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ON SINGLE-LAW DEFINITIONS OF GROUPS

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It will be proved that any mononomic variety of groups can be considered as a variety of (ρ,ϵ) or (ρ,τ) or (ν,ϵ)-algebras, or as a variety of n-groupoids—which satisfy a single law, where: $xy\rho = x.y^{-1}$, $x\tau = x^{-1}$, $xy\nu = x^{-1}.y^{-1}$, ϵ is the identity, and for certain interpretations of the n-ary operation. The problem is discussed for Ω-groups, too.

The problem of single-law definability of mononomic (that is finitely axiomatisable) varieties of groups is a very intriguing subject, not least because of the questions it raises in universal algebra—such as: when is it possible to adjoin a new operation, with some describable interpretation, to a language which defines a variety by a single law, and to preserve the property? This is not always possible: see [3]; on the other hand, it sometimes happens to be the case, as it will be shown below.

The notation is consistent with that of [2], [3] and [4]: lower case Greek letters denote operations, and capital letters other than A (which is reserved for a carrier) denote mappings of a considered carrier. Both operations and these mappings are written as right-hand operators.

For universal algebraic notions the reader is referred to [1].

It has been shown in [2] that the variety of groups satisfying the law $w = e$ (w is a term containing only the right-division operation $x.y^{-1}$, e the identity) is definable by the law

$$(i) \qquad xxx\rho w\rho y\rho z\rho xx\rho x\rho z\rho\rho\rho = y$$

in language (ρ) of type (2) with interpretation $xy\rho = x.y^{-1}$. A more general result will be proved here:

THEOREM 1. *Let ω be an n-ary group-polynomial which is capable of expressing basic group operations. Then any mononomic variety of groups is definable by a single law in language (λ) of type (n), with interpretation $\lambda = \omega$.*

PROOF: Let us express ω in terms of right-division, say by the equation $x_1 \cdots x_n\omega = t_\rho(x_1, \ldots, x_n)$, and let right division ρ be expressed via ω by the law $xy\rho = t_\omega(x,y)$.

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Let the variety of groups concerned be defined by $u = e$ where u is a term containing only the right division operation (every mononomic variety of groups is definable by such a law: see [2]). We define the term w to be $ux_1 \cdots x_n \omega t_\rho(x_1, \dots, x_n)\rho\rho$, where no variable x_i occurs in u. Now express the law (i) in terms of ω, by substituting $t_\omega(s,t)$ for $(st\rho)$, and replacing each occurrence of symbol ω by the symbol λ thus obtaining a (λ)-law; let us call it $(*)$. $(*)$ is the law for which we are looking. Indeed, let $A = (A, \lambda)$ be an n-groupoid such that $A \models (*)$. Then a new operation ρ on A is introduced by $xy\rho = t_\lambda(x,y)$ where t_λ is the term obtained from t_ω by replacing occurrences of ω by λ. Then we have $A^* = (A, \rho) \models (i)$; however, we cannot (yet) use the theorem from [2] because our w contains operation symbols other than ρ. As is easily seen from the proof of Theorem 3.2. of [2], the fact that w contains only ρ is used only to prove $w = e$ (by assigning $y_i = e$ for all its variables y_i, and using $ee\rho = e$). This can be avoided in the following way: let $L_x, R_y : A \to A$ be defined by $xy\rho = yL_x = xR_y$. Then one arrives at $ee\rho = e$ and $L_{ew\rho}R_z R_{ex\rho z\rho} L_x = I$ (the identity map) just as in [2]. Let $x = z = e$; then using $ee\rho = e$ we get $L_{ew\rho}R_e^2 L_e = I$. In particular, $eL_{ew\rho}R_e^2 L_e = e$; now since $e = ee\rho = eL_e = eR_e$, it follows that $eL_{ew\rho}R_e^2 L_e = eR_e^2 L_e$. But R_x, L_x are bijective (see[2]), and hence:

$$eL_{ew\rho} = e.$$

From $eL_{ew\rho} = wL_e R_e$ it follows that

$$wL_e R_e = eL_{ew\rho} = e = eL_e R_e$$

and again by bijectiveness of $L_e R_e$ we obtain

$$w = e.$$

Thus, proceeding as in [2], it follows that A^* is a group with $xy\rho = x.y^{-1}$. Now set $z_i = e$ for all variables z_i of u; this, by $ee\rho = e$, yields $u = e$ and hence $e = w = ex_1 \cdots x_n \lambda t_\rho(x_1, \dots, x_n)\rho\rho$ which implies $x_1, \dots, x_n \lambda = t_\rho(x_1, \dots, x_n)$, which is the desired interpretation: $\lambda = \omega$. $u = e$ follows in an obvious way and, consequently, the defined group belongs to the variety. It is easy to check that $(*)$ holds in any group which satisfies $u = e$, with interpretation $\lambda = \omega$—which finishes the proof. ∎

The observation that has just been made above has one more consequence:

THEOREM 2. *Let ω be an n-ary operation which is describable by a single law of group theory. Then any mononomic variety of groups is definable by a single law of language (ρ, π) of type $(2, n)$, with interpretation such that $xy\rho = x.y^{-1}$ and π satisfies the law which describes ω.*

PROOF: Let $t_1 = t_2$ be the law which describes (that is defines implicitly in a sense) ω. Put $w = us_1 s_2 \rho\rho$, where $u = e$ is the law defining the variety concerned, and

s_1, s_2 are (ρ, π)-terms obtained from t_1, t_2 respectively, by substituting occurrences of ω by π, and expressing basic group operations via right-division ρ. The law (i) defines this variety in the language (ρ, π) with $xy\rho = x.y^{-1}$ (since by the observation made in the proof of Theorem 1., we can use Theorem 3.2 of [2] now). Set $z_i = e$ for all variables z_i of u; then $u = e$ and hence every algebra of this variety satisfies $s_1 = s_2$ which proves that π has the desired property. The rest is trivial. ∎

In particular, if $\omega = e$ or $x\omega = x^{-1}$, Theorem 2. provides an affirmative answer to a question asked in [4]: whether there is a single law in language $(\rho, \varepsilon), (\rho, \tau)$ which defines mononomic varieties of groups with $xy\rho = x.y^{-1}$, $x\tau = x^{-1}$ and ε the identity. These laws are ($w = e$ defines the variety):

$$xxx\rho w a \tau a a \rho a \rho\rho\rho\rho y \rho z \rho x x \rho x \rho z \rho\rho\rho = y, xxx\rho w \varepsilon a a \rho\rho\rho\rho y \rho z \rho x x \rho x \rho z \rho\rho\rho = x.$$

3. One more question from [4] has an affirmative answer:

THEOREM 3. *A variety of grops which satisfy $w = e$ is defined by the law*

$$(ii) \qquad z\varepsilon y\nu\varepsilon t w\nu\nu t w'\nu\nu x\nu\nu\varepsilon z\nu y\nu\nu\nu = x$$

in language (ν, ε) of type $(2,0)$ with $xy\nu = x^{-1}.y^{-1}$ and ε the identity where w' is a term obtained from w by substituting a new variable x_i' for each x_i which occurs in w.

PROOF: By examining the proof of Theorem 1. of [4], the reader will see that the difference between the law (ii) and the law (1) of [4] only affects the proofs of identities (5)-(8) from [4]. These are:

$$(5) \qquad et\nu t\nu = e;$$

$$(6) \qquad T_e T_{ey\nu} S_{ez\nu y\nu} T_z = I, \text{ the identity map;}$$

$$(7) \qquad eT_e S_e = e;$$

$$(8) \qquad T_e = S_e$$

where $A = (A, \nu, \varepsilon) \models (ii)$ and, as in [4], $T_x, S_x : A \to A$ are defined by $xy\nu = yT_x = xS_y$, e is the interpretation of ε (this will turn out to be the identity hence we call it e). Since the law (ii) has $\varepsilon t w\nu\nu t w'\nu\nu$ instead of $\varepsilon t\nu t\nu$, we have to prove, in place of (5):

$$(5') \qquad etw\nu\nu tw'\nu\nu = e.$$

Now using maps T_x, S_x, (ii) can be written as

$$(iii) \qquad T_{etw\nu\nu tw'\nu\nu} T_{ey\nu} S_{ez\nu y\nu} T_z = I,$$

from which it follows, copying [4], that T_x, S_x are bijections for each x. Then the identity (iii) yields $T_{etw\nu\nu tw'\nu\nu} = T_z^{-1} S_{ezv y\nu}^{-1} T_{ey\nu}^{-1}$, and we see that $T_{etw\nu\nu tw'\nu\nu}$ does not depend on t, w, w'; hence the term $etw\nu\nu tw'\nu\nu$ does not depend on t, w, w', neither. Put $f = eT_e^{-1}, t = fS_w^{-1}, x_i = x_i'$; this means $ef\nu = e, tw\nu = f, w = w'$ and thus:

$$etw\nu\nu tw'\nu\nu = etw\nu\nu tw\nu\nu = ef\nu f\nu = ef\nu = e.$$

Therefore (5') holds. (6) follows immediately by (iii) and (5'). As in [4], one proves that $T_x S_x$ does not depend on x - let this permutation be denoted by K. Now choose in (5') $x_i = x_i'$ and $t = eS_w^{-1}$ (that is $w = w', tw\nu = e$); then:

$$e = etw\nu\nu tw'\nu\nu = etw\nu\nu tw\nu\nu = ee\nu e\nu = eT_e S_e = eK$$

which is (7). And finally, for any $a \in A$ let $x_i = x_i', t = aS_w^{-1}$. It follows that $ea\nu S_a = ae\nu S_a$, since:

$$\begin{aligned} ea\nu S_a = ea\nu a\nu &= e \quad , && \text{by (5') and our choice of } t, w, w' \\ &= eK \quad , && \text{by (7)} \\ &= eT_a S_a , && \text{since } K = T_e S_e = T_a S_a \\ &= ae\nu S_a . \end{aligned}$$

By the bijectiveness of S_a we obtain $ae\nu = ea\nu$, that is $T_e = S_e$, which is (8). The proof now proceeds as in [4], whereas A is a group with $xy\nu = x^{-1}.y^{-1}$, $\varepsilon = e$ the unity. To prove $A \models w = \varepsilon$, set $x_i' = e$, $t = e$; (5') then implies (by $ee\nu = e$):

$$e = eew\nu\nu ee\nu\nu = eew\nu\nu e\nu = w^{-1}, \text{ thus } w = e.$$

(ii) is easily seen to hold in any group which satisfies $w = e$, with this interpretation; this completes the proof. ∎

4. For the case of Ω-groups, the following is true (no proof will be given—it uses arguments similar to those in proofs of Theorem 1 and Theorem 2.)

THEOREM 4. *Any mononomic variety of (Ω, λ)-groups, such that nontrivial conditions are set on operators from Ω, is definable by $|\Omega| + 1$ laws of language $((\Omega, \lambda), \rho)$ with $xy\rho = x.y^{-1}$, where $|\Omega|$ is the number of operators in Ω. The condition is said to be trivial if it is of the form $e \dots e\omega = e$.*

Clearly, $|\Omega|$ laws are of the form $xx\rho \dots xx\rho\omega = xx\rho$ for $\omega \in \Omega$, and the remaining one defines ρ, λ and assures that the nontrivial laws for operators from Ω hold. The last law is constructed as in Theorem 1. or Theorem 2. In particular, for mononomic varieties of rings we have (by putting $\Omega = (\pi)$ of type (2), λ-the empty word, in Theorem 4.):

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Corollary 1. *A mononomic variety of rings defined by* $w' = 0$ *is defined by* (ρ, π) *laws:*

$$xx\rho xx\rho\pi = xx\rho \quad (R1)$$

$$xxx\rho w'jkk\rho k\rho\rho kjj\rho j\rho\rho\rho cab\rho\pi ca\pi cb\pi\rho\rho de\rho f\pi df\pi$$
$$\to ef\pi\rho\rho ghi\pi\pi gh\pi i\pi\rho\rho\rho\rho\rho\rho y\rho z\rho xx\rho x\rho z\rho\rho\rho = y \quad (R2)$$

where $xy\rho = x + (-y), xy\pi = x.y$.

A similar result has been announced in [5]: namely, it is easily seen that Theorem 1. of [5] is closely connected to our results. In particular, it yields a somewhat weaker (3 laws) result for the case of rings. However, the assertions that have been made in [5] have not received a published proof, as far as I know; also, the ring-laws (in fact, in [5] it was asserted that if $(R1)$ and another law hold then rings are single-law definable) were not given explicitly. Theorem 3. of [5] can be sharpened, too:

Corollary 2. *A mononomic variety of rings with unity which is defined by* $u = 0$, *is defined by* (ρ, π, ε) *laws:*

$$xxx\rho\pi = xx\rho \quad (RU1)$$

$$xx\rho\varepsilon\pi = xx\rho \quad (RU2)$$

and law (RU3), where (RU3) is the same as (R2) but with $w' = ut\varepsilon\pi t\rho\varepsilon s\pi s\rho\rho\rho$, *where* $xy\rho = x + (-y), xy\pi = x.y\varepsilon$ *and is the (multiplicative) identity.*

Proof: Let $A = (A, \rho, \pi, \varepsilon) \models (RU1)\&(RU2)\&(RU3)$. Then $(RU3)$ assures that (A, ρ) is a group in which $w = 0$, where w is the term which consists of the first 57 symbols following $xxx\rho$ in (R2) (the reader should note that we defined w in such a way that $(RU3)$ reduces to (i)). Put $v = 0$ for every $v \in [a, k]$, the closed interval of the alphabet. Then by (RU1) $00\pi = 0$, and by $00\rho = 0$, it follows $w' = 0$. Now set $t = 0, s = 0$; thus by (RU2) $0\varepsilon\pi = 0$, and by $(RU1)\,\varepsilon 0\pi = 0$, and hence we have:

$$0 = w' = u0\varepsilon\pi 0\rho\varepsilon 0\pi 0\rho\rho\rho = u00\rho 00\rho\rho\rho = u00\rho\rho = u0\rho = u.$$

Now $w' = 0$ yields $t\varepsilon\pi t\rho\varepsilon s\pi s\rho\rho = 0$, and therefore by $(RU1)$ putting $s = 0$ implies $t\varepsilon\pi t\rho = 0$, that is

$$t\varepsilon\pi = t.$$

It easily follows that $\varepsilon s\pi = s$ holds, too. Since $w' = 0(RU3)$ reduces to $(R2)$, thus $A \models (R1)\&(R2)$ and consequently (A, ρ, π) is a ring. By the above observations ε is the unity of this ring, and A belongs to the variety defined by $u = 0$. Laws

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(RU1)-(RU3) hold in any ring with unity with this interpretation, in which $u = 0$ holds. ∎

5. I do not know whether it is possible to improve Theorem 4.; another question is whether it is possible to define groups by a single law in language (ν, ε, π) with $xy\nu = x^{-1}.y^{-1}$, ε the identity and with π as some single-law-describable operation. It would suffice to prove that w' from Theorem 3. attains the value e for some valuation, without referring to operation symbols occuring in w'.

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