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The bifurcation of periodic orbits of one-dimensional maps

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Abstract. The bifurcation of C^1-continuous families of maps of the interval or circle is studied. It is shown, for example, that period-tripling cannot occur. This yields topological properties of the stratification of $C^1(I, I)$ induced by the Sarkovskii order, and corresponding bifurcation properties.

1. *Introduction*

This paper deals with the bifurcation theory of maps of the interval and of the circle. We are interested in how the least period of a periodic point can change in a family which is continuous with respect to the C^1 topology. More precisely, if a map f_s in a C^1-continuous family $\{f_t\}$ has a periodic point x_s of period k (which will always mean least period), we ask what may be the period of a periodic point x_t (of f_t) near x_s, where t is near s. If the period of x_t is $2k$, but f_s has no $2k$-periodic point near x_s, we say a period-doubling bifurcation occurs; if $3k$, period-tripling, etc. Period-doubling bifurcations have been studied extensively (see for example [**6**], [**7**]). The first theorem of this paper implies that for C^1-continuous families of maps of compact one-dimensional spaces, period-tripling, quadrupling, etc., bifurcations do not occur. This is easily seen to be false for families which are only C^0-continuous.

THEOREM 1. *Suppose (f_n) is a sequence of maps in $C^1(I, I)$ or $C^1(S^1, S^1)$ and (f_n) converges to f (in $C^1(I, I)$ or $C^1(S^1, S^1)$). Suppose each f_n has a periodic point x_n of period k (where k is a fixed positive integer). Suppose some subsequence of (x_n) converges to x. If k is odd, x is a periodic point of f of period k. If k is even, x is a periodic point of f of period k or $\frac{1}{2}k$.*

Using theorem 1, we show that the Sarkovskii ordering provides a stratification of $C^1(I, I)$ and gives topological restrictions on one parameter families in $C^1(I, I)$. Let $F(n)$ denote the set of maps in $C^1(I, I)$ which have a periodic point of period n. We will use the symbol $\lhd$ to denote the Sarkovskii order,

$$2^n \lhd 2^{n+1} \lhd \cdots \lhd 2^{n+1} \cdot 5 \lhd 2^{n+1} \cdot 3 \lhd \cdots 2^n \cdot 5 \lhd 2^n \cdot 3.$$

Sarkovskii showed that if $n \lhd m$ then $F(m) \subseteq F(n)$ ([**5**], [**10**], [**12**]). Let $G(n)$ denote those $f \in F(n)$ such that $f \notin F(m)$ if $n \lhd m$. We may include the symbol ∞ by defining $k \lhd \infty$ if $k = 2^j$, for some integer $j \geq 0$, and $\infty \lhd k$, otherwise. Those maps in $F(\infty)$, but not $F(m)$ for any m with $\infty \lhd m$, will be denoted $G(\infty)$.

The following corollaries follow immediately from theorem 1 and Sarkovskii's theorem. The second corollary was known to Misiurewicz [**8**].

COROLLARY 1. *If n is not a power of 2, $F(n)$ is a closed set.*

COROLLARY 2. *If n is a power of 2, the closure of $F(n)$ is contained in $F(\frac{1}{2}n)$.*

It follows at once from corollary 2 that $F(\infty)$ is closed. Hence the closure of the set of maps in $C^1(I, I)$ with positive topological entropy is contained in $F(\infty)$. The following result characterizes $F(\infty)$ as those maps whose set of periodic points is not closed.

THEOREM 2. *Let $f \in C^1(I, I)$ or $f \in C^1(S^1, S^1)$ and suppose the set of periodic points of f is a closed set. Then f has only finitely many periods.*

In [**11**], Nitecki gives an example of a continuous $f \in G(\infty)$ whose periodic points are closed; the above theorem shows this is a pathology of $C^0(I, I)$. Also, theorem 2 and [**11**] imply that for any $f \in G(\infty)$ which is C^1, there is a non-wandering point whose (infinite) orbit is 'separated to all orders.' If $f \in G(\infty)$ is C^3 and has negative Schwarzian derivative, the orbit structure of f is completely described by Misiurewicz in [**8**] under the assumption that f is unimodal.

Our final result uses theorem 1 and the stability theorem of [**1**] to obtain an intermediate value result for families in $C^1(I, I)$, in terms of the Sarkovskii ordering.

THEOREM 3. *Let f_s be a continuous arc in $C^1(I, I)$, $0 \le s \le 1$, with $f_0 \in G(n)$ and $f_1 \in G(m)$. Then for any k such that $n \lhd k \lhd m$, there exists s_k such that $f_{s_k} \in G(k)$, and $i < j$ implies $s_i < s_j$.*

Note that n, m or k may be ∞. It follows that any such arc with $h(f_0) = 0$ but $h(f_1) > 0$, where h denotes entropy, must pass through $G(\infty)$. Theorems 2 and 3 make the following strengthening of the remarks after corollary 2 seem reasonable.

CONJECTURE. *The closure of the maps in $C^1(I, I)$ which have positive entropy is exactly $F(\infty)$. The boundary of that set, and of the set of maps in $C^1(I, I)$ with only finitely many periods, is exactly $G(\infty)$.*

Of course, the above results do not exhaust the bifurcation behaviour of interval maps. The 'particle–antiparticle' bifurcation (which is sometimes called 'saddle-node' in higher dimensions) which occurs as a segment of the graph of f (or of f^k) moves across the diagonal is the most obvious additional phenomenon. The cluster of bifurcations which occurs as f_s moves into the positive-entropy region which results in homoclinic orbits is another (which we shall address in another paper).

2. *Proof of theorem* 1

We begin the proof of theorem 1 by proving two lemmas.

LEMMA 1. *Let $\{p_1, p_2, \ldots, p_k\}$ be a periodic orbit of $f \in C^1(I, I)$ of period k where $k > 2$ and $p_1 < p_2 < \cdots < p_k$. There are points y and z in the interval $[p_1, p_k]$ with $f'(y) > 0$ and $f'(z) \le -1$.*

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Proof. For some integer m with $1<m<k$, either $f(p_m)=p_1$, or $f(p_m)=p_k$. If $f(p_m)=p_1$, there is a point $y\in[p_m, p_{m+1}]$ with $f'(y)>0$. If $f(p_m)=p_k$, there is a point $y\in[p_{m-1}, p_m]$ with $f'(y)>0$.

Now, let i be the smallest element of $\{1,2,\ldots,k\}$ with $f(p_i)<p_i$. Then $i>1$, $f(p_i)\le p_{i-1}$, and $f(p_{i-1})\ge p_i$. Hence by the Mean Value Theorem, there is a point $z\in(p_{i-1}, p_i)$ with $f'(z)\le -1$. □

LEMMA 2. *Suppose (f_n) converges to f (in $C^1(I,I)$ or $C^1(S^1,S^1)$) and suppose that for each n, x_n is a periodic point of f_n of period k, where k is a fixed positive integer with $k>2$. If (x_n) converges to x then x is a fixed point of f^k but not a fixed point of f.*

Proof. By continuity, x is a fixed point of f^k, so it suffices to prove that x is not a fixed point of f. We assume that x is a fixed point of f and obtain a contradiction.

First suppose that (f_n) converges to f in $C^1(I,I)$. Let p_n denote the smallest element and q_n the largest element of the orbit of x_n. By taking subsequences, we can assume that (p_n) converges to p and (q_n) converges to q. There are positive integers i and j such that

$$(f_n)^i(x_n)=p_n \quad \text{and} \quad (f_n)^j(x_n)=q_n$$

for infinitely many n. Hence, by continuity,

$$f^i(x)=p \quad \text{and} \quad f^j(x)=q.$$

Since $f(x)=x$, we have $x=p=q$. By lemma 1, $f'(x)\ge 0$ and $f'(x)\le -1$, a contradiction.

Now suppose that (f_n) converges to f in $C^1(S^1,S^1)$. Since $f(x)=x$, there are proper closed intervals K and J on S^1 with $K\subset \text{int}(J)$, $f(K)\subset \text{int}(J)$, and $x\in \text{int}(K)$. For n sufficiently large, the orbit of x_n will be contained in K and $f_n(K)$ will be contained in J. Hence, we can look at the restrictions of f_n and f to the interval K and apply lemma 1, as in the preceding paragraph, to show that $f'(x)>0$ and $f'(x)\le -1$, a contradiction. □

Proof of theorem 1. We have three cases.

Case 1. k is odd and $k\ge 3$. If $k=3$ the conclusion follows from lemma 2. Proceeding by induction, we assume the conclusion is true for all odd numbers less than k. By lemma 2, x is a periodic point of f of period r where $1<r\le k$ and k is a multiple of r. Hence, $k=r\cdot s$ where s is an odd positive integer.

Let $g_n=(f_n)^r$ and $g=f^r$. Then (g_n) converges to g and x_n is a periodic point of g_n of period s (for each n), but x is a fixed point of g. By our induction hypothesis, $s=1$ and $k=r$.

Case 2. $k=2^s$ for some integer $s\ge 0$. If $k=1$ or $k=2$ the conclusion is immediate, so we assume $k\ge 4$. Let $g_n=(f_n)^{(\frac{1}{4}k)}$. Then for each n, x_n is a periodic point of g_n of period 4. By lemma 2, x is a periodic point of g of period 2 or 4. Hence, x is a periodic point of f of period k or $\frac{1}{2}k$.

Case 3. $k=m\cdot r$ where $r=2^s$ for some $s\ge 1$ and m is odd with $m\ge 3$. Note that the sequence $(f_n)^r$ converges to f^r and (for each n) x_n is a periodic point of $(f_n)^r$ of period m. Hence, by case 1, x is a periodic point of f^r of period m.

On the other hand, the sequence $(f_n)^m$ converges to f^m, so x is a periodic point of f^m of period r or $\frac{1}{2}r$ by case 2.

Let t denote the period of x as a periodic point of f. Then $\frac{1}{2}r$ and m are relatively prime, $\frac{1}{2}r$ and m divide t, and t divides k. Hence, $t=k$ or $t=\frac{1}{2}k$ and the proof of theorem 1 is complete. □

3. *Proof of theorem* 2

We need the following lemma for the proof of theorem 2.

LEMMA 3. *Let $f \in C^1(I,I)$ have a periodic orbit $\{x_1, x_2, \ldots, x_n\}$, where $x_i < x_{i+1}$ and $4 \leq n = 2^k$, for some k. Suppose this orbit is 'separated to first order'* [**10**] i.e. *if $f(x_i) = x_j$ then $i \leq \frac{1}{2}n$ if and only if $\frac{1}{2}n < j$. Then there are points y and z in the interval $[x_1, f(x_1)]$ with $f'(y) \leq -1$ and $f'(z) \geq 0$. (The same statement holds for $[f(x_n), x_n]$).*

Proof. Let $j = \frac{1}{2}n$. Then $f(x_j) \geq x_{j+1}$ and $f(x_{j+1}) \leq x_j$, by hypothesis, so $\exists y \in [x_j, x_{j+1}] \subseteq [x_1, f(x_1)]$ such that $f'(y) \leq -1$. Next let $f(x_i) = x_n$, so $1 \leq i \leq j$. If $i = 1$ the desired z exists by lemma 1. If $i > 1$, then $f(x_{i-1}) < f(x_i)$ implies such a point exists in $[x_{i-1}, x_i] \subseteq [x_1, f(x_1)]$. (The final remark follows by taking the 'mirror image' $g(x) = 1 - f(1-x)$.) □

Proof of theorem 2. We first consider the interval case, $f \in C^1(I,I)$. Since the set of periodic points is closed, every period is a power of 2 ([**4**], [**11**]). Therefore, [**3**], any periodic orbit of f^k is separated to order one under f^k, for any $k \geq 0$ (in [**3**] this is called 'simple').

Suppose now that all powers of 2 are periods of periodic points of f. Let p_n be the least point in an orbit of period 2^n, for each n. A subsequence of these must converge, say to p. Since the periodic points are a closed set, p is periodic of some period s (a power of 2). Let $g = f^s$, so $g(p) = p$, and note that for all n, p_n is an endpoint of a periodic orbit of g (separated to order one under g). Moreover, a subsequence of $\{g_n(p_n)\}$ converges to $g(p) = p$, but then (since $g \in C^1(I,I)$) lemma 3 implies that $g'(p) \leq -1$ and $g'(p) \geq 0$. Hence not all powers of 2 can be periods if the set of periodic points is closed, and so by the theorem of Sarkovskii, f has but finitely many periods.

We next take up maps of the circle, $f \in C(S^1, S^1)$. We may assume f has a fixed point (the set of periodic points of f and of f^k coincide). Were some period of f not a power of 2, f would have positive topological entropy [**5**], and the set of periodic points would therefore not be closed [**9**]. Hence every periodic point of f has a power of 2 as its period.

Choose an orientation of S^1, and let $\{p_1, \ldots, p_k\}$ be a periodic orbit, labelled in consecutive order so there are no points of that orbit interior to any of the intervals $M_1 = [p_1, p_2]$, $M_2 = [p_2, p_3], \ldots, M_k = [p_k, p_1]$. By theorem A_1 of [**2**], some M_j is not f-covered by any M_i for $i \neq j$, i.e. $M_j \neq f(L)$ for any interval $L \subseteq M_i$. By renumbering, we may assume $j = k$. Denote by I the complement (in S^1) of the interior of M_k. Then as in theorem A_2 of [**2**], f restricted to I is differentiably 'conjugate' to a map $g: I \to \mathbb{R}$. We may consider $\{p_1, \ldots, p_m\}$ to be a periodic orbit of g; it is separated to order one, and the 'endpoint' is well defined, so lemma 3 produces

y, z between p_1 and $f(p_1)$ with $f'(y) \leq -1$, $f'(z) \geq 0$. Proceeding as above, f has finitely many periods. □

4. *Proof of theorem* 3

We conclude by proving theorem 3. We will use a theorem of [**1**], which states that if $f \in C^0(I, I)$ and f has a point of period n, then there is a neighbourhood $N(f)$ in $C^0(I, I)$ such that for all $g \in N(f)$ and all k with $k \lhd n$, g has a periodic point of period k.

Proof. Let $t_k = \inf\{s \in [0, 1]: f_s \in F(k)\}$. Note that this set is non-empty by the theorem of Sarkovskii, since $n \lhd k \lhd m$.

First suppose that k is not a power of 2. Let $s_k = t_k$. By corollary 1 (or if $k = \infty$ by corollary 2), $f_{s_k} \in F(k)$, so by the theorem of [**1**], $f_{s_k} \in G(k)$.

Now suppose that $k = 2^j$ where $j \geq 1$. Let

$$r = t_{2k} = \inf\{s \in [0, 1]: f_s \in F(2k)\}.$$

Then $f_r \notin F(4k)$ (by the theorem of [**1**]), but $f_r \in F(k)$ (by corollary 2). If $f_r \in F(k) \backslash F(2)$ we let $s_k = r$. If $f_r \in F(2k)$ we choose $\varepsilon > 0$ sufficiently small that $f_{r-\varepsilon} \in F(k)$ (by the theorem of [**1**]), and let $s_k = r - \varepsilon$. In either case $f_{s_k} \in G(k)$. □

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