A NOTE ON THE DIOPHANTINE EQUATION $x^2+7=y^n$

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Abstract. In this note we prove that the equation $x^2+7=y^n$, $x,y,n \in \mathbb{N}$, $n>2$, has no solutions (x,y,n) with $2 \nmid y$. Moreover, all solutions (x,y,n) of the equation with $2 \mid y$ satisfy $n<5 \,.\, 10^6$ and $y<\exp\exp\exp 30$.

1. Introduction. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. In 1913, Ramanujan [**5**] conjectured that the equation

$$x^2+7=2^n, \qquad (x,n \in \mathbb{N}),$$

has only the solutions $(x,n)=(1,3)$, $(3,4)$, $(5,5)$, $(11,7)$ and $(181,15)$. In 1948, Nagell [**4**] verified the above conjecture. Let $k \in \mathbb{N}$ with $2 \nmid k$. Lewis [**2**] proved that the equation

$$x^2+7=k^n, \qquad x,n \in \mathbb{N}, n>2, \tag{1}$$

has at most two solutions (x,n). Moreover, if k is not a prime power, then (1) has no solution (x,n). In this note we discuss the solutions (x,y,n) of a general equation

$$x^2+7=y^n,\ x,y,n \in \mathbb{N},\ n>2. \tag{2}$$

We prove the following results:

THEOREM 1. *Equation* (2) *has no solutions* (x,y,n) *with* $2 \nmid y$.

THEOREM 2. *Equation* (2) *has only finitely many solutions* (x,y,n) *with* $2 \mid y$. *Moreover, all solutions* (x,y,n) *satisfy* $n<5 \,.\, 10^6$ *and* $y<\exp\exp\exp 30$.

2. Lemmas. Let α be an algebraic number with minimal polynomial

$$a_0z^d+a_1z^{d-1}+\ldots+a_d=a_0\prod_{i=1}^{d}(z-\sigma_i\alpha), \qquad a_0>0,$$

where $\sigma_1\alpha,\ldots,\sigma_d\alpha$ are all conjugates of α. Then

$$h(\alpha)=\frac{1}{d}\left(\log a_0+\sum_{i=1}^{d}\log\max(1,|\sigma_i\alpha|)\right)$$

is called the logarithmic absolute height of α.

LEMMA 1. *Let* α_1, α_2 *be non-zero algebraic numbers which are multiplicatively independent, and let* $\log\alpha_j$ $(j=1,2)$ *be any non-zero determination of the logarithm of* α_j. *Further let* D *be the degree of* $\mathbb{Q}(\alpha_1,\alpha_2)$, *and let*

$$A_j=\max\left(1, h(\alpha_j)+\log 2, \frac{2e\,|\log\alpha_j|}{D}\right), \qquad j=1,2.$$

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If $\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2 \neq 0$ *for some* $b_1, b_2 \in \mathbb{N}$ *with* $\max(b_1, b_2) \geq 10^6$, *then we have*

$$\log|\Lambda| \geq -43D^4 A_1 A_2 (1 + \log B + \log\log 2B)^2,$$

where $B = \max(b_1, b_2)$.

Proof. Since $B \geq 10^6$, by the definitions in [**3**], we get $G > 17{\cdot}4895$. Therefore, we may choose $\theta = 12$, $Z = 3$, $c = 9{\cdot}13$, $c_0 = 136{\cdot}89$, $c_1 = 2{\cdot}87$ and $C/Z^3 = 43$ by [**3**, Fig. 2]. Thus, by [**3**, Theorem 5.11], we obtain the lemma immediately.

For any algebraic number α, let $|\bar{\alpha}|$ be the maximum absolute value of the conjugates of α. Let K be an algebraic number field with the degree r, and let D_K, O_K be the discriminant and the ring of algebraic integers of K respectively. Further, let $F(X, Y) = a_0X^n + a_1X^{n-1}Y + \ldots + a_nY^n \in O_K[X, Y]$ be a binary form with the degree n.

LEMMA 2 ([**1**, Corollary]). *Let* $b \in O_K$. *If* $n \geq 3$ *and* $F(X, Y)$ *is irreducible in* K, *then all solutions* (X, Y) *of the equation*

$$F(X, Y) = b, \qquad X, Y \in O_K,$$

satisfy

$$\log\max(|\bar{X}|, |\bar{Y}|) < (25(n+3)nr)^{15(n+3)}(nr)^{2(nr+1)}n^{7n}(H^{r(n-1)}|D_k|)^{n/2} (\log(2H|D_k|))^{2nr}((H^{r(n-1)}|D_k|)^{n/2} + \log|\bar{b}|),$$

where $H = \max(|\bar{a}_0|, |\bar{a}_1|, \ldots, |\bar{a}_n|)$.

3. Proof of Theorem 1. Let $K = \mathbb{Q}(\sqrt{-7})$, and let h_k, O_K be the class number and the ring of algebraic integers of K respectively. Then we have $h_K = 1$ and

$$O_K = \left\{\frac{a + b\sqrt{-7}}{2} \,\middle|\, a, b \in \mathbb{Z},\ a \equiv b \pmod 2\right\}. \tag{3}$$

Let (x, y, n) be a solution of (2) with $2 \nmid y$. If $2 \mid n$, then we get $y^{n/2} + x = 7$ and $y^{n/2} - x = 1$, whence we obtain $y^{n/2} = 4$, a contradiction. So we have $2 \nmid n$.

From (2), we have

$$(x + \sqrt{-7})(x - \sqrt{-7}) = y^n. \tag{4}$$

Since $2 \nmid y$ and $h_K = 1$, we get $\gcd(x + \sqrt{-7}, x - \sqrt{-7}) = 1$, and by (4),

$$x + \sqrt{-7} = (a_1 + b_1\sqrt{-7})^n, \tag{5}$$

where $a_1, b_1 \in \mathbb{Z}$ satisfy

$$a_1^2 + 7b_1^2 = y, \qquad \gcd(a_1, b_1) = 1. \tag{6}$$

From (5), we have

$$1 = b_1 \sum_{i=0}^{(n-1)/2} \binom{n}{2i+1} a_1^{n-2i-1}(-7b_1^2)^i,$$

whence we get $b_1 = \pm 1$ and

$$\pm 1 = \sum_{i=0}^{(n-1)/2} \binom{n}{2i+1} (-7)^i a_1^{n-2i-1} = \sum_{i=0}^{(n-1)/2} \binom{n}{2i} (-7)^{(n-1)/2-i} a_1^{2i}. \tag{7}$$

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Since $2 \nmid y$ and $-7\equiv 1 \pmod 4$, we see from (6) and (7) that $2 \parallel a_1$ and

$$\sum_{i=0}^{(n-1)/2} \binom{n}{2i}(-7)^{(n-1)/2-i}a_1^{2i}=1. \tag{8}$$

Let $2^\alpha \parallel a_1$. If $n\equiv 3 \pmod 4$, then we have

$$2^3 \Bigg\| (-7)^{(n-1)/2}-1, \qquad 2^{2\alpha} \Bigg\| \sum_{i=1}^{(n-1)/2} \binom{n}{2i}(-7)^{(n-1)/2-i}a_1^{2i}.$$

It implies that (8) is impossible in this case. If $n\equiv 1 \pmod 4$, let $2^\beta \parallel n-1$, then we have

$$2^{\beta+2} \| (-7)^{(n-1)/2}-1, \qquad 2^{2\alpha+\beta-1} \| \binom{n}{2}(-7)^{(n-3)/2}a_1^2. \tag{9}$$

Let $2^{v_i} \parallel 2i$ for any $i\in\mathbb{N}$. Since $v_i\le(\log 2i)/\log 2\le 2(i-1)$ if $i>1$, we get

$$\binom{n}{2i}(-7)^{(n-1)/2-i}a_1^{2i}=n(n-1)(-7)^{(n-1)/2-i}a_1^2\binom{n-2}{2i-2}\frac{a_1^{2(i-1)}}{2i(2i-1)}\equiv 0 \pmod{2^{2\alpha+\beta}}. \tag{10}$$

Since $\beta+2\neq 2\alpha+\beta-1$, we find from (9) and (10) that (8) is impossible. Thus, the equation (2) has no solutions (x,y,n) with $2\nmid y$.

4. Proof of Theorem 2. Let (x,y,n) be a solution of (2) with $2\mid y$. By [**4**], it suffices to prove the theorem while y is not a power of 2. Therefore, by the proof of Theorem 1, we have $2\nmid n$.

From (2), we get

$$\left(\frac{x+\sqrt{-7}}{2}\right)\left(\frac{x-\sqrt{-7}}{2}\right)=\frac{y^n}{4}.$$

Since $\gcd((x+\sqrt{-7})/2, (x-\sqrt{-7})/2)=1$, we see from (3) and (11) that

$$\left(\frac{3+\lambda\sqrt{-7}}{2}\right)\left(\frac{x+\sqrt{-7}}{2}\right)=\frac{1}{2}\left(\frac{3x-7\lambda}{2}+\frac{\lambda x+3}{2}\sqrt{-7}\right)=\left(\frac{a+b\sqrt{-7}}{2}\right)^n, \tag{12}$$

where $\lambda\in\{-1,1\}$ which make $(3x-7\lambda)/2$, $(\lambda x+3)/2\in\mathbb{Z}$ with $\gcd((3x-7\lambda)/2, (\lambda x+3)/2)=1$, and $a,b\in\mathbb{Z}$ satisfy

$$a^2+7b^2=4y, \qquad \gcd(a,b)=1. \tag{13}$$

Let

$$\varepsilon=\frac{a+b\sqrt{-7}}{2}, \qquad \bar{\varepsilon}=\frac{a-b\sqrt{-7}}{2}. \tag{14}$$

We get from (12) that

$$\left(\frac{3-\lambda\sqrt{-7}}{2}\right)\varepsilon^n-\left(\frac{3+\lambda\sqrt{-7}}{2}\right)\bar{\varepsilon}^n=8\sqrt{-7}. \tag{15}$$

Notice that $n \geq 3$, $[K:\mathbb{Q}]=2$ and $D_K=-7$. By Lemma 2, we obtain from (13), (14) and (15) that

$$\sqrt{y}=|\varepsilon|=\max(\lceil\varepsilon\rceil, \lceil\bar{\varepsilon}\rceil)<\exp((50n(n+3))^{15(n+3)}(2n)^{2(2n+1)}n^{7n}(2^{2(n-1)}7)^{n/2} \\ (\log 28)^{4n}(2^{2(n-1)}7)^{n/2}+\log 8\sqrt{7}). \tag{16}$$

Let $i=\sqrt{-1}$,

$$\frac{3-\lambda\sqrt{-7}}{2}=2e^{\phi_1 i}, \qquad \frac{3+\lambda\sqrt{-7}}{2}=2e^{-\phi_1 i}, \tag{17}$$

$$\varepsilon=\sqrt{y}\,2e^{\phi_2 i}, \qquad \bar{\varepsilon}=\sqrt{y}\,e^{-\phi_2 i}, \tag{18}$$

We see from (15) that

$$\sin(\phi_1+n\phi_2)=\frac{2\sqrt{7}}{y^{n/2}} \tag{19}$$

and

$$\left|\log\frac{3+\lambda\sqrt{-7}}{3-\lambda\sqrt{-7}}-n\log\frac{\varepsilon}{\bar{\varepsilon}}\right|=2\,|\phi_1+n\phi_2|. \tag{20}$$

By (17), (18) and (19), we may choose ϕ_1, ϕ_2 such that

$$|\phi_1|=\arctan\tfrac{1}{3}, \qquad 0<\phi_1+n\phi_2<\pi. \tag{21}$$

Then, by (19), (20) and (21), we have

$$\left|\log\frac{3+\lambda\sqrt{-7}}{3-\lambda\sqrt{-7}}-n\log\frac{\varepsilon}{\bar{\varepsilon}}\right|<\frac{8\sqrt{7}}{y^{n/2}}. \tag{22}$$

Let $\alpha_1=(3+\lambda\sqrt{-7})/(3-\lambda\sqrt{-7})$ and $\alpha_2=\varepsilon/\bar{\varepsilon}$. By (13) and (14), α_1 and α_2 satisfy $4\alpha_1^2-\alpha_1+4=0$ and $y\alpha_2^2-(a^2-7b^2)\alpha_2/2+y=0$ respectively. So we have $h(\alpha_1)=\log 2$ and $h(\alpha_2)=\log\sqrt{y}$. Since $[K:\mathbb{Q}]=2$, by Lemma 1, if $n\geq 10^6$, then we have

$$\left|\log\frac{3+\lambda\sqrt{-7}}{3-\lambda\sqrt{-7}}-n\log\frac{\varepsilon}{\bar{\varepsilon}}\right|\geq\exp(-43\,.\,2^4(\log 4)(\log 2\sqrt{y})(1+\log n+\log\log 2n)^2). \tag{23}$$

The combination of (22) and (23) yields

$$\log 8\sqrt{7}+953{\cdot}8(\log 2\sqrt{y})(1+\log n+\log\log 2n)^2>n\log\sqrt{y}. \tag{24}$$

Since y is not a power of 2, we have $y\geq 22$, and by (24),

$$2+1381{\cdot}6(1+\log n+\log\log 2n)^2>n. \tag{25}$$

We conclude from (25) that

$$n<5\,.\,10^6.$$

Substitute (26) into (16), we get $y<\exp\exp\exp 30$. The theorem is proved.

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