COINCIDENCE PRODUCING MAPS ONTO TREES

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1. Introduction. Holsztyński [1] called a map $f: X \to Y$ from a space X into a space Y 'universal for all maps of X into Y' if for all maps $g: X \to Y$ there exists a point $x \in X$ such that $f(x) = g(x)$, i.e., if f has a coincidence with all maps from X into Y. As the word 'universal' is already widely used with different meanings, we prefer the more precise term 'coincidence producing' for these maps. Such maps must clearly be surjective.

Questions concerning coincidence producing maps are related to fixed point questions. Coincidence producing maps from X onto Y can exist only if Y has the fixed point property, and the identity: $Y \to Y$ is in this case coincidence producing for all maps from Y into itself. Criteria for coincidence producing maps onto the n-cell have been established in [1] and [2].

This paper is concerned with maps onto (generalized) trees. Simple examples show that an arbitrary map onto a tree need not be coincidence producing, but we establish a sufficient condition for coincidence producing maps from a continuum onto a tree (Theorem 1). This condition is, in particular, satisfied by maps which are either monotone, quasi-monotone, or open. Hence our result generalizes one found by Wallace [3] for monotone transformations.

It is easy to construct examples which show that a coincidence producing map onto a tree need not satisfy the assumptions of Theorem 1. The problem of finding a condition which is both necessary and sufficient for a coincidence producing map remains open.

2. Definitions and Results. All spaces in this section are assumed to be compact Hausdorff, and all maps to be surjective.

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By a continuum we mean a connected compact Hausdorff space. A locally connected space is a space such that for every point x and every open neighbourhood N of x the component of N to which x belongs is a neighbourhood of x.

We use the word combinatorial tree for a connected finite linear graph without one-cycles, the word tree according to

DEFINITION 1. A tree T is a locally connected continuum which is acyclic in the sense that for every finite open cover $\mathcal{U}$ of T there exists a finite open refinement $\mathcal{V} \subset \mathcal{U}$ such that the nerve $N(\mathcal{V})$ is a combinatorial tree.

The following lemma shows that the local connectivity of a tree can be expressed in terms of finite open covers.

LEMMA 1. A continuum X is locally connected if and only if for every open cover $\mathcal{U}$ of X there exists a finite open cover $\mathcal{V} \subset \mathcal{U}$ such that all $V \in \mathcal{V}$ are connected.

The lemma is an easy consequence of results given in Wilder ([5], pp. 40-41, 106-108). Our definition of a tree is hence equivalent to the one in Wallace [3].

DEFINITION 2. A map $f: X \to Y$ from a continuum X onto a space Y is called weakly monotone if for any continuum $C \subset Y$ with non-empty interior, each component of $f^{-1}(C)$ is mapped onto C under f.

We now state our main result.

THEOREM 1. If a map $f: X \to T$ from a continuum X onto a tree T is weakly monotone, then it is coincidence producing.

If $f: X \to Y$ is a monotone map ([4], p. 127) then the inverse of every connected subspace of Y is connected. Hence we have the following corollary.

COROLLARY 1. A monotone map $f: X \to T$ from a continuum onto a tree is coincidence producing.

This result was already proved by Wallace [3].

Two further special cases which can easily be obtained from

Theorem 1 concern quasi-monotone and open maps.

DEFINITION 3. A map $f: X \to Y$ from a continuum X onto a space Y is called <u>quasi-monotone</u> if for any continuum $C \subset Y$ with non-empty interior, $f^{-1}(C)$ has only a finite number of components and each of these maps onto C under f (compare Whyburn [4], p. 151). Hence quasi-monotone maps are weakly monotone. This yields

COROLLARY 2. A quasi-monotone map $f: X \to T$ from a continuum onto a tree is coincidence producing.

Using the fact that every open map from a compact Hausdorff space is weakly monotone[1], we obtain a further corollary.

COROLLARY 3. An open map: $X \to T$ from a continuum onto a tree is coincidence producing.

3. <u>Some Lemmas.</u> The following two properties of trees are needed in the proof of Theorem 1.

LEMMA 2. Any finite open cover $\mathcal{U}$ of a tree has a finite refinement $\mathcal{G} \subset \mathcal{U}$ such that each $G \in \mathcal{G}$ is a continuum with non-empty interior and that the nerve $N(\mathcal{G})$ is a combinatorial tree.

LEMMA 3. The intersection of two continua of a tree is again a continuum.

Lemma 3 is proved in [3]. It is further shown there that any finite open cover $\mathcal{U}$ of a tree has a finite refinement $\mathcal{G}$ of connected closed sets such that the nerve $N(\mathcal{G})$ is a combinatorial tree. A study of the proof shows that the interior of the sets $G \in \mathcal{G}$ can be assumed to be non-empty.

Next, we prove a result on finite open covers for coincidence free maps.

LEMMA 4. Let X and Y be compact Hausdorff spaces

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[1] The proof of Theorem 7.5 on p. 148 of [4] remains valid in the non-metric case.

and $f, g : X \to Y$ be two maps such that $f(x) \neq g(x)$ for all $x \in X$. Then there exists a finite open cover $\mathcal{U} = \{U_i\}$ of Y such that for no $x \in X$ are $f(x)$ and $g(x)$ contained in the same U_i.

Proof. Consider the product map $f \times g : X \to Y \times Y$ determined by $f, g : X \to Y$. As the image $C = f \times g(X)$ and the diagonal D of $Y \times Y$ are disjoint closed sets in $Y \times Y$, there exists an open set V such that $D \subset V \subset Y \times Y$ and $C \cap V = \phi$. By definition of the product topology, $V = \bigcup_{\lambda \in \Lambda, \mu \in M} U_\lambda \times U_\mu$, where U_λ, U_μ are open sets in Y. For every point $\bar{x} = (x, x) \in D$, we therefore have

$$\bar{x} \in U_{\lambda(x)} \times U_{\mu(x)} \subset V$$

for some $\lambda(x) \in \Lambda$, $\mu(x) \in M$.

Let $U(x) = U_{\lambda(x)} \cap U_{\mu(x)}$. Then $U(x)$ is an open subset of Y, and

$$\bar{x} \in U(x) \times U(x) \subset V ,$$

hence

$$D \subset \bigcup_{x \in X} U(x) \times U(x) \subset V .$$

As D is compact in $Y \times Y$, there exists a finite subcollection $\{U_i, i = 1, \dots, n\}$ of the $\{U(x), x \in X\}$ such that

$$D \subset \bigcup_{i=1}^{n} U_i \times U_i \subset V .$$

But $C \cap V = \phi$, therefore $C \cap (\bigcup_{i=1}^{n} U_i \times U_i) = \phi$, so that $\mathcal{U} = \{U_i\}$ is the desired finite open cover of Y.

Finally, we prove a result which is the counterpart of Theorem 1 in the case of simplicial maps from (finite or infinite) polyhedra onto combinatorial trees. The (closed) star of the vertex a of a polyhedron is denoted by $st(a)$.

LEMMA 5. Let $u : M \to N$ be a simplicial map of a polyhedron M onto a combinatorial tree N such that $st[u(a)] = u[st(a)]$ for all vertices a of M. If $v : M \to N$ is an arbitrary vertex transformation, then there exist two (not necessarily distinct) neighbouring vertices a', a'' of M such that $u(a'a'')$ is contained in the chain joining $v(a')$ to $v(a'')$.

Proof. If $u(a) = v(a)$ for any vertex $a \in M$, the result is trivial. Otherwise, let a_1 be an arbitrary vertex of M, let $b_1 = u(a_1)$, and let b_2 be the next vertex after b_1 on the chain in N from b_1 to $v(a_1)$. As $st[u(a_1)] = u[st(a_1)]$, there exists an edge a_1a_2 of M such that $u(a_2) = b_2$. Continuing in this way we obtain a chain $a_1, a_2, a_3, \ldots$ in M and a chain $b_1, b_2, b_3, \ldots$ in N such that $u(a_i) = b_i$ and b_{i+1} is the next vertex after b_i on the chain from b_i to $v(a_i)$. As N is finite, there exist two indices i and k, $i < k$, such that $b_i = b_k$, but $b_i \neq b_j$ if $i < j < k$. By definition of the b_i we have $k > i+1$. It is not possible that $k > i+2$, as otherwise $b_i, b_{i+1}, b_{i+2}, \ldots, b_k$ would be a cycle in the combinatorial tree N. Hence $k = i+2$, and the vertices $a' = a_i$, $a'' = a_{i+1}$ satisfy the conditions of Lemma 5.

4. Proof of Theorem 1. Let $f : X \to T$ be a map from a continuum X onto a tree T which is weakly monotone. In order to prove that f is coincidence producing we show that the existence of a map $g : X \to T$ with $f(x) \neq g(x)$ for all $x \in X$ leads to a contradiction.

Assume that $g : X \to T$ is such a map. It follows from Lemma 4 that we can find a finite open cover $\mathcal{U} = \{U_i\}$ of T such that for no $x \in X$ are both $f(x)$ and $g(x)$ contained in the same U_i. According to Lemma 2 we choose a finite cover $\mathfrak{G} \subset \mathcal{U}$ such that each $G \in \mathfrak{G}$ is a continuum with non-empty interior and that the nerve $N = N(\mathfrak{G})$ is a combinatorial tree. Let $\mathfrak{F} = \{F\}$ be the cover of X consisting of all the components of all the $f^{-1}(G)$. Then all $F \in \mathfrak{F}$ are continua, and the nerve $M = M(\mathfrak{F})$ is a one-dimensional polyhedron.

A vertex transformation from M to N is defined as follows: Let the vertex $a \in M$ corespond to $F_a \in \mathfrak{F}$. Take

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$G_{u(a)} \in \mathcal{G}$ such that F_a is a component of $f^{-1}(G_{u(a)})$. Clearly this determines a simplicial map $u: M \to N$.

The relation $u[st(a)] \subseteq st[u(a)]$ is true for an arbitrary simplicial map. But here we have $st[u(a)] \subseteq u[st(a)]$ as well: Let $u(a) = b$, and take any $b' \in st(b)$. It follows from Lemma 3 that $G_b \cap G_{b'}$ is a continuum, and as f is weakly monotone, we have that $f(F_a) = G_b$, hence at least one component K of $f^{-1}(G_b \cap G_{b'})$ is contained in F_a. As $K \subset f^{-1}(G_{b'})$, we have $F_a \cap f^{-1}(G_{b'}) \neq \phi$, i.e. for at least one component $F_{a'}$ of $f^{-1}(G_{b'})$ is it true that $F_a \cap F_{a'} \neq \phi$. Hence $a' \in st(a)$ with $u(a') = b'$.

Therefore $u: M \to N$ is a simplicial map such that $st[u(a)] = u[st(a)]$ for all vertices $a \in M$.

We further define a vertex transformation $v: M \to N$ by assigning to each $a \in M$ a vertex $v(a) \in N$ such that $g(F_a) \cap G_{v(a)} \neq \phi$.

By Lemma 5, there exist two neighbouring vertices a', a'' of M such that $u(a'a'')$ is contained in the chain joining $v(a')$ to $v(a'')$. Hence we find a chain of sets

$$G_{v(a'')}, \ldots, G_{b_i}, \ldots, G_{u(a')}, G_{u(a'')}, \ldots, G_{b_j}, \ldots, G_{v(a')} .$$

Let

$$D = G_{v(a'')} \cup \ldots \cup G_{b_i} \cup \ldots \cup G_{u(a')} ,$$

$$E = G_{u(a'')} \cup \ldots \cup G_{b_j} \cup \ldots \cup G_{v(a')} .$$

Then D and E are continua with $D \cap E = G_{u(a')} \cap G_{u(a'')} = f(F_{a'}) \cap f(F_{a''})$. As $F_{a'} \cap F_{a''} \neq \phi$, it follows that $C = g(F_{a'}) \cup g(F_{a''})$ is a continuum, and by definition of v, both $C \cap G_{v(a')} \neq \phi$ and $C \cap G_{v(a'')} \neq \phi$. As $C \cap (D \cup E)$ is a continuum by Lemma 3 and non-empty, we have $C \cap (D \cap E) \neq \phi$, or

$$[g(F_{a'}) \cup g(F_{a''})] \cap [f(F_{a'}) \cap f(F_{a''})] \neq \phi .$$

Hence

$$g(F_{a_o}) \cap f(F_{a_o}) \neq \phi$$

for at least either $a_o = a'$ or $a_o = a''$.

Now select an $x \varepsilon X$ for which $g(x) \varepsilon g(F_{a_o}) \cap f(F_{a_o})$. Then both $f(x)$ and $g(x)$ are in $f(F_{a_o})$. By construction, $f(F_{a_o}) \subset U_i$ for some $U_i \varepsilon \mathcal{U}$, so that both $f(x)$ and $g(x)$ are contained in the same U_i, in contradiction to the choice of $\mathcal{U}$. Therefore it is not possible that $f(x) \neq g(x)$ for all $x \varepsilon X$, and f is coincidence producing.

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