On a Difference Equation due to Stirling.

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1. *Introduction.*

In 1730 there was published Stirling's *Methodus Differentialis*, and in it (Prop. VIII., p. 44) he considers the Difference Equation

$$y(z) - \frac{z-m}{z} y(z+1) = \frac{1}{z-n}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and shews that it is satisfied by an inverse factorial series

$$y_1(z) = \frac{1}{m} + \frac{n}{z} \frac{1}{m+1} + \frac{n(n+1)}{z(z+1)} \frac{1}{m+2} + \frac{n(n+1)(n+2)}{z(z+1)(z+2)} \frac{1}{m+3} + \ldots \quad (2)$$

This result of Stirling's is the starting-point of the present paper: it will be convenient to modify his equations thus:

Taking

$$\frac{n-1}{z-1} y(z) = u(z)$$

and $\quad u(z+1, n+1) = v(z)$

and making the necessary changes in (1) and (2), we obtain the result that the Difference Equation

$$z\,v(z) - (z-m+1)\,v(z+1) = \frac{n}{z-n} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

is satisfied by

$$v_1(z) = \frac{n}{z}\frac{1}{m} + \frac{n(n+1)}{z(z+1)} \frac{1}{m+1} + \frac{n(n+1)(n+2)}{z(z+1)(z+2)} \frac{1}{m+2} + \ldots \quad \ldots (4)$$

We shall consider the difference equation in this latter form; its general solution is

$$v(z) = C \frac{\Gamma(z)}{\Gamma(z-m+1)} - \frac{\Gamma(z)}{\Gamma(z-m+1)} \sum_z \frac{\Gamma(z-m+1)}{\Gamma(z+1)} \frac{n}{z-n}, \quad \ldots (5)$$

where Σ, as usual, denotes the operation inverse to Δ; and C is a quantity independent of z.

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We shall prove that $v_1(z)$ and other solutions of the difference-equation (3) are also solutions of difference-equations with respect to the variables m, n, and we shall find relations connecting the various solutions.

2. *Difference Equation in m.*

From (5) we have

$$v(z) = C\,\frac{\Gamma(z)}{\Gamma(z-m+1)} - \frac{\Gamma(z)}{\Gamma(z-m+1)}\sum_z \frac{\Gamma(z-m+1)}{\Gamma(z+m)}\,\frac{z}{z-n}.$$

Let us now consider this as a function of m and denote it by $\theta(m)$, supposing for the present that C is independent of m,

i.e. $$\theta(m) = C\,\frac{\Gamma(z)}{\Gamma(z-m+1)} - \frac{\Gamma(z)}{\Gamma(z-m+1)}\sum_z \frac{\Gamma(z-m+1)}{\Gamma(z+1)}\,\frac{n}{z-n}.$$

Then

$$\theta(m+1) = C\,\frac{\Gamma(z)}{\Gamma(z-m)} - \frac{\Gamma(z)}{\Gamma(z-m)}\sum_z \frac{\Gamma(z-m)}{\Gamma(z+m)}\,\frac{n}{z-n}.$$

Therefore we have

$$(z-m)\,\theta(m) - (n-m)\,\theta(m+1)$$

$$= C\,\frac{\Gamma(z)}{\Gamma(z-m)}(1-n+m) - \frac{\Gamma(z)}{\Gamma(z-m)}$$

$$\sum_z \left\{\frac{\Gamma(z-m+1)}{\Gamma(z+1)} - (n-m)\frac{\Gamma(z-m)}{\Gamma(z+1)}\right\}\frac{n}{z-n}$$

$$= C\,\frac{\Gamma(z)}{\Gamma(z-m)}(1-n+m) - \frac{\Gamma(z)}{\Gamma(z-m)}\sum_z \frac{\Gamma(z-m)}{\Gamma(z+1)}\left\{\overline{z-m} - \overline{n-m}\right\}\frac{n}{z-n}$$

$$= C\,\frac{\Gamma(z)}{\Gamma(z-m)}(1-n+m) - \frac{n\,\Gamma(z)}{\Gamma(z-m)}\sum_z \frac{\Gamma(z-m)}{\Gamma(z+1)}\,\frac{z-n}{z-n}$$

$$= C\,\frac{\Gamma(z)}{\Gamma(z-m)}(1-n+m) - \frac{\Gamma(z)\,n}{\Gamma(z-m)}\sum_z \frac{\Gamma(z-m)}{\Gamma(z-1)}.$$

Now consider

$$\underset{z}{\Delta}\,\frac{\Gamma(z-m)}{\Gamma(z)} = \frac{\Gamma(z-m+1)}{\Gamma(z+1)} - \frac{\Gamma(z-m)}{\Gamma(z)}$$

$$= \frac{\Gamma(z-m)}{\Gamma(z+1)}\,(z-m-z).$$

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Applying the operator Σ,

$$\frac{\Gamma(z-m)}{\Gamma(z)} = -m\sum_z \frac{\Gamma(z-m)}{\Gamma(z+1)},$$

$\therefore$ we have

$$(z-m)\,\theta(m)-(n-m)\,\theta(m+1)$$

$$= C\frac{\Gamma(z)}{\Gamma(z-m)}(1-n+m)+\frac{n}{m}\frac{\Gamma(z)}{\Gamma(z-m)}\frac{\Gamma(z-m)}{\Gamma(z)}$$

$$= C\frac{\Gamma(z)}{\Gamma(z-m)}(1-n+m)+\frac{n}{m},$$

which is a difference equation in m.

i.e. Every particular solution of the difference equation (3) *in z is a solution of the difference equation in m,*

$$(m-z)\,\theta(m)-(m-n)\,\theta(m+1) = -C\frac{\Gamma(z)}{\Gamma(z-m)}(1-n+m)-\frac{n}{m}, \quad (6)$$

where C is the arbitrary constant in that particular solution.

We now require to find the difference equation in m which is satisfied by the particular solution (4) of the original difference equation

$$\theta(m)=v_1(z)=\frac{n}{z}\frac{1}{m}+\frac{n(n+1)}{z(z+1)}\frac{1}{m+1}+\frac{n(n+1)(n+2)}{z(z+1)(z+2)}\frac{1}{m+2}+\ldots$$

$$\therefore\ \theta(m+1)=\frac{n}{z}\frac{1}{m+1}+\frac{n(n+1)}{z(z+1)}\frac{1}{m+2}+\frac{n(n+1)(n+2)}{z(z+1)(z+3)}\frac{1}{m+3}+\ldots$$

$$\therefore\ (m-z)\,\theta(m)=\frac{n}{z}+\frac{n(n+1)}{z(z+1)}+\frac{n(n+1)(n+2)}{z(z+1)(z+2)}+\ldots$$

$$-\frac{n}{m}-\frac{n(n+1)}{z}\frac{1}{m+1}-\frac{n(n+1)(n+2)}{z(z+1)}\frac{1}{m+2}-\ldots$$

$$\therefore\ (m-n)\,\theta(m+1)=\frac{n}{z}+\frac{n(n+1)}{z(z+1)}+\ldots$$

$$-\frac{n(n+1)}{z}\frac{1}{m+1}-\frac{n(n+1)(n+2)}{z(z+1)}\frac{1}{m+2}-\ldots.$$

Substituting in (6) we have

$$-C\frac{\Gamma(z)}{\Gamma(z-m)}(1-n+m)-\frac{n}{m}=(m-z)\,\theta(m)-(m-n)\,\theta(m+1)$$

$$=-\frac{n}{m},$$

$$\therefore \quad C=0,$$

i.e. $v_1(z)$ *is a solution of the difference equation*

$$(m-z)\,\theta(m)-(m-n)\,\theta(m+1)=-\frac{n}{m}. \qquad \ldots\ldots\ldots (7)$$

3. *Solution of the equation*

$$(m-z)\,\theta(m)-(m-n)\,\theta(m+1)=-\frac{n}{m}.$$

Assume as a solution of this an inverse factorial series in m.

$$\theta(m)=a_0+\frac{a_1}{m}+\frac{a_2}{m(m+1)}+\frac{a_3}{m(m+1)(m+2)}+\ldots.$$

$$\therefore\ \theta(m+1)=a_0+\frac{a_1}{m+1}+\frac{a_2}{(m+1)(m+2)}+\frac{a_3}{(m+1)(m+2)(m+3)}+\ldots$$

$$\therefore\ (m-z)\,\theta(m)=(m-z)\,a_0+a_1+\frac{a_2}{m+1}+\frac{a_3}{(m+1)(m+2)}+\ldots$$

$$-\frac{z\,a_1}{m}-\frac{z\,a_2}{m(m+1)}-\frac{z\,a_3}{m(m+1)(m+2)}-\ldots$$

$$\therefore\ (m-n)\,\theta(m+1)=(m-n)\,a_0+a_1+\frac{a_2}{m+1}+\frac{a_3}{(m+1)(m+2)}+\ldots$$

$$-\frac{(n+1)\,a_1}{m+1}-\frac{(n+2)\,a_2}{(m+1)(m+2)}\,\frac{(n+3)\,a_3}{(m+1)(m+2)(m+3)}+\ldots$$

$$\therefore\ -\frac{n}{m}=(m-z)\,\theta(m)-(m-n)\,\theta(m+1)$$

$$=(n-z)\,a_0+\left(\frac{n+1}{m+1}-\frac{z}{m}\right)a_1+\left(\frac{n+2}{m+2}-\frac{z}{m}\right)\frac{a_2}{m+1}$$

$$+\left(\frac{n+3}{m+3}-\frac{z}{m}\right)\frac{a_3}{(m+1)(m+2)}+\ldots$$

$$=(n-z)\,a_0+\frac{(n-z+1)(m+1)-(n+1)}{m(m+1)}a_1+\frac{(n-z+2)(m+2)-2(n+2)}{m(m+1)(m+2)}a_2$$

$$+\frac{(n-z+3)(m+3)-3(n+3)}{m(m+1)(m+2)(m+3)}a_3+\ldots$$

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$$= (n-z)\,a_0 + \frac{(n-z+1)\,a_1}{m} + \frac{(n-z+2)\,a_2 - 1\,(n+1)\,a_1}{m\,(m+1)}$$

$$+ \frac{(n-z+3)\,a_3 - 2\,(n+2)\,a_2}{m\,(m+1)\,(m+2)} + \frac{(n-z+4)\,a_4 - 3\,(n+3)\,a_3}{m\,(m+1)\,(m+2)\,(m+3)} + \dots .$$

Equating coefficients of $\frac{1}{m}$, $\frac{1}{m\,(m+1)}$, ..., we obtain

$$a_0 = 0$$

$$a_1 = -\frac{n}{n-z+1}$$

$$a_2 = \frac{1\,.\,(n+1)}{n-z+2}\,a_1 = -\frac{1\,.\,n\,(n+1)}{(n-z+1)\,(n-z+2)}$$

$$a_3 = \frac{2\,(n+2)}{n-z+3}\,a_2 = -\frac{1\,.\,2\,n\,(n+1)\,(n+2)}{(n-z+1)\,(n-z+2)\,(n-z-3)}$$

$$a_4 = \frac{3\,(n+3)}{n-z+4}\,a_3 = -\frac{1\,.\,2\,.\,3\,n\,(n+1)\,(n+2)\,(n+3)}{(n-z+1)\,(n-z+2)\,(n-z+3)\,(n-z+4)}$$

....

∴ We have

$$\theta_1(m) = -\frac{n}{m}\,\frac{1}{n-z+1} - \frac{n\,(n+1)}{m\,(m+1)}\,\frac{1\,!}{(n-z+1)\,(n-z+2)}$$

$$-\frac{n\,(n+1)\,(n+2)}{m\,(m+1)\,(m+2)}\,\frac{2\,!}{(n-z+1)\,(n-z+2)\,(n-z+3)} - \dots \quad (8)$$

is a solution of the difference equation in m.

We can also find the *general* solution of the equation

$$(m-z)\,\theta(m) - (m-n)\,\theta(m+1) = -\frac{n}{m}.$$

First suppose that the term $-\frac{n}{m}$ is absent, then

$$\theta(m+1) = \frac{m-z}{m-n}\,\theta(m),$$

a solution of which is obviously

$$\theta(m) = \alpha\,\frac{\Gamma(m-z)}{\Gamma(m-n)},$$

where α is an arbitrary constant.

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Now assume

$$\theta(m) = u(m) \frac{\Gamma(m-z)}{\Gamma(m-n)}$$

$$\therefore \quad \theta(m+1) = u(m+1) \frac{(m-z)\,\Gamma(m-z)}{(m-n)\,\Gamma(m-n)}$$

$$\therefore \quad \frac{\Gamma(m-z+1)}{\Gamma(m-n)} \{u(m) - u(m+1)\} = -\frac{n}{m}$$

$$\therefore \quad \Delta u(m) = \frac{n}{m} \frac{\Gamma(m-n)}{\Gamma(m-z+1)}$$

$$\therefore \quad u(m) = C' + \sum_m \frac{n}{m} \frac{\Gamma(m-n)}{\Gamma(m-z+1)}$$

where C' is an arbitrary constant.

$$\therefore \quad \theta(m) = C' \frac{\Gamma(m-z)}{\Gamma(m-n)} + \frac{\Gamma(m-z)}{\Gamma(m-n)} \sum_m \frac{n}{m} \frac{\Gamma(m-n)}{\Gamma(m-n+1)}, \quad \ldots\ldots (9)$$

which is the general solution of the difference equation in m.

4. If now we consider $\theta(m)$, as given by (9), as a function of z, we can find a difference equation involving the arbitrary constant C' in z, which it must satisfy. Then, exactly as in § 2, we find that the difference equation in z which is satisfied by the series (8) is

$$z\,v(z) - (z-m+1)\,v(z+1) = \frac{n}{z+n},$$

which is the original difference equation again; *i.e.* $\theta_1(m)$ is a solution of both

$$z\,v(z) - (z-m+1)\,v(z+1) = \frac{n}{z-n}$$

and $$(m-z)\,\theta(m) - (m-n)\,\theta(m+1) = -\frac{n}{m}.$$

$\therefore$ We see that *both* $v_1(z)$ *and* $\theta_1(m)$ *are solutions of each of the two equations*

$$z\,v(z) - (z-m+1)\,v(z+1) = \frac{n}{z-n}$$

and $$(m-z)\,\theta(m) - (m-n)\,\theta\,m+1) = -\frac{n}{m}.$$

5. *Difference Equation in n.*

We have

$$v(z) = C\frac{\Gamma(z)}{\Gamma(z-m+1)} - \frac{\Gamma(z)}{\Gamma(z-m-1)}\sum_z \frac{\Gamma(z-m+1)}{\Gamma(z+1)}\frac{n}{z-n}.$$

Let us consider this as a function of n and denote it by $\phi(n)$, assuming for the present that C is independent of n,

$$i.e.\quad \phi(n) = C\frac{\Gamma(z)}{\Gamma(z-m+1)} - \frac{\Gamma(z)}{\Gamma(z-m+1)}\sum_z \frac{\Gamma(z-m+1)}{\Gamma(z+1)}\frac{n}{z-n}.$$

We could then find a difference equation involving $\phi(n)$, but it is simpler to deal with the function

$$\psi(n) = \frac{\phi(n)}{n},$$

$$i.e.\quad \psi(n) = C\frac{1}{n}\frac{\Gamma(z)}{\Gamma(z-m+1)} - \frac{\Gamma(z)}{\Gamma(z-m+1)}\sum_z \frac{\Gamma(z-m+1)}{\Gamma(z+1)}\frac{1}{z-n}.$$

$$\therefore\ \psi(n+1) = C\frac{1}{1+n}\frac{\Gamma(z)}{\Gamma(z-m+1)} - \frac{\Gamma(z)}{\Gamma(z-m+1)}\sum_z \frac{\Gamma(z-m+1)}{\Gamma(z+1)}\frac{1}{z-n-1}$$

We shall now find a difference equation involving $\psi(n)$.

Consider

$$\underset{z}{\Delta}\frac{\Gamma(z-m+1)}{\Gamma(z+1)}\frac{1}{z-n-1}$$

$$= \frac{\Gamma(z-m+2)}{\Gamma(z+1)}\frac{1}{z-n} - \frac{\Gamma(z-m+1)}{\Gamma(z)}\frac{1}{z-n-1}$$

$$= \frac{\Gamma(z-m+1)}{\Gamma(z+1)}\left\{\frac{z-m+1}{z-n} - \frac{z}{z-n-1}\right\}$$

$$= \frac{\Gamma(z-m+1)}{\Gamma(z+1)}\left\{\frac{n-m+1}{z-n} - \frac{n+1}{z-n-1}\right\}.$$

$$\therefore\quad \frac{\Gamma(z-m+1)}{\Gamma(z)}\frac{1}{z-n-1} = \sum_z \frac{\Gamma(z-m+1)}{\Gamma(z+1)}\left\{\frac{n-m+1}{z-n} - \frac{n+1}{z-n-1}\right\}$$

$$\therefore\quad \frac{1}{z-n-1}\frac{\Gamma(z)}{\Gamma(z-m+1)} = \sum_z \frac{\Gamma(z-m+1)}{\Gamma(z+1)}\left\{\frac{n-m+1}{z-n} - \frac{n+1}{z-n-1}\right\}$$

$$= -(n-m+1)\psi(n) + \frac{n-m+1}{n}C\frac{\Gamma(z)}{\Gamma(z-m+1)}$$

$$+ (n+1)\psi(n+1) - C\frac{\Gamma(z)}{\Gamma(z-m+1}.$$

$$\therefore (n-m+1)\,\psi(n)-(n+1)\,\psi(n+1)=-\frac{m-1}{n}\,C\,\frac{\Gamma(z)}{\Gamma(z-m+1)}+\frac{1}{n-z-1}$$

If now we wish to find the difference equation which is satisfied by the particular solution $v_1(z)$, we have only to substitute the series

$$\frac{1}{z}\,\frac{1}{m}+\frac{n+1}{z(z+1)}\,\frac{1}{m+1}+\frac{(n+1)(n+2)}{z(z+1)(z+2)}\,\frac{1}{m+2}+\dots$$

for $\psi(n)$ in this equation, and we find immediately that $C=0$.

Therefore the required difference equation in n is

$$(n-m+1)\,\psi(n)-(n+1)\,\psi(n+1)=\frac{1}{n-z+1}. \quad\text{......(10)}$$

6. *Solution of the equation*

$$(n-m+1)\,\psi(n)-(n+1)\,\psi(n+1)=\frac{1}{n-z+1}.$$

Assume as a solution a series of inverse factorials in $n-z+1$,

i.e. $$\psi(n)=a_0+\frac{a_1}{n-z+1}+\frac{a_2}{(n-z+1)(n-z+2)}+\dots$$

$$\therefore\quad \psi(n+1)=a_0+\frac{a_1}{n-z+2}+\frac{a_2}{(n-z+2)(n-z+3)}+\dots$$

$$\therefore\quad (n-m+1)\,\psi(n)$$

$$=(n+1)\,a_0+a_1+\frac{a_2}{n-z+2}+\frac{a_3}{(n-z+2)(n-z+3)}+\dots$$

$$-m\,a_0+\frac{z-m}{n-z+1}\,a_1+\frac{(z-m)\,a_2}{(n-z+1)(n-z+2)}+\frac{(z-m)\,a_3}{(n-z+1)(n-z+2)(n-z+3)}+\dots$$

$$(n+1)\,\psi(n+1)$$

$$=(n+1)\,a_0+a_1+\frac{a_2}{n-z+2}+\frac{a_3}{(n-z+2)(n-z+3)}+\dots$$

$$+\frac{(z-1)\,a_1}{n-z+2}+\frac{(z-2)\,a_2}{(n-z+2)(n-z+3)}+\frac{(z-3)\,a_3}{(n-z+2)(n-z+3)(n-z+4)}+\dots$$

$$\therefore \quad \frac{1}{n-z+1} = (n-m+1)\,\psi(n) - (n+1)\,\psi(n+1)$$

$$= -m a_0 + \left(\frac{z-m}{n-z+1} - \frac{z-1}{n-z+2}\right) a_1$$

$$+ \left(\frac{z-m}{n-z+1} - \frac{z-2}{n-z+3}\right) \frac{a_2}{n-z+2}$$

$$+ \left(\frac{z-m}{n-z+1} - \frac{z-3}{n-z+4}\right) \frac{a_3}{(n-z+2)(n-z+3)}$$

$$+ \cdots\cdots\cdots$$

$$= -m a_0 + \frac{-(m-1)(n-z+2)+z-1}{(n-z+1)(n-z+2)} a_1$$

$$+ \frac{-(m-2)(n-z+3)+2(z-2)}{(n-z+1)(n-z+2)(n-z+3)} a_2$$

$$+ \frac{-(m-3)(n-z+4)+3(z-3)}{(n-z+1)(n-z+2)(n-z+3)(n-z+4)} a_3 + \ldots$$

$$= -m a_0 - \frac{(m-1)\,a_1}{n-z+1} - \frac{(m-2)\,a_2 - (z-1)\,a_1}{(n-z+1)(n-z+2)}$$

$$- \frac{(m-3)\,a_3 - 2(z-2)\,a_2}{(n-z+1)(n-z+2)(n-z+3)}$$

$$- \frac{(m-4)\,a_4 - 3(z-3)}{(n-z+1)(n-z+2)(n-z+3)(n-z+4)} + \ldots$$

$\therefore$ Equating coefficients of $\dfrac{1}{n-z+1}$, $\dfrac{1}{(n-z+1)(n-z+2)}$, $\ldots$,

we have

$$a_0 = 0$$

$$a_1 = -\frac{1}{m-1}$$

$$a_2 = \frac{z-1}{m-2}\,a_1 = -\frac{z-1}{(m-1)(m-2)}$$

$$a_3 = \frac{2(z-2)}{m-3}\,a_2 = \frac{1\,.\,2\,(z-1)(z-2)}{(m-1)(m-2)(m-3)}$$

$$a_4 = \frac{3(z-3)}{m-4}\,a_3 = \frac{1\,.\,2\,.\,3\,.\,(z-1)(z-2)(z-3)}{(m-1)(m-2)(m-3)(m-4)}$$

$$\cdots\cdots\cdots$$

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$$\therefore\ \psi_1(n) = -\frac{1}{m-1}\frac{1}{n-z+1} - \frac{z-1}{(m-1)(m-2)}\frac{1!}{(n-z+1)(n-z+2)}$$
$$-\frac{(z-1)(z-2)}{(m-1)(m-2)(m-3)}\frac{2!}{(n-z+1)(n-z+2)(n-z+3)}$$
$$-\frac{(z-1)(z-2)(z-3)}{(m-1)(m-2)(m-3)(m-4)}\frac{3!}{(n-z+1)(n-z+2)(n-z+3)(n-z+4)}$$
$$-\ \ldots\ldots$$

$$\therefore\ \phi_1(n) = n\,\psi_1(n)$$
$$= -\frac{n}{m-1}\frac{1}{n-z+1} - \frac{n(z-1)}{(m-1)(m-2)}\frac{1!}{(n-z+1)(n-z+2)}$$
$$-\frac{n(z-1)(z-2)}{(m-1)(m-2)(m-3)}\frac{2!}{(n-z+1)(n-z+2)(n-z+3)}$$
$$-\frac{n(z-1)(z-2)(z-3)}{(m-1)(m-2)(m-3)(m-4)}\frac{3!}{(n-z+1)(n-z+2)(n-z+3)(n-z+4)}$$
$$-\ \ldots\ldots \quad (11)$$

To find the general solution of the equation we first assume that the term $\dfrac{1}{n-z+1}$ is absent,

then
$$\psi(n+1) = \frac{n-m+1}{n+1}\psi(n),$$
a solution of which is
$$\psi(n) = \beta\,\frac{\Gamma(n-m+1)}{\Gamma(n+1)}$$
where β is an arbitrary constant.

Now assume that $\psi(n) = v(n)\dfrac{\Gamma(n-m+1)}{\Gamma(n+1)}$,

$$\therefore\ \psi(n+1) = v(n+1)\frac{(n-m+1)\,\Gamma(n-m+1)}{(n+1)\,\Gamma(n+1)}.$$

$$\therefore\ \frac{1}{n-z+1} = (n-m+1)\,\psi(n) - (n+1)\,\psi(n+1)$$
$$= \frac{\Gamma(n-m+2)}{\Gamma(n+1)}\{v(n) - v(n+1)\}.$$

$$\therefore\ \Delta v(n) = -\frac{\Gamma(n+1)}{\Gamma(n-m+2)}\frac{1}{n-z+1}.$$

$$\therefore\ v(n) = C'' - \sum_n \frac{\Gamma(n+1)}{\Gamma(n-m+2)}\frac{1}{n-z+1},$$
where C'' is an arbitrary constant.

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$$\therefore\ \psi(n) = C''\frac{\Gamma(n-m+1)}{\Gamma(n+1)} - \frac{\Gamma(n-m+1)}{\Gamma(n+1)}\sum_n \frac{\Gamma(n+1)}{\Gamma(n-m+2)}\,\frac{1}{n-z+1}$$

$$\therefore\ \phi(n) = n\,\psi(n)$$

$$= C''\frac{\Gamma(n-m+1)}{\Gamma(n+1)} - \frac{\Gamma(n-m+1)}{\Gamma(n)}\sum_n \frac{\Gamma(n+1)}{\Gamma(n-m+2)}\,\frac{1}{n-z+1}. \quad (12)$$

7. **Now**

$$\sum_t f(t) = \text{constant} + f(t-1) + f(t-2) + f(t-3)\ldots$$
$$= \text{constant} - f(t) - f(t+1) - f(t+2)\ldots\ldots .$$

Therefore equation (5) gives

$$v(z) = C_1\frac{\Gamma(z)}{\Gamma(z-m+1)} - \frac{\Gamma(z)\,n}{\Gamma(z-m+1)}\left\{\frac{\Gamma(z-m)}{\Gamma(z)}\,\frac{1}{z-n-1}\right.$$
$$\left. + \frac{\Gamma(z-m-1)}{\Gamma(z-1)}\,\frac{1}{z-n-2} + \frac{\Gamma(z-m-2)}{\Gamma(z-2)}\,\frac{1}{z-n-3} + \ldots\right\}$$

$$= C_1\frac{\Gamma(z)}{\Gamma(z-m+1)} - \left\{\frac{1}{z-m}\,\frac{n}{z-n-1} + \frac{z-1}{(z-m)(z-m-1)}\,\frac{n}{z-n-2}\right.$$
$$\left. + \frac{(z-1)(z-2)}{(z-m)(z-m-1)(z-m-2)}\,\frac{n}{z-n-3} + \ldots\right\}$$

and

$$v(z) = C_2\frac{\Gamma(z)}{\Gamma(z-m+1)} + \frac{n\,\Gamma(z)}{\Gamma(z-m+1)}\left\{\frac{\Gamma(z-m+1)}{\Gamma(z+1)}\,\frac{1}{z-n}\right.$$
$$\left. + \frac{\Gamma(z-m+2)}{\Gamma(z+2)}\,\frac{1}{z-n+1} + \frac{\Gamma(z-m+3)}{\Gamma(z+3)}\,\frac{1}{z-n+2} + \ldots\right\}$$

$$= C_2\frac{\Gamma(z)}{\Gamma(z-m+1)} + \frac{1}{z}\,\frac{n}{z-n} + \frac{z-m+1}{z(z+1)}\,\frac{n}{z-n+1}$$
$$+ \frac{(z-m+1)(z-m+2)}{z(z+1)(z+2)}\,\frac{n}{z-n+2} + \ldots .$$

Also equation (9) gives

$$\theta(m) = C_1'\frac{\Gamma(m-z)}{\Gamma(m-n)} + \frac{\Gamma(m-z)}{\Gamma(m-n)}\,n\left\{\frac{\Gamma(m-n-1)}{\Gamma(m-z)}\,\frac{1}{m-1}\right.$$
$$\left. + \frac{\Gamma(m-n-2)}{\Gamma(m-z-1)}\,\frac{1}{m-2} + \frac{\Gamma(m-n-3)}{\Gamma(m-z-2)}\,\frac{1}{m-3} + \ldots\right\}$$

$$= C_1'\frac{\Gamma(m-z)}{\Gamma(m-n)} + \frac{1}{m-n-1}\,\frac{n}{m-1} + \frac{(m-z-1)}{(m-n-1)(m-n-2)}\,\frac{n}{m-2}$$
$$\left. + \frac{(m-z-1)(m-z-2)}{(m-n-1)(m-n-2)(m-n-3)}\,\frac{n}{m-3} + \ldots\ldots\right\}$$

and

$$\theta(m) = C_2' \frac{\Gamma(m-z)}{\Gamma(m-n)} - \frac{\Gamma(m-z)}{\Gamma(m-n)} n \left\{ \frac{\Gamma(m-n)}{\Gamma(m-z+1)} \frac{1}{m} \right.$$

$$\left. + \frac{\Gamma(m-n+1)}{\Gamma(m-z+2)} \frac{1}{m+1} + \frac{\Gamma(m-n+2)}{\Gamma(m-z+3)} \frac{1}{m+2} + \ldots\ldots \right\}$$

$$= C_2' \frac{\Gamma(m-z)}{\Gamma(m-n)} - \left\{ \frac{1}{m-z} \frac{n}{m} + \frac{m-n}{(m-z)(m-z+1)} \frac{n}{m+1} \right.$$

$$\left. + \frac{(m-n)(m-n+1)}{(m-z)(m-z+1)(m-z+2)} \frac{n}{m+2} + \ldots\ldots \right\}$$

Also equation (12) gives

$$\phi(n) = C_1'' \frac{\Gamma(n-m+1)}{\Gamma(n)} - \frac{\Gamma(n-m+1)}{\Gamma(n)} \left\{ \frac{\Gamma(n)}{\Gamma(n-m+1)} \frac{1}{n-z} \right.$$

$$\left. + \frac{\Gamma(n+1)}{\Gamma(n-m)} \frac{1}{n-z-1} + \frac{\Gamma(n-2)}{\Gamma(n-m-1)} \frac{1}{n-z-2} + \ldots\ldots \right\}$$

$$= C_1'' \frac{\Gamma(n-m+1)}{\Gamma(n)} - \left\{ \frac{1}{n-z} + \frac{n-m}{n-1} \frac{1}{n-z-1} \right.$$

$$\left. + \frac{(n-m)(n-m-1)}{(n-1)(n-2)} \frac{1}{n-z-2} + \ldots\ldots \right\}$$

and

$$\phi(n) = C_2'' \frac{\Gamma(n-m+1)}{\Gamma(n)} + \frac{\Gamma(n-m+1)}{\Gamma(n)} \left\{ \frac{\Gamma(n+1)}{\Gamma(n-m+2)} \frac{1}{n-z+1} \right.$$

$$\left. + \frac{\Gamma(n+2)}{\Gamma(n-m+3)} \frac{1}{n-z+2} + \frac{\Gamma(n+3)}{\Gamma(n-m+4)} \frac{1}{n-z+3} + \ldots\ldots \right\}$$

$$= C_2'' \frac{\Gamma(n-m+1)}{\Gamma(n)} + \frac{n}{n-m+1} \frac{1}{n-z+1} + \frac{n(n+1)}{(n-m+1)(n-m+2)} \frac{1}{n-z+2}$$

$$+ \frac{n(n+1)(n+2)}{(n-m+1)(n-m+2)(n-m+3)} \frac{1}{n-z+3} + \ldots\ldots .$$

Dropping the arbitrary constant in each case, we obtain from these and the series 4, 8, 11, the following nine series :—

$$
\left.
\begin{aligned}
v_1(z) &= \frac{n}{z}\,\frac{1}{m} + \frac{n(n+1)}{z(z+1)}\,\frac{1}{m+1} + \frac{n(n+1)(n+2)}{z(z+1)(z+2)}\,\frac{1}{m+2} + \dots \quad \dots \\
v_2(z) &= \frac{1}{z-m}\,\frac{n}{n-z+1} + \frac{z-1}{(z-m)(z-m-1)}\,\frac{n}{n-z+2} + \frac{(z-1)(z-2)}{(z-m)(z-m-1)(z-m-2)}\,\frac{n}{n-z+3} + \dots \quad \dots \\
v_3(z) &= \frac{1}{z}\,\frac{n}{z-n} + \frac{(z-m+1)}{z(z+1)}\,\frac{n}{z-n+1}\,\frac{(z-m+1)(z-m+2)}{z(z+1)(z+2)}\,\frac{n}{z-n+2} + \dots \quad \dots \\
\theta_1(m) &= -\frac{n}{m}\,\frac{1}{n-z+1} - \frac{n(n+1)}{m(m+1)}\,\frac{1\,!}{(n-z+1)(n-z+2)} - \frac{n(n+1)(n+3)}{m(m+1)(m+2)}\,\frac{2\,!}{(n-z+1)(n-z+2)(n-z+3)} + \dots \quad \dots \\
\theta_2(m) &= \frac{1}{m-n-1}\,\frac{n}{m-1} + \frac{m-z-1}{(m-n-1)(m-n-2)}\,\frac{n}{m-2} + \frac{(m-z-1)(m-z-2)}{(m-n-1)(m-n-2)(m-n-3)}\,\frac{n}{m-3} + \dots \quad \dots \\
\theta_3(m) &= -\frac{1}{m-z}\,\frac{n}{m} - \frac{m-n}{(m-z)(m-z)(m-z+1)}\,\frac{n}{m+1} - \frac{(m-n)(m-n+1)}{(m-z)(m-z+1)(m-z+2)}\,\frac{n}{m+2} + \dots \quad \dots \\
\phi_1(n) &= -\frac{n}{m-1}\,\frac{1}{n-z+1} - \frac{n(z-1)}{(m-1)(m-2)}\,\frac{1\,!}{(n-z+1)(n-z+2)} - \frac{n(z-1)(z-2)}{(m-1)(m-2)(m-3)}\,\frac{2\,!}{(n-z+1)(n-z+2)(n-z+3)} + \dots \\
\phi_2(n) &= \frac{1}{z-n} + \frac{n-m}{n-1}\,\frac{1}{z-n+1} + \frac{(n-m)(n-m-1)}{(n-1)(n-2)}\,\frac{1}{z-n+2} + \dots \quad \dots \\
\phi_3(n) &= \frac{n}{n-m+1}\,\frac{1}{n-z+1} + \frac{n(n+1)}{(n-m+1)(n-m+2)}\,\frac{1}{n-z+2} + \frac{n(n+1)(n+2)}{(n-m+1)(n-m+2)(n-m+3)}\,\frac{1}{n-z+3} + \dots \quad \dots
\end{aligned}
\right\}
\qquad (13)
$$

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8. *Equality of certain series.*

Now denote $v_1(z)$ by $v(z, m, n)$.

$$\therefore \quad v_2(z) = \frac{n}{z} v(m-z, n-z+1, -z)$$

$$v_3(z) = \frac{n}{z-m} v(z, z-n, z-m)$$

$$\theta_2(z) = \frac{n}{z-m} v(n-m+1, 1-m, z-m)$$

$$\theta_3(z) = \frac{-n}{m-n-1} v(m-z, m, m-n-1)$$

$$\phi_2(z) = \frac{n}{n-m+1} v(-n, z-n, m-n-1)$$

$$\phi_3(z) = v(n-m+1, n-z+1, n).$$

Now it was proved by Stirling that $v_1 = v_3$,

$$i.e. \quad v(z, m, n) = \frac{n}{z-m} v(z, z-n, z-m)$$

$$\therefore \quad v(m-z, n-z+1, -z) = \frac{-z}{m-n-1} v(m-z, m, m-n-1)$$

$$\therefore \quad \frac{n}{z} v(m-z, n-z+1, -z) = \frac{-n}{m-n-1} v(u-z, m, m-n-1),$$

$$i.e. \quad v_2 = \theta_3.$$

Also

$$v(n-m+1, 1-m, z-m) = \frac{z-m}{n} v(n-m+1, n-z+1, n)$$

$$\therefore \frac{n}{z-m} v(n-m+1, 1-m, z-m) = v(n-m+1, n-z+1, n),$$

$$i.e. \quad \theta_2 = \phi_3.$$

Also

$$v(-n, z-n, m-n-1) = \frac{m-n-1}{-z} v(-n, 1-m, -z)$$

$$\therefore \frac{n}{n-m+1} v(-n, z-n, m-n-1) = \frac{n}{z} v(-n, 1-m, -z),$$

$$i.e. \quad \phi_2$$

$$= \frac{n}{z}\left\{\frac{-z}{-n}\frac{1}{1-m} + \frac{-z(1-z)}{-n(1-)n}\frac{1}{2-m} + \frac{-z(1-z)(2-z)}{-n(1-n)(2-n)}\frac{1}{3-m} + \dots\right\}$$

$$= -\frac{1}{m-1} - \frac{z-1}{n-1}\frac{1}{m-2} - \frac{(z-1)(z-2)}{(n-1)(n-2)}\frac{1}{m-2} - \dots,$$

which connects certain of the series.

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9. *Relation between the series.*

$v_1(z)$ and $v_2(z)$ are both solutions of the difference equation in z.

Therefore they are both particular cases of the general solution

$$v(z)=C\frac{\Gamma(z)}{\Gamma(z-m+1)}-\frac{\Gamma(z)}{\Gamma(z-m+1)}\sum_z\frac{\Gamma(z-m+1)}{\Gamma(z+1)}\frac{n}{z-n}.$$

Therefore their difference must be of the form

$$C\frac{\Gamma(z)}{\Gamma(z-m+1)}.$$

Up till now we have assumed C to be an arbitrary constant, but it may also be a periodic function of z.

For, suppose it involves z.

$$v(z)=C(z)\frac{\Gamma(z)}{\Gamma(z-m+1)}-\frac{\Gamma(z)}{\Gamma(z-m+1)}\sum_z\frac{\Gamma(z-m+1)}{\Gamma(z+1)}\frac{n}{z-n}$$

$$v(z+1)=C(z+1)\frac{\Gamma(z+1)}{\Gamma(z-m+2)}-\frac{\Gamma(z+1)}{\Gamma(z-m+2)}\sum_z\frac{\Gamma(z-m+2)}{\Gamma(z+2)}\frac{n}{z-n+1}$$

$$\therefore\quad zv(z)-(z-m+1)\,v(z+1)$$

$$=\frac{\Gamma(z+1)}{\Gamma(z-m+1)}\left\{C(z)-C(z+1)\right\}+\frac{n}{z-n}.$$

Therefore, in order that the difference equation may be satisfied, we must have

$$C(z)-C(z+1)\equiv 0,$$

which is true, provided $C(z)$ is a periodic function of z.

$$\therefore\quad v_1-v_2=C\frac{\Gamma(z)}{\Gamma(z-m+1)},\qquad\text{(14)}$$

where C is either a constant or a periodic function of z.

But $v_2=\theta_3$.

Therefore v_2 is a solution of the difference equation in m, of which v_1 is also a solution.

$$\therefore\quad v_1-v_2=C'\frac{\Gamma(m-z)}{\Gamma(m-n)},\qquad\text{(15)}$$

where C does not involve m, or else is a periodic function of m.

Also, v_2 is a solution of the difference equation in n.

For, if $\psi(n)=\dfrac{1}{z-m}\dfrac{1}{n-z+1}+\dfrac{z-1}{(z-m)(z-m-1)}\dfrac{1}{n-z+2}+\ldots$

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then $(n-m+1)\,\psi(n)-(n+1)\,\psi(n+1)$

$$=\frac{1}{z-m}\left(\frac{n-m+1}{n-z+1}-\frac{n+1}{n-z+2}\right)+\frac{(z-1)}{(z-m)(z-m-1)}\left(\frac{n-m+1}{n-z+2}-\frac{n+1}{n-z+3}\right)$$
$$+\frac{(z-1)(z-2)}{(z-m)(z-m-1)(z-m-2)}\left(\frac{n-m+1}{n-z+3}-\frac{n+1}{n-z+4}\right)+\ldots$$
$$=\frac{1}{z-m}\left(\frac{z-m}{n-z+1}-\frac{z-1}{n-z+2}\right)+\frac{z-1}{(z-m)(z-m-1)}\left(\frac{z-m-1}{n-z+2}-\frac{z-2}{n-z+3}\right)$$
$$+\frac{(z-1)(z-2)}{(z-m)(z-m-1)(z-m-2)}\left(\frac{z-m-2}{n-z+3}-\frac{z-3}{n-z+4}\right)+\ldots$$
$$=\frac{1}{n-z+1}-\frac{z-1}{z-m}\,\frac{1}{n-z+2}+\frac{z-1}{z-m}\,\frac{1}{n-z+2}-\frac{(z-1)(z-2)}{(z-m)(z-m-1)}\,\frac{1}{n-z+3}$$
$$+\frac{(z-1)(z-2)}{(z-m)(z-m-1)}\,\frac{1}{n-z+3}-\ldots$$
$$=\frac{1}{n-z+1},$$

and v_1 is also a solution of the same equation.

$$\therefore\quad v_1-v_2=C''\,\frac{\Gamma(n-m+1)}{\Gamma(n)},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

where C'' is either independent of n or else is a periodic function of n.

Therefore, combining these three results (14), (15), (16), we have

$$C\,\frac{\Gamma(z)}{\Gamma(z-m+1)}=C'\,\frac{\Gamma(m-z)}{\Gamma(m-n)}=C''\,\frac{\Gamma(n-m+1)}{\Gamma(n)}.$$

$$\therefore\quad C'=\frac{\Gamma(z)\,\Gamma(m-n)}{\Gamma(z-m+1)\,\Gamma(m-z)}\,C$$
$$=\Gamma(z)\,\Gamma(m-n)\,\frac{\sin\pi(m-z)}{\pi}\,C.$$

Now C' is periodic in m.

$$\therefore\quad C=\Gamma(n-m+1)\,\beta,$$

where β is periodic in m; for, if so, then

$$C'=\Gamma(z)\,\Gamma(m-n)\,\Gamma(n-m+1)\,\frac{\sin\pi(m-z)}{\pi}\,\beta$$
$$=\Gamma(z)\,\frac{\sin\pi(m-z)}{\sin\pi(m-n)}\,\beta,$$

which is periodic in m.

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Also $$C'' = \frac{\Gamma(z)\,\Gamma(n)}{\Gamma(z-m+1)\,\Gamma(n-m+1)}\,C$$
$$= \frac{\Gamma(z)\,\Gamma(n)}{\Gamma(z-m+1)}\,\beta.$$

C'' is periodic in n.

$$\therefore\quad \beta = \frac{1}{\Gamma(n)}\,\alpha,$$

where α is periodic in n; for, if so, then

$$C'' = \frac{\Gamma(z)}{\Gamma(z-m+1)}\,\alpha,$$

which is periodic in n.

$$\therefore\quad C = \frac{\Gamma(n-m+1)}{\Gamma(n)}\,\alpha,$$

where α is periodic in z, m, n.

$$\therefore\quad v_1 - v_2 = \alpha\,(z, m, n)\,\frac{\Gamma(z)\,\Gamma(n-m+1)}{\Gamma(n)\,\Gamma(z-m+1)} \quad\ldots\ldots\ldots\ldots (17)$$

where α is a periodic function of its three arguments.

If the value of α is known for any particular value z_1 of the argument z, we can at once write down the relation between v_1 and v_2 for values of the argument

$$z_1+1,\ z_1+2,\ z_1+3,\ z_1+4,\ +\ldots,$$
$$z_1-1,\ z_1-2,\ z_1-3\ \ldots .$$

In exactly the same way we may shew that the difference of any two of the series must be of the same form.

10. *Some particular cases.*

Stirling, in the work already referred to, points out that if n is a negative integer, the series v_1 gives the sum of the series v_3 *exactly*.

We will now shew that if z is a positive integer, we can find sum of the series v_1 by means of the hypergeometric function.

First, let $z=1$.

Then

$$v_1(z) = \frac{n}{1}\,\frac{1}{m} + \frac{n(n+1)}{1\,.\,2}\,\frac{1}{m+1} + \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}\,\frac{1}{m+2} + \ldots$$
$$= \frac{1}{m-1}\left\{\frac{n}{1}\,\frac{m-1}{m} + \frac{n(n+1)}{1\,.\,2}\,\frac{(m-1)m}{1\,.\,2\,.\,3}\right.$$
$$\left. + \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}\,\frac{(m-1)m(m+1)}{m(m+1)(m+2)} + \ldots\right\}$$

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$$=\frac{1}{m-1}\left\{F(n,\, m-1,\, m,\, 1)-1\right\}$$

where F is the hypergeometric function,

$$=\frac{1}{m-1}\left\{\frac{\Gamma(m)\,\Gamma(1-n)}{\Gamma(1)\,\Gamma(m-n)}-1\right\}$$

$$=\frac{1}{1-m}+\frac{\Gamma(m-1)\,\Gamma(1-n)}{\Gamma(m-n)}.$$

Also $\qquad v_2(z)=\frac{1}{1-m}\,\frac{n}{n}.$

$$\therefore\ v_1(z)-v_2(z)=\frac{\Gamma(m-1)\,\Gamma(1-n)}{\Gamma(m-n)}.$$

$\therefore$ For $z=1$,

$$\alpha(z,\, m,\, n)=\frac{\Gamma(z)\,\Gamma(n-m+1)}{\Gamma(n)\,\Gamma(z-m+1)}=\frac{\Gamma(m-1)\,\Gamma(1-n)}{\Gamma(m-n)},$$

i.e. $\qquad \alpha\,\frac{\Gamma(1)\,\Gamma(n-m+1)}{\Gamma(n)\,\Gamma(2-m)}=\frac{\Gamma(m-1)\,\Gamma(1-u)}{\Gamma(m-n)},$

i.e. $\qquad \alpha=\frac{\Gamma(m-1)\,\Gamma(1-n)\,\Gamma(n)\,\Gamma(2-m)}{\Gamma(m-n)\,\Gamma(n-m+1)}.$

$$=\frac{\pi\sin(m-n)\pi}{\sin n\pi\sin(m-1)\pi}$$

$$=\frac{-\pi\sin(m-n)\pi}{\sin m\pi\sin n\pi},$$

which can be easily calculated from a table of sines.

Now α is unaltered if z be replaced by $z+1$; *i.e.* α has this value for all positive integral values of z.

i.e. If z is a positive integer, then

$$\frac{n}{z}\,\frac{1}{m}+\frac{n(n+1)}{z(z+1)}\,\frac{1}{m+1}+\frac{n(n+1)(n+2)}{z(z+1)(z+2)}\,\frac{1}{m+2}+\dots$$

$$=\frac{-\pi\sin(m-n)\pi}{\sin m\pi\sin n\pi}\qquad\frac{\Gamma(z)\,\Gamma(n-m+1)}{\Gamma(n)\,\Gamma(z-m+1)}+v_2$$

$$=\frac{-\pi}{\sin m\pi}\,\frac{\Gamma(z)}{\Gamma(z-m+1)}\,\frac{\Gamma(1-n)}{\Gamma(m-n)}+\frac{1}{z-m}\,\frac{n}{n-z+1}$$

$$+\frac{z-1}{(z-m)(z-m-1)}\,\frac{n}{n-z+2}$$

$$+\frac{(z-1)(z-2)}{(z-m)(z-m-1)(z-m-2)}\frac{n}{n-z+3}+\ldots$$

$$+\frac{(z-1)(z-2)\ldots 2.1}{(z-m)(z-m-1)\ldots(1-m)}\frac{n}{n} \ldots\ldots\ldots\ldots\ldots\ldots \ldots\ldots (18)$$

which contains only a finite number of terms.

Again, if $m=z+1$,

$$v_1=\frac{n}{z(z+1)}+\frac{n(n+1)}{z(z+1)(z+2)}+\ldots$$

$$=\frac{n}{z}\left\{\frac{1}{z+1}+\frac{n+1}{(z+1)(z+2)}+\ldots\right\}$$

$$=\frac{n}{z}\frac{1}{z-n}.$$

$$v_2=-n\left\{\frac{1}{n-z+1}+\frac{1-z}{1\ 2}\frac{1}{n-z+2}+\frac{(1-z)(2-z)}{1.2.3}\frac{1}{n-z+3}+\ldots\right\}$$

$$=\frac{n}{z(n-z)}\left\{1+\frac{-z}{1}\frac{n-z}{n-z+1}+\frac{-z(1-z)}{1.2}\frac{(n-z)(n-z+1)}{(n-z+1)(n-z+2)}+\ \ldots -1\right\}$$

$$=\frac{n}{z(n-z)}\{F(-z,\ n-z,\ n-z+1,\ 1)-1\}$$

$$=\frac{n}{z(n-z)}\frac{\Gamma(n-z+1)\Gamma(1+z)}{\Gamma(1)\Gamma(1+n)}+\frac{n}{z(z-n)}$$

$$=\frac{\Gamma(n-z)\Gamma(z)}{\Gamma(n)}+\frac{n}{z(z-n)}$$

$$\therefore\quad v_2-v_1=\frac{\Gamma(n-z)\Gamma(z)}{\Gamma(n)}\text{ for } m=z+1.$$

But

$$v_2-v_1=C'\frac{\Gamma(m-z)}{\Gamma(m-n)}$$

where C' is periodic in m

$$\therefore\quad \frac{\Gamma(n-z)\Gamma(z)}{\Gamma(n)}=C'\frac{\Gamma(1)}{\Gamma(z-n+1)}$$

$$\therefore\quad C'=\Gamma(z-n+1)\Gamma(n-z)\frac{\Gamma(z)}{\Gamma(n)}$$

for $m-z=1$,

and therefore for $m-z=$ any positive integer;

i.e. If $m-z$ is a positive integer, then

$$v_3 = v_1 + \frac{\Gamma(z-n+1)\,\Gamma(n-z)\,\Gamma(z)\,\Gamma(m-z)}{\Gamma(m-n)\,\Gamma(n)}$$

$$= v_3 + \frac{\Gamma(z)\,\Gamma(m-z)}{\Gamma(n)\,\Gamma(m-n)}\,\frac{\pi}{\sin\pi(n-z)}$$

i.e.
$$\frac{1}{z-m}\,\frac{n}{n-z+1} + \frac{z-1}{(z-m)(z-m-1)}\,\frac{n}{n-z+2} + \frac{(z-1)(z-2)}{(z-m)(z-m-1)(z-m-2)}\,\frac{n}{n-z+3} + \dots$$

$$= \frac{\Gamma(z)}{\Gamma(n)}\,\frac{\Gamma(m-z)}{\Gamma(m-n)}\,\frac{\pi}{\sin\pi(n-z)} + \frac{1}{z}\,\frac{n}{z-n} + \frac{z-m+1}{z(z+1)}\,\frac{n}{z-n+1} + \frac{(z-m+1)(z-m+2)}{z(z+1)(z+2)}\,\frac{n}{z-n+2} + \dots + \frac{(z-m+1)(z-m+2)\dots(-2)(-1)}{z(z+1)-\ \dots\ -(m-1)}\,\frac{n}{m-n-1}. \quad \dots\dots (19)$$

In exactly the same way, if $n-m$ is a positive integer, we can calculate v_1 by means of θ_3, using the fact that

$$v_1 - v_3 = C''\,\frac{\Gamma(n-m+1)}{\Gamma(n)}$$

where C'' is a periodic function of n.

11. By giving various values to z, m, n in the series which are equal to one another, we obtain identities in z.

e.g. if in v_1 and v_3 we put

$$m = z$$
$$m = z-1$$

we find the identity

$$(z-1)\left\{\frac{1}{z^2} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^3} + \dots\right\}$$

$$= (z-1)\left\{\frac{1}{z} + \frac{1}{z(z+1)}\,\frac{1}{2} + \frac{1}{z(z+1)(z+2)}\,\frac{1.2}{3} + \frac{1}{z(z+1)(z+2)(z+3)}\,\frac{1.2.3}{4} + \dots\right\}$$

$$\therefore\quad \frac{1}{z^2} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \dots$$

$$= \frac{1}{z} + \frac{1}{z(z+1)}\,\frac{1}{2} + \frac{1}{z(z+1)(z+2)}\,\frac{1.2}{3} + \dots \quad \dots\dots\dots\dots (20)$$

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Now, put $z = 14$ and we find

$$\frac{1}{14^2}+\frac{1}{15^2}+\ldots = \cdot 074040270,$$

only 13 terms being required for its computation. But we find from Barlow's tables that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ \ldots\ \frac{1}{13^2}=1\cdot 570893798.$$

$$\therefore \quad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots .$$

$$= \quad 1\cdot 570893798$$

$$+ \ \cdot 074040270$$

$$= \quad 1\cdot 64493407,$$

which gives the sums of the squares of the reciprocals of the natural numbers—a series which, in its original form, is only very slowly convergent.