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TOPOLOGICAL SPACES WITH A UNIQUE COMPATIBLE QUASI-UNIFORMITY

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1. **Introduction.** In [2] P. Fletcher proved that a finite topological space has a unique compatible quasi-uniformity; C. Barnhill and P. Fletcher showed in [1] that a topological space $(X, \mathcal{T})$, with $\mathcal{T}$ finite, has a unique compatible quasi-uniformity. In this note we give some necessary conditions for unique quasi-uniformizability.

2. **Preliminaries**

DEFINITION. Let X be a nonempty set, a *quasi-uniformity* $\mathbf{U}$ for X is a filter of reflexive subsets of $X \times X$ with the property that if $U \in \mathbf{U}$, there exists a $V \in \mathbf{U}$ such that $V \circ V \subset U$.

A source of facts on quasi-uniform spaces is the monograph of Murdeshwar and Naimpally [6].

DEFINITION. A relation δ on $\mathbf{P}(X)$ is a *quasi-proximity* [7, 9] for X iff it satisfies (a) $A \not\delta \phi$, $\phi \not\delta A$ for each A in $\mathbf{P}(X)$; (b) $C \,\delta\, A \cup B$ iff $C \,\delta\, A$ or $C \,\delta\, B$, and $A \cup B \,\delta\, C$ iff $A \,\delta\, C$ or $B \,\delta\, C$; (c) $\{x\} \,\delta\, \{x\}$, for each $x \in X$; (d) if $A \not\delta B$, then there exist C, D with $C \cap D = \phi$, $A \not\delta X - C$, and $X - D \not\delta B$.

We say a topological space $(X, \mathcal{T})$ is *uqu* (*uqp*) iff it has a unique compatible quasi-uniformity (quasi-proximity). For any space $(X, \mathcal{T})$, $\{S_G = G \times G \cup (X - G) \times X : G \in \mathcal{T}\}$ is a subbase for a totally bounded quasi-uniformity which we will denote by $\mathbf{U}_P$ [8].

DEFINITION. A *Q-cover* of X is an open cover $\mathscr{C}$ of X such that for each $x \in X$, $A_x^{\mathscr{C}} = \bigcap \{C \in \mathscr{C} : x \in C\}$ is open.

If α denotes the collection of all Q-covers of a given space $(X, \mathcal{T})$ and $U_{\mathscr{C}} = \bigcup \{\{x\} \times A_x^{\mathscr{C}} : x \in X\}$, then $\{U_{\mathscr{C}} : \mathscr{C} \in \alpha\}$ is a subbase for a quasi-uniformity $\mathbf{U}_Q$ which is compatible with $\mathcal{T}$ [3]. We say a space is *Q-finite* iff each Q-cover is finite. Since for each $G \in \mathcal{T}$, $\mathscr{C} = \{G, X\}$ is a Q-cover and $S_G = U_{\mathscr{C}}$, $\mathbf{U}_Q \supset \mathbf{U}_P$.

DEFINITION. A topological space $(X, \mathcal{T})$ is *supercompact* iff each subset of X is compact.

DEFINITION. An *ascending* (*descending*) *open sequence* is a collection $\{G_n \in \mathcal{T} : n \in N\}$ such that $G_n \subset G_{n+1}$ ($G_n \supset G_{n+1}$) for all $n \in N$.

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2.1. THEOREM. *A topological space* $(X, \mathcal{T})$ *is supercompact iff every ascending open sequence is finite.*

Proof. If $\{G_n : n \in N\}$ is an infinite ascending open sequence, then $\bigcup\{G_n : n \in N\}$ is not compact. Conversely if A is a noncompact subset of X, there exists an open cover $\mathscr{C}$ of A having no finite subcover. From $\mathscr{C}$ we may select $\{C_n : n \in N\}$ such that $\{\bigcup\{C_i : i = 1, 2, \ldots, n\} : n \in N\}$ is an infinite ascending open sequence.

2.2. COROLLARY. *If* $(X, \mathcal{T})$ *is Q-finite, it is supercompact.*

Proof. If $\mathscr{C}$ is an open ascending sequence, then $\mathscr{C} \cup \{X\}$ is a Q-cover.

In the sequel, we will make frequent use of the following theorems found in [4] and [5] respectively.

2.3. THEOREM. *Let* (X, δ) *be a quasi-proximity space. The collection* $\{X \times X - A \times B : A \,\bar{\delta}\, B\}$ *is a subbase for a totally bounded quasi-uniformity* $\mathbf{U}_\beta$ *which is compatible with* δ. *Moreover,* $\mathbf{U}_\beta$ *is the coarsest quasi-uniformity compatible with* δ *and is the only totally bounded quasi-uniformity compatible with* δ.

2.4. THEOREM. *If* $(X, \mathcal{T})$ *is supercompact, then it is uqp.*

3. Uniquely quasi-uniformizable topological spaces

3.1. We offer here a simplified proof of the following theorem proved in [1] by C. Barnhill and P. Fletcher.

THEOREM. *If* $(X, \mathcal{T})$ *is a topological space with* $\mathcal{T}$ *finite, then* $(X, \mathcal{T})$ *is uqu.*

Proof. $\mathcal{T}$ finite implies $(X, \mathcal{T})$ is supercompact and hence *uqp*. Thus, in view of Theorem 2.3, it suffices to show that $\mathbf{U}_W$ (the universal quasi-uniformity) is totally bounded. This is immediate since Theorem 3.3 [2] yields $\mathbf{U}_P = \mathbf{U}_W$.

3.2. Fletcher conjectures in [2] that if a space $(X, \mathcal{T})$ is *uqu* then $\mathcal{T}$ is finite. In [5], the author proved that the conjecture is true for R_1 topological spaces, and further showed that the real numbers with cofinite topology is a *uqu* space. We give here a simpler example of a *uqu* space $(X, \mathcal{T})$ with $\mathcal{T}$ infinite.

Let $X = [0, 1)$ and $\mathcal{T} = \{\phi, [0, 1/n) : n \in N\}$. $(X, \mathcal{T})$ is supercompact hence *uqp*. Thus to prove $(X, \mathcal{T})$ is *uqu*, we need only show that each quasi-uniformity $\mathbf{U}$ compatible with $\mathcal{T}$ is totally bounded. Take $V, U \in \mathbf{U}$ with $V \circ V \subset U$; if $[0, 1/m) = \operatorname{int}(V(0))$, set $A_1 = [0, 1/m)$, $A_2 = [1/m, 1/(m-1)), \ldots, A_m = [\frac{1}{2}, 1)$. Then $\bigcup\{A_i : i = 1, 2, \ldots, m\} = X$ and $\bigcup\{A_i \times A_i : i = 1, 2, \ldots, m\} \subset V \circ V \subset U$; hence $\mathbf{U}$ is totally bounded.

3.3. THEOREM. *Let* $(X, \mathcal{T})$ *be a topological space with* $\mathbf{U}_Q$ *totally bounded* (*i.e.* $\mathbf{U}_Q = \mathbf{U}_P$), *then* $(X, \mathcal{T})$ *is Q-finite.*

Proof. Let $\mathscr{C}$ be a Q-cover; for each $x \in X$, set $G_x = A_x^{\mathscr{C}}$, $U = U_{\mathscr{C}}$, $\mathbf{G} = \{G_x : x \in X\}$, $H_y = \{z : G_z = G_y\}$, and $\mathbf{H} = \{H_y ; y \in X\}$. Note that $\mathbf{H}$ is a partition of X and

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card (**H**) = card (**G**). Since $\mathbf{U}_Q$ is totally bounded, there exists a collection $\{A_i: i=1, 2, \ldots, n\}$ such that $\bigcup \{A_i : i=1, 2, \ldots, n\} = X$ and $\bigcup \{A_i \times A_i : i=1, 2, \ldots, n\} \subset U$. If **H** (equivalently **G**) is infinite, there exist A_i, H_x, H_y such that $H_x \neq H_y$, and there exist $w \in A_i \cap H_x$ and $z \in A_i \cap H_y$. Since $(w, z) \in A_i \times A_i \subset U$, $z \in U(w) = G_w$; hence $G_z \subset G_w$. Since $(z, w) \in A_i \times A_i \subset U$, $w \in U(z) = G_z$; hence $G_w \subset G_z$ and $G_w = G_z$. $w \in H_x$ implies $G_w = G_x$; $z \in H_y$ implies $G_z = G_y$. Finally $G_x = G_w = G_z = G_y$ implies $H_x = H_y$ which is a contradiction. Hence we have proved **H** is finite; thus **G** is finite. Since for each $G \in \mathscr{C}$, $G = \bigcup \{G_x : x \in G\}$, $\mathscr{C}$ must be finite as well.

3.4. Theorem. *If $(X, \mathscr{T})$ is a uqu topological space, then*

(a) *$(X, \mathscr{T})$ is Q-finite;*

(b) *if* **G** *is a descending open sequence and* $\bigcap \mathbf{G} \in \mathscr{T}$, *then* **G** *is finite;*

(c) *every ascending open sequence is finite;*

(d) *$(X, \mathscr{T})$ is supercompact;*

(e) *if $(X, \mathscr{T})$ is Hausdorff, then X is finite.*

Proof. (a) Since $(X, \mathscr{T})$ is *uqu*, $\mathbf{U}_Q = \mathbf{U}_P$ and Theorem 3.3 applies. (b) $\{X\} \cup \mathbf{G}$ is a Q-cover. (c) and (d) follow from Corollary 2.2 and (a). (e) Every supercompact Hausdorff space is finite.

3.5. Corollary. *A topological space $(X, \mathscr{T})$ is uqu iff* $\mathbf{U}_P = \mathbf{U}_W$.

Proof. If $\mathbf{U}_P = \mathbf{U}_W$, then $\mathbf{U}_P = \mathbf{U}_Q$ and $(X, \mathscr{T})$ is Q-finite. In light of Corollary 2.2 and Theorem 2.4, $(X, \mathscr{T})$ is *uqp*. Now Theorem 2.3 applies to yield $(X, \mathscr{T})$ is *uqu*.

3.6. Corollary. *If each quasi-uniformity compatible with $\mathscr{T}$ is totally bounded, then $(X, \mathscr{T})$ is uqu.*

3.7. Corollary. *If $(X, \mathscr{T})$ is a topological space, with the property that $\mathscr{T}$ is a Q-cover, then $(X, \mathscr{T})$ is uqu iff $\mathscr{T}$ is finite.*

4. **The problem of characterizing *uqu* topological spaces.** The problem of characterizing *uqu* topological spaces is clearly related to the following: For which spaces does Q-finite imply *uqu*? For which spaces is it true that $\mathbf{U}_Q = \mathbf{U}_W$?

4.1. Theorem. *If $(X, \mathscr{T})$ is a topological space with the property that $\mathscr{T}$ is a Q-cover, then* $\mathbf{U}_Q = \mathbf{U}_W$.

Proof. Since $\mathscr{T}$ is a Q-cover, $\{U_{\mathscr{T}}\}$ is a base for $\mathbf{U}_Q$ and Theorem 3.3 of [2] yields $\mathbf{U}_Q = \mathbf{U}_W$.

4.2. Example. Let $X = (0, 1)$, $\mathscr{T} = \{\phi, (0, 1/n) : n \in N\}$. $(X, \mathscr{T})$ is supercompact, hence *uqp*. By Theorem 4.1, $\mathbf{U}_Q = \mathbf{U}_W$. $(X, \mathscr{T})$ is a subspace of the *uqu* space given in 3.2; nonetheless Corollary 3.7 implies $(X, \mathscr{T})$ is not *uqu*.

Added in proof. It has only recently come to the author's attention that supercompact spaces are discussed extensively by A. H. Stone in Hereditarily compact spaces, Amer. J. Math. **82** (1960), 900–916.

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