be the real number determining this cut. The relation $\bigcup_{\mu<\lambda(x)} E(\mu) = E(\lambda(x))$ shows that x non $\epsilon\, E(\lambda(x))$ and hence that $\lambda(x)$ belongs to the lower section. We conclude that $x \epsilon E(\lambda(x))' \cap \bigcap_{\mu>\lambda(x)} E(\mu) = P(\lambda(x))$, $\bigcup_{x\epsilon X} P(\lambda(x)) = X$. We also see that a real function f can be defined in the manner described in the theorem, and that this function is indeed given also by the relation $f(x) = \lambda(x)$. In order to verify that $E(\lambda) = E_f(\lambda)$, we note that $f(x) = \mu < \lambda$ implies $x \epsilon P(\mu) \subset E(\lambda)$ while $x \epsilon E(\lambda)$ implies $x \epsilon E(\mu)$ for $\mu \geqq \lambda$ and hence x nonϵ $P(\mu) \subset E(\mu)'$ for $\mu \geqq \lambda$, a relation which requires that $f(x) < \lambda$.

A result useful in applications may be noted here.

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THEOREM 3. *If $H(\lambda)\subset X$ is defined for all λ in an everywhere dense subset Λ of Σ and has the properties $\bigcap_{\lambda\epsilon\Lambda} H(\lambda)=O$, $\bigcup_{\lambda\epsilon\Lambda} H(\lambda)=X$, $H(\lambda_1)\subset H(\lambda_2)$ when $\lambda_1<\lambda_2$, then the set $E(\lambda)=\bigcup_{\mu\epsilon\Lambda,\,\mu<\lambda} H(\mu)$, $-\infty<\lambda<+\infty$, has the properties enumerated in Theorem 2. In order that $E(\lambda)=H(\lambda)$ for $\lambda\epsilon\Lambda$ it is necessary and sufficient that $H(\lambda)=\bigcup_{\mu\epsilon\Lambda,\,\mu<\lambda} H(\mu)$ for $\lambda\epsilon\Lambda$.*

Various simple properties of functions are reflected by simple properties of their spectral families. Thus we see immediately that the following statements are true.

THEOREM 4. *In order that $\alpha\leqq f(x)\leqq\beta$ for all x in X it is necessary and sufficient that $E_f(\lambda)=O$ for $\lambda\leqq\alpha$ and $E_f(\lambda)=X$ for $\lambda>\beta$.*

THEOREM 5. *In order that $f\leqq g$ it is necessary and sufficient that $E_f(\lambda)\supset E_g(\lambda)$ for all λ.*

When X is a topological space there are connections between topological properties of a function and topological properties of its spectral family. A very useful instance of this general remark is the following proposition.

THEOREM 6. *When X is a topological space, a necessary and sufficient condition for the continuity of the function f is that $E_f(\lambda_1)$ be strongly contained in $E_f(\lambda_2)$ for $\lambda_1<\lambda_2$—in other words, that the closure of $E_f(\lambda_1)$ be interior to $E_f(\lambda_2)$: $E_f(\lambda_1)^-\subset E_f(\lambda_2)'^{-\prime}$.*

Proof. When f is continuous, the sets $E_f(\lambda_1)$ and $E_f(\lambda_2)$ are open while the set $\{x; f(x)\leqq\frac{1}{2}(\lambda_1+\lambda_2)\}$ is closed. When $\lambda_1<\lambda_2$ the relations $E_f(\lambda_1)\subset\{x; f(x)\leqq\frac{1}{2}(\lambda_1+\lambda_2)\}\subset E_f(\lambda_2)$ hold, and the condition of the theorem follows from them. On the other hand, when this condition holds, the relations $E_f(\lambda)=\bigcup_{\mu<\lambda}E_f(\mu)\subset\bigcup_{\mu<\lambda}E_f(\mu)^-\subset\bigcup_{\mu<\lambda}E_f(\frac{1}{2}(\lambda+\mu))'^{-\prime}\subset\bigcup_{\mu<\lambda}E_f(\frac{1}{2}(\lambda+\mu))\subset E_f(\lambda)$ are valid and show that $E_f(\lambda)=\bigcup_{\mu<\lambda}E_f(\frac{1}{2}(y+\mu))'^{-\prime}$ is an open set. In the same way the set $H_f(\lambda)=\{x; f(x)>\lambda\}$ is seen to be open, since $H_f(\lambda)=\bigcup_{\mu>\lambda}E_f(\mu)'\subset\bigcup_{\mu>\lambda}E_f(\mu)'^-\subset\bigcup_{\mu>\lambda}E_f(\frac{1}{2}(\lambda+\mu))^{-\prime}\subset\bigcup_{\mu>\lambda}E_f(\frac{1}{2}(\lambda+\mu))'\subset H_f(y)$. Thus the set $\{x;\alpha<f(x)<\beta\}=H_f(\alpha)\cap E_f(\beta)$ is open and the function f is continuous.

In actual constructions the following variant of Theorem 3 is often useful.

THEOREM 7. *If, when X is a topological space, the set $H(\lambda)\subset X$ is defined for all λ in an everywhere dense subset Λ of Σ and has the properties stated in Theorem 3 with the modification that $H(\lambda_1)$ is strongly contained in $H(\lambda_2)$ for $\lambda_1<\lambda_2$, then the family of sets $E(\lambda)=\bigcup_{\mu\epsilon\Lambda,\,\mu<\lambda}H(\mu)$ is the spectral family for a continuous function f.*

Proof. We have to verify that $E_f(\lambda)=E(\lambda)$ satisfies the condition of the preceding theorem. When λ_1 and λ_2 are given, $\lambda_1<\lambda_2$, we choose μ_1 and μ_2 in Λ so that $\lambda_1<\mu_1<\mu_2<\lambda_2$. We then have $E(\lambda_1)\subset H(\mu_1)$, $H(\mu_2)\subset E(\lambda_2)$ and hence $E(\lambda_1)^-\subset H(\mu_2)^-\subset H(\mu_2)'^{-\prime}\subset E(\lambda_2)'^{-\prime}$, as we wished to show.

We may digress briefly to show how Theorem 7 yields the crucial property of normal spaces—namely, the existence of a continuous function f vanishing throughout one closed set of a disjoint pair A, B and taking on the value 1 throughout the other. We choose Λ as the set of all dyadically rational numbers; and for $\lambda \epsilon \Lambda$, we put $H(\lambda) = O$, $H(\lambda) = A$, $H(\lambda) = B'$, $H(\lambda) = X$ according as $\lambda < 0, \lambda = 0, \lambda = 1, \lambda > 1$. We then determine $H(\lambda)$ for $\lambda = \frac{p}{2^n}$, $0 \leqq \lambda \leqq 1$, by induction on n, starting from the case $n = 0$: if we have found $H\left(\frac{p}{2^n}\right)$ for $p = 0, 1, 2, \ldots, 2^n$ so that $H\left(\frac{p}{2^n}\right)$ is strongly contained in $H\left(\frac{p+1}{2^n}\right)$, we can then choose $H\left(\frac{2p+1}{2^{n+1}}\right)$ as an open set containing $H\left(\frac{p}{2^n}\right)^-$ and strongly contained in $H\left(\frac{p+1}{2^n}\right)^{\prime-\prime}$, by virtue of the normality of the space X. Theorem 7 can now be applied to the family $H(\lambda)$, and shows that the required function f exists. Moreover $0 \leqq f(x) \leqq 1$ for all x in X in accordance with Theorem 4.

We close this section with a result of some intrinsic interest in the light of later investigations.

THEOREM 8. *In order that, when X is a topological space, the set $\{x; f(x) \neq g(x)\}$ be of the first category (that is, be the union of a denumerable family of nowhere dense sets) it is necessary and sufficient that the symmetric difference $E_f(\lambda) + E_g(\lambda) = (E_f(\lambda)' \cap E_g(\lambda)) \cup (E_f(\lambda) \cap E_g(\lambda)')$ be a set of first category for every λ. Indeed this condition remains sufficient if λ be restricted to an everywhere dense subset Λ of Σ.*

Proof. It is clear that $A \subset \{x; f(x) = g(x)\}$ if and only if $E_f(\lambda) \cap A = E_g(\lambda) \cap A$ for all λ. Thus if $A \subset \{x; f(x) = g(x)\}$ and $B = A' \supset \{x; f(x) \neq g(x)\}$, where B is of the first category, we must have $B \supset E_f(\lambda) + E_g(\lambda)$ and can conclude that the latter set is of the first category. On the other hand, if $E_f(\lambda) + E_g(\lambda)$ is of the first category for all λ in an everywhere dense subset Λ of Σ, we determine a denumerable everywhere dense subset M of Λ and put $B = \bigcup_{\lambda \epsilon \mathrm{M}} (E_f(\lambda) + E_g(\lambda))$. Here B is clearly a set of the first category. If $A = B'$ we have $E_f(\lambda) \cap A = E_g(\lambda) \cap A$ for $\lambda \epsilon \mathrm{M}$. However this means that we must also have for arbitrary λ

$$E_f(\lambda) \cap A = \bigcup_{\mu \epsilon \mathrm{M},\, \mu < \lambda} E_f(\mu) \cap A = \bigcup_{\mu \epsilon \mathrm{M},\, \mu < \lambda} E_g(\mu) \cap A = E_g(\lambda) \cap A$$

by virtue of Theorem 1. Accordingly we see that $B \supset \{x; f(x) \neq g(x)\}$ and the latter set is of the first category.

2. The main investigation. Let $C(X)$ be the partially ordered system of all continuous real functions on the topological space X. Of course, $C(X)$ is a lattice. The following stronger boundedness conditions are, as is well known, equivalent:

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($A_\aleph$) *every non-void part of* $C(X)$ *which has an upper bound and at most* $\aleph$ *members has a least upper bound;*

($B_\aleph$) *every non-void part of* $C(X)$ *which has a lower bound and at most* $\aleph$ *members has a greatest lower bound.*

Naturally if the cardinal number of $C(X)$ does not exceed $\aleph$ these conditions become respectively:

(A) *every non-void part of* $C(X)$ *which has an upper bound has a least upper bound;*

(B) *every non-void part of* $C(X)$ *which has a lower bound has a greatest lower bound.*

It is our object to determine topological properties of the space X which are equivalent to ($A_\aleph$) and ($B_\aleph$). In doing so there is no essential loss of generality in assuming that X is completely regular: for the study of the real continuous functions on a general topological space X can always be reduced to the study of corresponding continuous functions on a certain completely regular continuous image of X which is a topological invariant of X. On the other hand, the interests of simplicity are often served by making this assumption, as we shall do here at our convenience.

At certain points it will also be convenient to introduce the hypothesis that

(C) *every non-void open subset of* X *is of the second category* (*i.e. is not of the first category*).

A simple condition for a completely regular space X to have the property (C) is that X be locally $\aleph_0$-compact in the following sense: every point of X has such a neighbourhood U that the relations $\bigcap_{n=1}^{\infty} X_n = O$, $X_n \subset U$ for closed sets X_n imply $X_1 \cap \ldots \cap X_m = O$ for some m. We sketch the proof. Let Y be a non-void open set, $\{Y_n\}$ a sequence of nowhere dense sets. For a point of Y let U be such a neighbourhood as was described above. Let $X_0 = X$, and suppose that closed sets $X_1, \ldots, X_{n-1}$ have been determined so that $O \neq X_k'^{-\prime} \subset X_k \subset Y \cap U \cap Y_1^{-\prime} \cap \ldots \cap Y_k^{-\prime} \cap X_{k-1}'^{-\prime}$ for $k = 1, \ldots n-1$. Since Y_n is nowhere dense the open set $Y_n^{-\prime} \cap X_{n-1}'^{-\prime}$ is non-void. Hence, by the complete regularity of X, there exists a continuous function f_n, $0 \leqq f_n \leqq 1$, which vanishes at some point of this open set and assumes the value 1 on its complement. The set $X_n = \{x; f_n(x) \leqq \frac{1}{2}\}$ is then a closed set such that $O \neq X_n'^{-\prime} \subset X_n \subset Y \cap U \cap Y_1^{-\prime} \cap \ldots \cap Y_n^{-\prime} \cap X_{n-1}'^{-\prime}$. Since $X_1 \cap \ldots \cap X_m = X_m \neq O$ and $X_n \subset U$, we see that $\bigcap_{n=1}^{\infty} X_n \neq O$ and hence that Y contains at least one point not in $\bigcup_{n=1}^{\infty} Y_n$.

We now assume that the equivalent conditions ($A_\aleph$), ($B_\aleph$) are satisfied for a fixed infinite cardinal number $\aleph$ and a given completely regular space X; and we investigate the consequences for X and $C(X)$. As a first step let $\{f_n\}$ be a sequence in $C(X)$ with a lower bound—and hence, by hypothesis, a greatest

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lower bound, f. The sequence of functions $g_n = \min(f_1 - f, \ldots, f_n - f)$ is in $C(X)$, it decreases with $1/n$, and it has in $C(X)$ the greatest lower bound 0 (the constant everywhere equal to zero). Since $g_n(x)$ decreases with $1/n$, and is non-negative, the limit $g(x) = \lim_{n \to \infty} g_n(x)$ exists and defines a function $g \geqq 0$ which is not necessarily continuous. We now observe that the set $\{x; g(x) > 0\}$ is expressible in the form $\bigcup_{p=1}^{\infty} X_p$ where $X_p = \{x; g(x) \geqq 1/p\} = \bigcap_{n=1}^{\infty} \{x; g_n(x) \geqq 1/p\}$, and is therefore an F_σ-set—that is, is the union of a sequence of closed sets. If the interior of X_p were non-void, we could derive a contradiction, as follows. Since X is completely regular, the existence of a point x_0 in $X_p'^{-'}$ would imply the existence of a function h belonging to $C(X)$, bounded between 0 and 1, assuming the value 1 at x_0, and vanishing on the closed set $X_p'^{-}$. It would then be true that $g \geqq (1/p)h$ and hence that $g_n \geqq (1/p)h$ for $n = 1, 2, 3, \ldots$. Consequently, $0 \geqq (1/p)h$—in contradiction with the fact that $(1/p)h(x_0) = 1/p > 0$. Inasmuch as X_p is closed and contains no interior point it must be nowhere dense. Hence the set $\{x; g(x) > 0\}$ is a set of the first category. In terms of the original functions f_n, f, this means that the continuous function f is uniquely determined as the maximum element in $C(X)$ which is everywhere less than or equal to $\inf_n f_n(x)$; and that the set $\{x; f(x) < \inf_n f_n(x)\}$ is of the first category. From this general result we can now infer that, if the sequence of continuous functions f_n is bounded in $C(X)$, then $\limsup_{n \to \infty} f_n$ (and similarly $\liminf_{n \to \infty} f_n$) exists and differs from a continuous function only on a set of first category. Indeed, the result just established shows that $\lim_{n \to \infty} \max(f_n, \ldots, f_{n+p}) \leqq g_n$ where g_n is the minimal element in $C(X)$ with this property; and that $Y_n = \{x; \lim_{p \to \infty} \max(f_n, \ldots, f_{n+p}) < g_n\}$ is a set of the first category. Moreover the characterization of g_n shows that $\lim_{p \to \infty} \max(f_m, \ldots, f_{m+p}) \leqq g_n$ when $m \geqq n$ and hence that $g_m \leqq g_n$. The lower bounds of $\{f_n\}$ are also lower bounds of $\{g_n\}$. Now $\lim_{n \to \infty} g_n$ exists and differs from a continuous function f only on a set of the first category; and at the same time $\lim_{n \to \infty} g_n$ and $\limsup_{n \to \infty} f_n$ differ only on the set $Y = \bigcup_{n=1}^{\infty} Y_n$, which is of the first category. Hence $\limsup_{n \to \infty} f_n$ differs from f, a continuous function, only on a set of the first category. As a corollary of the facts established above, we remark that if a sequence $\{f_n\}$ bounded in $C(X)$ converges everywhere then its limit differs only on a set of the first category from a continuous function which is bounded by the bounds of $\{f_n\}$. In fact we can generalize this result to the family of all functions of Baire on X, as follows:

THEOREM 9. *For a completely regular space X the conditions* $(A_{\aleph})$, $(B_{\aleph})$ *imply that every real function of Baire which is bounded by continuous functions* *differs only on a set of the first category from a continuous function (which under the hypothesis (C) is uniquely determined); and these conditions further imply that every (finite) real function of Baire differs only on a set of the first category from the quotient of two continuous functions (with denominator vanishing only on a nowhere dense set when hypothesis (C) is valid).*

Proof. It is necessary to proceed by transfinite induction based on the classical procedure of Baire. Let Ω be the first ordinal number such that the class of all preceding ordinal numbers is non-denumerable. The class of all ordinal numbers α such that $0 \leqq \alpha < \Omega$ is the substratum for the recursive definition of the Baire classes: the continuous functions on X are taken as constituting the 0th Baire class; and, when the βth Baire class has been defined for every $\beta < \alpha < \Omega$, then those functions which are limits of sequences of functions in the available classes without being members of any such class are taken as constituting the αth Baire class. The union of all the Baire classes is the family of Baire functions. We have already seen above that every function of the first Baire class which is bounded with respect to $C(X)$ differs from a continuous function only on a set of the first category. Suppose we have proved the like result for every function in the βth Baire class, for $\beta < \alpha$. Let f be in the αth Baire class—specifically let f_n be in the β_nth Baire class, $\beta_n < \alpha$, and let the sequence $\{f_n\}$ be bounded with respect to $C(X)$ and converge to f. By hypothesis there exists a continuous function g_n which differs from f_n only on a set of the first category. We may suppose that the sequence $\{g_n\}$ is bounded in $C(X)$: otherwise we could replace g_n by the function max $(h, \min (g_n, k))$ where h and k are continuous functions such that $h \leqq f_n \leqq k$ for all n. Thus we see that $\limsup_{n\to\infty} g_n$ exists and differs from $f = \lim_{n\to\infty} f_n = \limsup_{n\to\infty} f_n$ only on a set of the first category. The results proved above show that there is a continuous function g which differs from $\limsup_{n\to\infty} g_n$, and hence also from f, only on a set of the first category. Now (C) implies the uniqueness of g: for if g_1 and g_2 are continuous functions each differing from f only on a set of the first category, then the set $\{x;\, g(x_1) \neq g(x_2)\}$ is open and of the first category, hence void; and $g_1 = g_2$. The part of the theorem dealing with functions of Baire which are bounded by continuous functions now follows in accordance with the principle of transfinite induction. When f is a general (finite) function of Baire, the functions $f/(1 + f^2)$ and $1/(1 + f^2)$ are bounded functions of Baire, equal except on a set of the first category to continuous functions g and h respectively. It is evident that $f = g/h$ except on a set of the first category. When (C) holds we see that the closed set $\{x;\, h(x) = 0\}$, being of the first category, has void interior and is therefore nowhere dense.

An immediate consequence of the property just established for the Baire functions is this:

THEOREM 10. *Let ϕ be any real function of Baire of the real variable λ on a domain which includes the range of a given continuous function f on a completely* *regular space X; and let the function $\phi \times f$, where $(\phi \times f)(x) = \phi(f(x))$, be bounded in $C(X)$. Then the conditions $(A_{\aleph})$, $(B_{\aleph})$ imply that $\phi \times f$ differs only on a set of the first category from a continuous function (which is uniquely determined under hypothesis (C)).*[2]

The next step in our investigation depends essentially upon a particular case of Theorems 9 and 10.

THEOREM 11. *For a completely regular space X, the conditions $(A_{\aleph})$, $(B_{\aleph})$ imply that the spectral family $\{E_f(\lambda)\}$ for a continuous real function f on X has the properties:*

(1) *$E_f(\lambda)^-$ is open as well as closed;*

(2) *$E_f(\lambda)$ is expressible as the union of a sequence of closed-and-open sets, through the specific formula $E_f(\lambda) = \bigcup_{\mu \epsilon \Lambda,\, \mu < \lambda} E_f(\lambda)^-$, where Λ is a denumerable everywhere dense subset of Σ.*

Proof. The function ϕ_λ defined in Theorem 1 is a function of Baire, being the limit of the increasing sequence of continuous functions $\phi_{\lambda, n}$ where $\phi_{\lambda, n}(\mu) = 1$ for $\mu \leqq \lambda - 1/n$, $\phi_{\lambda, n}(\mu) = n(\lambda - \mu)$ for $\lambda - 1/n \leqq \mu \leqq \lambda$, and $\phi_{\lambda, n}(\mu) = 0$ for $\mu \geqq \lambda$. The function $\phi_\lambda \times f$ is therefore equal except on a set of the first category to a continuous function g which can be taken as the least upper bound in $C(X)$ of the family of continuous functions $\phi_{\lambda, n} \times f$. For this function it is evident that $0 \leqq g \leqq 1$ and that $g(x) = 1$ when $x \epsilon E_f(\lambda)$. The continuity of g implies that $g(x) = 1$ for $x \epsilon E_f(\lambda)^-$. Now let $x_0 \epsilon E_f(\lambda)^{-\prime}$. In accordance with the fact that X is completely regular, there exists a continuous function h such that $0 \leqq h \leqq 1$, $h(x_0) = 0$, $h(x) = 1$ for $x \epsilon E_f(\lambda)^-$. The inequality $\phi_{\lambda, n} \times f \leqq h$ is valid and implies that $g \leqq h$. In particular, $g(x_0) = 0$. Thus the continuous function g is a characteristic function, being equal to 1 on $E_f(\lambda)^-$ and to 0 on its complement; and both these sets are therefore closed-and-open sets. The remainder of the present theorem follows from the relations $E_f(\lambda) = \bigcup_{\mu \epsilon \Lambda,\, \mu < \lambda} E_f(\mu)$, $E_f(\mu)^- \subset E_f(\lambda)$, noted in Theorems 1 and 6.

Concerning the subsets of X which are both closed and open we can now make the following general statements.

THEOREM 12. *For a completely regular space the conditions $(A_{\aleph})$, $(B_{\aleph})$ imply that the closed-and-open subsets of X are an additive basis for the open subsets of X, and that they are an $\aleph$-additive family in the following sense: the union of at most $\aleph$ closed-and-open sets has a closure which is both closed and open.*

Proof. If U is an open set and x_0 a point of U, there exists a continuous function f bounded between 0 and 1 which vanishes at x and assumes the value 1 on U'. The set $E_f(\lambda)^-$ is closed-and-open, in accordance with Theorem 11. Moreover $x \epsilon E_f(\frac{1}{2})^- \subset \{x; f(x) \leqq \frac{1}{2}\} \subset U$. It follows that U is the union of closed-and-open sets. In order to establish the $\aleph$-additivity asserted above,

[2]The proof is supplied by observing that $\phi \times f$ is a function of Baire. For implications of this theorem, see Stone, footnote 4.

we consider the union G of a family of closed-and-open sets G_ρ, where the index ρ belongs to a class of cardinal number at most $\aleph$. The characteristic function g for G is defined in terms of the characteristic functions g_ρ for the sets G_ρ by the formula $g(x) = \sup_\rho g_\rho(x)$. The functions g_ρ are continuous and bounded by the constant 1. Hence they have in $C(X)$ a least upper bound, f. Obviously the relations $g_\rho \leqq f$ imply $g(x) \leqq f(x)$ for every x; in particular $0 \leqq f(x)$ everywhere and $1 \leqq f(x)$ for $x \epsilon G$. Since $g_\rho \leqq 1$, $g_\rho \leqq f$, we must have $g_\rho \leqq \min\ (1, f)$ and hence $f \leqq \min\ (1, f) \leqq 1$. It follows that $f(x)$ is equal to 1 for every x in G—and by continuity even for every x in G^-. On the other hand, we can show that $f(x)$ vanishes for every x in the open set $U = G^{-\prime}$. Let x_0 be in U. Then there exists a continuous function h bounded between 0 and 1, vanishing at x_0, and assuming the value 1 on $U' = G^-$. Since $g_\rho \leqq g \leqq h$ for every ρ, we see that $g_\rho \leqq \min\ (f, h) \leqq f$ for every ρ and hence that $f \leqq \min\ (f, h)$. Thus $f = \min\ (f, h)$ and $0 \leqq f(x_0) = \min\ (f(x_0), h(x_0)) \leqq h(x_0) = 0$, $f(x_0) = 0$. We now see that f is the characteristic function of G^-, this set therefore being open as well as closed. This completes the proof of the theorem.

By combining some of the necessary conditions obtained in Theorems 9-12 we can now exhibit a set of conditions on X and $C(X)$ which implies $(A_\aleph)$ and $(B_\aleph)$ and which is therefore equivalent to each of the latter conditions taken separately. Indeed, some of the properties assumed for the space X are consequences of this set of conditions, so that we can even state a somewhat stronger theorem, as follows.

THEOREM 13. *If X is a topological space, the conditions*

(1) *the closed-and-open sets in X have the $\aleph$-additivity property of the preceding theorems;*

(2) *for any continuous function f, the spectral set $E_f(\lambda)$ is the union of at most $\aleph$ closed-and-open sets*

jointly imply that X has the properties $(A_\aleph)$ and $(B_\aleph)$; and also that (2) actually assumes the stronger form stating that $E_f(\lambda)$ is the union of at most $\aleph_0$ closed-and-open sets. If X is completely regular, then (2) implies that the closed-and-open sets constitute an additive basis for X (a condition which is sufficient to imply complete regularity).

Proof. The final assertion of the theorem requires no detailed examination. Clearly (1) and (2) together imply that $E_f(\lambda)^-$ is open as well as closed. As in the proof of Theorem 11, it then follows that (2) assumes the stronger form stating that $E_f(\lambda)$ is the union of at most $\aleph_0$ closed-and-open sets. Now let us consider a family of at most $\aleph$ continuous functions f_ρ with a lower bound in $C(X)$. The function h such that $h(x) = \inf_\rho f_\rho(x)$ is then defined for all x in X but is not necessarily a continuous function. It is clear that $E_h(\lambda) = \bigcup_\rho E_{f_\rho}(\lambda)$. Hence $E_h(\lambda)$ is the union of at most $\aleph$ closed-and-open sets by (2); and its

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closure is an open set by (1). Putting $E(\lambda) = \bigcup_{\mu<\lambda} E_h(\mu)^-$, we see that $E(\lambda) \subset E_h(\lambda)^-$. If $\lambda_1 < \lambda_2$, then $E(\lambda_1)^- \subset E_h(\lambda_1)^- \subset E(\lambda_2)$; and since $E_h(\lambda_1)^-$ is closed-and-open, it follows that $E(\lambda_1)$ is strongly contained in $E(\lambda_2)$. Theorem 6 now shows that $E(\lambda)$ is the spectral family of a continuous function f, $E(\lambda) = E_f(\lambda)$. Since $E_{f_\rho}(\lambda) \subset E_h(\lambda) = \bigcup_{\mu<\lambda} E_h(\mu) \subset E(\lambda)$ for all ρ, it follows that $f \leqq f_\rho$ for every ρ, in accordance with Theorem 5. On the other hand, if g is a continuous function such that $g \leqq f_\rho$ for all ρ, we have $E_g(\lambda) \supset E_{f_\rho}(\lambda)$ for all ρ and hence $E_g(\lambda) \supset \bigcup_\rho E_{f_\rho}(\lambda) \supset E_h(\lambda)$. Since g is continuous, it is true that $E_g(\lambda) \supset E_g(\mu)^-$ when $\lambda > \mu$. Hence $E_g(\lambda) \supset E_g(\mu)^- \supset E_h(\mu)^-$ for $\lambda > \mu$ and $E_g(\lambda) \supset \bigcup_{\mu<\lambda} E_h(\mu)^- = E_f(\lambda)$. Theorem 5 then shows that $g \leqq f$. Hence the function f is the greatest lower bound in $C(X)$ for the family of functions f_ρ. This completes the proof that the property $(B_{\aleph})$ is true for X and $C(X)$. The property $(A_{\aleph})$ is, of course, equivalent to $(B_{\aleph})$.

When the cardinal number of $C(X)$ does not exceed $\aleph$ the results obtained above may be expressed in the following terms:

THEOREM 14. *For a completely regular space X, the conditions (A) and (B) hold if and only if every open set has a closure which is open. For a topological space X, the latter condition implies both (A) and (B).*

Proof. If X is a completely regular space for which (A) or (B) is valid, then Theorem 12 shows that every open set is the union of closed-and-open sets and, as such, has a closure which is an open set. When X is a topological space in which every open set has a closure which is open, the condition (1) of Theorem 13 is obviously verified. As for condition (2), we observe that, if f is a continuous function and $\lambda > \mu$, then $E_f(\mu) \subset E_f(\mu)^- \subset E_f(\lambda)$ and $E_f(\lambda) = \bigcup_{\mu<\lambda} E_f(\mu)^-$ where the set $E_f(\mu)^-$ is open as well as closed.

There is no corresponding simplification in the other extreme case, $\aleph = \aleph_0$, unless X is assumed to be a normal space. We can then assert:

THEOREM 15. *In order that a normal Hausdorff space X have the properties $(A_{\aleph_0})$, $(B_{\aleph_0})$ it is necessary that every open F_σ-set in X have a closure which is open; and in order that a normal space X have the properties $(A_{\aleph_0})$, $(B_{\aleph_0})$ it is sufficient that the latter condition hold.*

Proof. It is well known that in a normal space an open set is an F_σ-set if and only if it is a spectral set $E_f(\lambda)$ for some continuous function f. Hence in a normal Hausdorff space with the properties $(A_{\aleph_0})$, $(B_{\aleph_0})$ we see by Theorem 11 (1) that every open F_σ-set has a closure which is open. In a normal space with the latter property, the condition (2) follows from the fact that $E_f(\lambda) = \bigcup_{n=1}^{\infty} E_f(\lambda - 1/n)^-$ where $E_f(\lambda - 1/n)^-$ is open as well as closed.

Theorems 14 and 15 have been given by H. Nakano in a paper published subsequently to our original announcement.[3]

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[3]H. Nakano, *Proc. Imp. Acad. Tokyo*, vol. 17 (1941), 308-310.

3. The case of a compact Hausdorff space. If we now concentrate our attention upon the case of a compact Hausdorff space X, we can simplify the results developed in Sec. **2**. The simplification results from the following facts.

THEOREM 16. *If X is a compact Hausdorff space, then*

(1) *X is completely regular;*

(2) *no non-void open set is of the first category;*

(3) *if f is a continuous function, then the spectral set $E_f(\lambda)$ is the union of at most $\aleph_0$ sets chosen from any specified additive basis for the open sets in X.*

Proof. (1) is well known, and (2) follows by application of the sufficient condition given at the time when (2) was introduced as hypothesis (C). To prove (3) we note first that $E_f(\lambda)$ is an F_σ-set, being the union of the closed sets $F_n = \{x; f(x) \leqq \lambda - 1/n\}$. From any specified additive basis for the open sets of X, let us select a family of open sets having $E_f(\lambda)$ as their union. This family is a covering for F_n and therefore contains a finite subcovering for F_n, by virtue of the compactness of X. We thus obtain a sequence of such finite coverings. Taken together they clearly provide a countable exact covering of $E_f(\lambda)$ by sets from the specified basis.

In view of Theorem 14 and the results of Sec. **2**, we now have the following result.

THEOREM 17. *In order that a compact Hausdorff space X have the equivalent properties $(A_\aleph)$, $(B_\aleph)$ it is necessary and sufficient that*

(1) *the closed-and-open sets constitute a basis for X;*

(2) *the closed-and-open sets have the $\aleph$-additivity property of Theorem* 12.

On any such space every bounded Baire function differs from a continuous function only on a set of the first category.

If we take into account the known results of the theory of Boolean algebras,[4] we can put the above result in the following form.

THEOREM 18. *A compact Hausdorff space has the equivalent properties $(A_\aleph)$, $(B_\aleph)$ if and only if it is the representative Boolean space for an $\aleph$-additive Boolean algebra.*

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[4]See Stone, *Bull. Amer. Math. Soc.*, vol. 44 (1938), 807-816; *Trans. Amer. Math. Soc.*, vol. 40 (1936), 37-111, and vol. 41 (1937), 375-481. In Sec. 7 of the Bulletin paper (which is a brief general survey) the union of any non-void subclass of a Boolean ring is defined. The property of $\aleph$-additivity is the property that the union of every subclass of at most $\aleph$ members exists. In the topological representation (Theorem 67 of the first Transactions paper, and Theorems 1, 2, 4 of the second) this property is equivalent to the following: the union G of any family of at most $\aleph$ closed-and-open sets is contained in a smallest closed-and-open set, necessarily the closure of G.