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CANONICALLY ISOMORPHIC SPACES OF BOUNDED SOLUTIONS OF $\Delta u = Pu$

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Let R be a hyperbolic Riemann surface and P, Q nonnegative C^1 2-forms on R. The space of bounded solutions of $\Delta u = Pu$ ($\Delta u = Qu$, respectively) on R is denoted by $PB(R)$ ($QB(R)$, respectively). A vector space isomorphism S between $PB(R)$ and $QB(R)$ is called *canonical* if for each $u \in PB(R)$, there is a potential p_u on R with $|u - Su| \leqq p_u$. The canonical isomorphism theme in the study of the equation $\Delta u = Pu$ was introduced in H. Royden's paper [**9**] on the order comparison condition. A variety of work giving sufficient conditions for the canonical isomorphism followed (see [**6**; **3**; **4**; **5**; **2**; and **7**], among others). The first necessary and sufficient condition for the existence of the canonical isomorphism was given by M. Nakai [**8**] and the author [**1**]. This condition is expressed as follows. Let R^* be the Wiener compactification of R and δ the harmonic boundary. Define

$$\delta^P = \left\{ p \in \delta \,|\, p \text{ has a neighborhood } U^* \text{ in } R^* \text{ with } \int_{U^* \cap R} G_R(\cdot, z) P < +\infty \text{ for some } z \in R \right\}.$$

Here $G_R(\cdot, z)$ is the harmonic Green's function of R with pole at z. Then $\delta^P = \delta^Q$ if and only if $PB(R)$ and $QB(R)$ are canonically isomorphic.

The purpose of this note is to give a necessary and sufficient condition for the existence of the canonical isomorphism which can be expressed without the Wiener compactification theory. However, in order to give a simple proof we use the compactification theory in our arguments. We shall use the notations and results of [**1**] here as well as the result of [**8**, Theorem 9] that δ^P is compact and open in δ.

We shall call a subset $K \subset R$ *PB-negligible* if there is a continuous superharmonic function s on R such that $s|K = 1$, $0 \leqq s \leqq 1$ and $u \leqq s$ for $u \in PB(R)$ only if $u \leqq 0$. This is a refinement of a notion of negligibility introduced in [**7**].

THEOREM. *The spaces $PB(R)$, $QB(R)$ are canonically isomorphic if and only if there is a subset K of R which is both PB- and QB-negligible such that*

$$(*) \qquad \int_{R \setminus K} G_R(\cdot, z)|P - Q| < +\infty,$$

for some $z \in R$.

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Assume that K is PB- and QB-negligible and that (*) holds. Let s be the function in the definition of PB-negligibility. We claim that $s|\delta^P$ must be 0. In fact if $s(p) > 0$ for some $p \in \delta^P$, then there is a neighborhood U^* of p in R^* with $U^* \cap \delta^P \subset \delta^P$ and $s|U^* \geqq \epsilon$ for some constant $\epsilon > 0$. Consequently, there is a function f in the Wiener algebra such that $0 \leqq f \leqq 1$, $\operatorname{supp} f \subset U^*$ and $f(p) = 1$. Therefore there is a function $u \in PB(R)$ with $u|\delta = f|\delta$ (cf. [**1**, Theorem 4]). Since $s - u$ is bounded and superharmonic on R and $s - u|\delta \geqq 0$, we conclude that $s - u \geqq 0$. The fact that $u > 0$ is a contradiction and the claim is established. The continuity of s on R^* implies that $\bar{K} \cap \delta^P = \emptyset$. This means that $R^*\backslash\bar{K}$ is a neighborhood of δ^P. Similarly $R^*\backslash\bar{K}$ is a neighborhood of δ^Q. From the definition and (*) it now follows that $\delta^P = \delta^Q$ which is equivalent to the desired conclusion.

Conversely, assume $\delta^P = \delta^Q$. For each $p \in \delta^P$ we choose $U_p{}^*$ an open neighborhood of p with

$$\int_{U_p{}^* \cap R} G_R(\cdot, z_p)P < +\infty,$$

for some $z_p \in R$. By the Harnack inequality the finiteness of the integrals does not depend on the choice of z_p. Therefore take $z_p = z$ for some fixed $z \in R$. By the compactness of δ^P there is a finite collection $U_{p_1}{}^*, \ldots, U_{p_n}{}^*$ such that $\delta^P \subset U^* = \bigcup_{i=1}^n U_{p_i}{}^*$. Also $\int_{U^* \cap R} G_R(\cdot, z)P < +\infty$. Similarly there is an open neighborhood V^* of δ^Q with $\int_{V^* \cap R} G_R(\cdot, z)P < +\infty$. Set $W^* = U^* \cap V^*$. Then

$$(**) \quad \int_{W^* \cap R} G_R(\cdot, z)(P + Q) < +\infty.$$

The desired subset K of R is $R\backslash W^*$. Trivially (**) implies (*). The PB- and QB-negligibility of K follows from the fact that $\bar{K} \cap \delta^P = \emptyset$. Take a function f in the Wiener algebra with $f|\bar{K} = 1$, $f|\delta^P = 0$, $0 \leqq f \leqq 1$. Let s be the function in the Wiener algebra with $s|\bar{K} \cup \delta = f|\bar{K} \cup \delta$ and s harmonic on $R\backslash\bar{K}$. Then s is continuous and superharmonic on R with $0 \leqq s \leqq 1$. If $u \in PB(R)$ and $u \leqq s$ then $u|\delta^P \leqq s|\delta^P = 0$. This implies that $u \leqq 0$ (cf. [**1**, Corollary 3]).

An interesting consequence of the theorem is that the order comparison condition, $cP \leqq Q \leqq c^{-1}P$, for some $c > 0$, implies the integral comparison condition: there is a PB- and QB-negligible set K such that (*) holds. Actually, it is sufficient that $cP \leqq Q \leqq c^{-1}P$ hold outside a set K which is PB- and QB-negligible. This sort of observation has been made by A. Lahtinen [**2**].

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