Canad. Math. Bull. Vol. **51** (4), 2008 pp. 535–544

On the Simple $\mathbb{Z}_2$-homotopy Types of Graph Complexes and Their Simple $\mathbb{Z}_2$-universality

Péter Csorba

Abstract. We prove that the neighborhood complex $\mathrm{N}(G)$, the box complex $\mathrm{B}(G)$, the homomorphism complex $\mathrm{Hom}(K_2, G)$ and the Lovász complex $\mathrm{L}(G)$ have the same simple $\mathbb{Z}_2$-homotopy type in the sense of Whitehead. We show that these graph complexes are simple $\mathbb{Z}_2$-universal.

1 Introduction

The topological method in graph theory was initiated by Lovász [10] to prove Kneser's conjecture [9]. He defined the neighborhood complex $\mathrm{N}(G)$ and the so called Lovász complex $\mathrm{L}(G)$. For similar reasons other complexes assigned to graphs were studied such as the box complex $\mathrm{B}(G)$ [12] and the homomorphism complex $\mathrm{Hom}(K_2, G)$, which was invented by Lovász as well [1]. We will refer to these complexes as *graph complexes*. The $\mathbb{Z}_2$-homotopy equivalence of these complexes have been studied in several papers [3, 4, 11, 14]. The neighborhood complex does not admit a free $\mathbb{Z}_2$-action. By slightly abusing the notation we will say that it is $\mathbb{Z}_2$-homotopy equivalent to the other graph complexes, meaning only homotopy equivalence.

We will show that something more can be said about these complexes. We prove that these graph complexes have the same simple $\mathbb{Z}_2$-homotopy type in the sense of Whitehead [13]. It was independently proven by Kozlov [8] that $\mathrm{N}(G)$, $\mathrm{L}(G)$ and $\mathrm{Hom}(K_2, G)$ are simple homotopy equivalent. Here we give simpler and $\mathbb{Z}_2$-proofs. It is known that graph complexes are universal [3]. We extend it into simple $\mathbb{Z}_2$-universality. We show that for any $\mathbb{Z}_2$-simplicial complex there is a graph G such that the given complex and the graph complexes assigned to G are simple $\mathbb{Z}_2$-homotopy equivalent.

2 Preliminaries

In this section we recall some basic facts about graphs, simplicial complexes, and posets, to fix notation. The interested reader is referred to [11] or [2] for details.

Any graph G considered will be assumed to be finite, simple, connected, and undirected, *i.e.,* G is given by a finite set $V(G)$ of *vertices* and a set of *edges* $E(G) \subseteq \binom{V(G)}{2}$.

Received by the editors May 26, 2006; revised February 6, 2007.
Supported by grants from NSERC and the Canada Research Chairs program.
AMS subject classification: Primary: 57Q10; secondary: 05C10, 55P10.
Keywords: graph complexes, simple $\mathbb{Z}_2$-homotopy, universality.

https://doi.org/10.4153/CMB-2008-053-9 Published online by Cambridge University Press

The *common neighborhood* of $A \subseteq V(G)$ is

$$\mathrm{CN}(A) = \{v \in V(G)\colon \{a, v\} \in E(G) \text{ for all } a \in A\}.$$

We define $\mathrm{CN}(\varnothing) := V(G)$. For two disjoint sets of vertices $A, B \subseteq V(G)$ we define $G[A, B]$ as the (not necessarily induced) subgraph of G with $V(G[A, B]) = A \cup B$ and $E(G[A, B]) = \{\{a, b\} \in E(G)\colon a \in A, b \in B\}$.

A *simplicial complex* K is a finite hereditary set system. We denote its vertex set by $V(\mathsf{K})$ and its barycentric subdivision by $\mathrm{sd}(\mathsf{K})$.

For sets A, B define $A \uplus B := \{(a, 1)\colon a \in A\} \cup \{(b, 2)\colon b \in B\}$. An important construction in the category of simplicial complexes is the *join operation*. For two simplicial complexes K and L, the join $\mathsf{K} * \mathsf{L}$ is defined as $\mathsf{K} * \mathsf{L} := \{A \uplus B \,|\, A \in \mathsf{K} \text{ and } B \in \mathsf{L}\}$.

A $\mathbb{Z}_2$*-space* is a pair (X, ν) where X is a topological space and $\nu\colon X \to X$, called the $\mathbb{Z}_2$-action, is a homeomorphism such that $\nu^2 = \nu \circ \nu = \mathrm{id}_X$.

The *neighborhood complex* [10] is $\mathrm{N}(G) = \{S \subseteq V(G)\colon \mathrm{CN}(S) \neq \varnothing\}$.

The *Lovász complex* [10] is $\mathrm{L}(G) := \mathrm{CN}(\mathrm{sd}(\mathrm{N}(G)))$. CN is a free $\mathbb{Z}_2$-action on $\mathrm{L}(G)$.

The *box complex* $\mathrm{B}(G)$ of a graph G (the one introduced by Matoušek and Ziegler [12]) is defined by

$$\mathrm{B}(G) := \Big\{ A \uplus B : A, B \subseteq V(G),\ A \cap B = \varnothing,$$
$$G[A, B] \text{ is complete bipartite}, CN(A) \neq \varnothing \neq \mathrm{CN}(B) \Big\}$$

The vertices of the box complex are

$$V_1 := \{v \uplus \varnothing\colon v \in V(G)\} \text{ and}$$
$$V_2 := \{\varnothing \uplus v\colon v \in V(G)\}.$$

The subcomplexes of $\mathrm{B}(G)$ induced by V_1 and V_2 are disjoint subcomplexes of $\mathrm{B}(G)$ such that both are isomorphic to the neighborhood complex $\mathrm{N}(G)$. We refer to these two copies as *shores* of the box complex. The box complex is endowed with a $\mathbb{Z}_2$-action which interchanges the shores.

The *shore subdivision* [4] of $\mathrm{B}(G)$ is the complex obtained by barycentricly subdividing the shores of $\mathrm{B}(G)$.

$$\mathrm{ssd}(\mathrm{B}(G)) := \Big\{ \mathrm{sd}(\sigma \cap V_1) * \mathrm{sd}(\sigma \cap V_2)\colon \sigma \in \mathrm{B}(G) \Big\}.$$

The *homomorphism complex* $\mathrm{Hom}(K_2, G)$, or actually its barycentric subdivision $\mathrm{sd}(\mathrm{Hom}(K_2, G))$, can be defined as a subcomplex of $\mathrm{sd}(\mathrm{B}(G))$ induced by the vertices $A \uplus B$ such that $A \neq \varnothing \neq B$. This definition gives the barycentric subdivision of the original definition of the homomorphism complex $\mathrm{Hom}(K_2, G)$ (see [1]).

Examples For the complete graph K_n, its neighborhood complex $\mathrm{N}(K_n)$ is the boundary complex of the $n-1$ dimensional simplex. $\mathrm{L}(K_n)$ is the barycentric subdivision of the $n-1$ dimensional simplex. Its box complex $\mathrm{B}(K_n)$ is the boundary complex of the n-dimensional cross polytope with two opposite facets removed.

Definition 1 Let K be a simplicial complex. Let $\sigma, \tau \in \mathsf{K}$ such that

(i) $\tau \subset \sigma$,

(ii) σ is a maximal simplex, and no other maximal simplex contains τ.

A (simplicial) *collapse* of K is the removal of all simplices γ, such that $\tau \subseteq \gamma \subseteq \sigma$. If in addition $\dim \tau = \dim \sigma - 1$, then this is called an *elementary* collapse.

When Y is a simplicial subcomplex of X, we say that X *collapses* onto Y if there exists a sequence of elementary collapses leading from X to Y. The reverse of an elementary collapse is called an elementary *expansion*. A sequence of elementary collapses and elementary expansions leading from a complex X to the complex Y is called a *formal deformation*. If such a sequence exists, then the simplicial complexes X and Y are said to have the same *simple homotopy type*, see [13].

The definition of the $\mathbb{Z}_2$-*collapse* and *simple $\mathbb{Z}_2$-homotopy type* is self-evident. Since we are dealing with free $\mathbb{Z}_2$-complexes, it just means that the collapses can be performed in pairs equivariantly.

It is well known, see *e.g.*, [8], that for a simplicial complex X the subdivisions $\mathrm{sd}(\mathsf{X})$ and $\mathrm{ssd}(\mathsf{X})$ have the same simple homotopy type as X, since they can be obtained by repeating stellar subdivision. This extends to simple $\mathbb{Z}_2$-homotopy type for free $\mathbb{Z}_2$-complexes. In the $\mathbb{Z}_2$ case, collapses and expansions corresponding to the stellar subdivision can be performed equivariantly.

We recall that a *partially ordered set*, or *poset* for short, is a pair $(P, \preceq)$, where P is a set and $\preceq$ is a binary relation on P that is reflexive ($x \preceq x$), transitive ($x \preceq y$ and $y \preceq z$ imply that $x \preceq z$), and weakly antisymmetric ($x \preceq y$ and $y \preceq x$ imply $x = y$). When the order relation $\preceq$ is understood, we say only "a poset P." The *order complex* of a poset P is the simplicial complex $\Delta(P)$ whose vertices are the elements of P and whose simplices are all chains (*i.e.*, $x_1 \prec x_2 \prec \cdots \prec x_k$) in P.

We need the following theorem of Kozlov to prove collapsibility.

Theorem 2 **([7, Theorem 2.1])** *Let P be a poset, and let ϕ be a descending closure operator. Then $\Delta(P)$ collapses onto $\Delta(\phi(P))$. By symmetry the same is true for an ascending closure operator.*

Actually we need the $\mathbb{Z}_2$-modification of this theorem.

Definition 3 A poset $(P, \preceq)$ is *involutive* if it is equipped with an involution $\varphi\colon P \to P$ which is either monotone or antimonotone and $\varphi^2 = \mathrm{id}_P$. Instead of involutive we also say that $(P, \preceq)$ admits a $\mathbb{Z}_2$-*action* or that $(P, \preceq)$ is a $\mathbb{Z}_2$-*poset*. We will call a $\mathbb{Z}_2$-poset $(P, \preceq, \varphi)$ *free* if φ is a free $\mathbb{Z}_2$-action on its order complex.

Theorem 4 *Let P be a poset with a free involution, and let a $\mathbb{Z}_2$-map ϕ be a descending closure operator. Then $\Delta(P)$ $\mathbb{Z}_2$-collapses onto $\Delta(\phi(P))$. By symmetry the same is true for a $\mathbb{Z}_2$-ascending closure operator.*

Proof The same argument works as in [7, Theorem 2.1]. ■

We introduce the basics of Discrete Morse Theory, which was invented by Forman [5]. It provides a convenient language for describing sequences of elementary collapses.

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Definition 5 Let P be a poset with the order relation $\succ$.

- We define a *partial matching* on P to be a set $\Sigma \subseteq P$, and an injective map $\mu\colon \Sigma \to P \setminus \Sigma$, such that $\mu(x) \succ x$, for all $x \in \Sigma$.
- The elements of $P \setminus (\Sigma \cup \mu(\Sigma))$ are called *critical*.
- Additionally, such a partial matching μ is called *acyclic* if there exists no sequence of distinct elements $x_1, \dots, x_t \in \Sigma$ with $t \geq 2$ satisfying $\mu(x_1) \succ x_2$, $\mu(x_2) \succ x_3$, $\dots, \mu(x_t) \succ x_1$.

The partial acyclic matchings and elementary collapses are closely related, as the next proposition shows.

Proposition 6 **([6, Proposition 5.4])** *Let Δ be a regular CW complex and Δ' a subcomplex of Δ. Then the following are equivalent:*

(a) *there is a sequence of elementary collapses leading from Δ to Δ';*
(b) *there is a partial acyclic matching on the face poset of Δ with the set of critical cells being exactly the simplices of Δ'.*

Remark 7. We will use the $\mathbb{Z}_2$-version of this theorem. In our settings $\Delta \supset \Delta'$ are free $\mathbb{Z}_2$-simplicial complexes, and the acyclic matching μ respects the $\mathbb{Z}_2$-action ν (*i.e.,* $\nu(\mu(x)) = \mu(\nu(x))$). In this case we only use that Δ $\mathbb{Z}_2$-collapses to Δ' (the critical cells of this $\mathbb{Z}_2$ symmetric matching are the simplices of Δ'). The same argument as in [6, Proposition 5.4] proves this $\mathbb{Z}_2$ variation.

3 Simple $\mathbb{Z}_2$-homotopy Equivalences of Graph Complexes

In this section we will prove that $\mathrm{B}(G)$ collapses to $\mathrm{N}(G)$, $\mathrm{sd}(\mathrm{B}(G))$ $\mathbb{Z}_2$-collapses to $\mathrm{sd}(\mathrm{Hom}(K_2, G))$, and $\mathrm{ssd}(\mathrm{B}(G))$ $\mathbb{Z}_2$-collapses to $\mathrm{L}(G)$.

Theorem 8 *$\mathrm{B}(G)$ collapses to $\mathrm{N}(G)$.*

Proof We will collapse $\mathrm{B}(G)$ to its first shore, which is isomorphic to $\mathrm{N}(G)$. Let $\sigma \in \mathrm{B}(G)$ be a simplex such that it has a vertex from the second shore. Then $\sigma = \{v_1 \uplus \varnothing, \dots, v_l \uplus \varnothing; \varnothing \uplus w_1, \dots, \varnothing \uplus w_k\}$. The set $\varnothing \neq \{w_1, \dots, w_k\}$ has a common neighbor by the properties of the box complex. We denote the smallest[1] common neighbor by x_σ. We define the matching μ by

$$\mu(\sigma) := \begin{cases} \sigma \setminus (x_\sigma \uplus \varnothing) & \text{if } (x_\sigma \uplus \varnothing) \in \sigma, \\ \sigma \cup (x_\sigma \uplus \varnothing) & \text{if } (x_\sigma \uplus \varnothing) \notin \sigma. \end{cases}$$

This matching is well defined since σ and $\mu(\sigma)$ have the same vertex set from the second shore, so $x_{\mu(\sigma)} = x_\sigma$. We show that μ is acyclic. If we go up by the matching ($\sigma \subset \mu(\sigma)$) then we should delete a vertex $v \uplus \varnothing$ from the first shore (we can never add a vertex to the second shore). If we do not delete $x_\sigma \uplus \varnothing$, then $\mu(\sigma) \setminus (v \uplus \varnothing)$ is matched down. The critical cells of μ are the simplices of the first shore, which completes the proof. ∎

[1] Now we have to fix a linear order on $V(G)$ to be consistent with our choices.

Theorem 9 $\operatorname{sd}(\mathrm{B}(G))$ *$\mathbb{Z}_2$-collapses to* $\operatorname{sd}(\operatorname{Hom}(K_2, G))$.

Proof $\operatorname{sd}(\operatorname{Hom}(K_2, G))$ is a subcomplex of $\operatorname{sd}(\mathrm{B}(G))$. The extra vertices are vertices on the shores of the box complex $\operatorname{sd}(\mathrm{B}(G))$. (They are in the form $\varnothing \uplus A$ and $B \uplus \varnothing$.) We work only with the first shore: the $B \uplus \varnothing$ part of $\operatorname{sd}(\mathrm{B}(G))$. On the other shore every collapse $\mathbb{Z}_2$-pair is done. We describe an acyclic matching on the face poset of $\operatorname{sd}(\mathrm{B}(G))$. Let $\sigma \in \operatorname{sd}(\mathrm{B}(G))$ be a simplex. We assume that σ has a vertex from the first shore. Its vertices form a chain

$$A_1 \uplus \varnothing \subset \cdots \subset A_n \uplus \varnothing \subset A_{n+1} \uplus B_1 \subset \cdots \subset A_{n+m} \uplus B_m,$$

where $n \geq 1$ and $B_1 \neq \varnothing$. We set $B_0 = \varnothing$ and consider the vertex $\mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$. Let i be the maximal index such that $A_{n+i} \uplus B_i \subseteq \mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$. We note that $A_n \uplus B_0 \subseteq \mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$, so such an i exists.

If $i = m$, then we can have $A_{n+m} \uplus B_m = \mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$. In this case we match σ with $\sigma \setminus (\mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n))$. Otherwise, we match σ with $\sigma \cup (\mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n))$.

If $i \neq m$, then we consider $X \uplus Y := A_{n+i+1} \uplus B_{i+1} \cap \mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$. If $(X \uplus Y) \in \sigma$, then we match σ with $\sigma \setminus (X \uplus Y)$. If $(X \uplus Y) \notin \sigma$, then we match σ with $\sigma \cup (X \uplus Y)$.

Next we show that the obtained matching μ is acyclic. Assume that there exists a sequence $\sigma_0, \ldots, \sigma_t \in \operatorname{sd}(\mathrm{B}(G))$ such that all σ_i are different, with the exception $\sigma_0 = \sigma_t$, and such that $\mu(\sigma_i) \succ \sigma_{i+1}$ for $0 \leq i \leq t-1$. Assume that $\mu(\sigma_0) = A_1 \uplus \varnothing \subset \cdots \subset A_n \uplus \varnothing \subset A_{n+1} \uplus B_1 \subset \cdots \subset A_{n+m} \uplus B_m$. If σ_0 were $\mu(\sigma_0) \setminus (A_{n+m} \uplus B_m)$, then since $\sigma_0 \neq \sigma_1$ it would be not possible to match σ_1 upwards unless we delete $A_n \uplus \varnothing$. But matched pairs contain the same number of vertices in type $A \uplus \varnothing$, so it can not be a member of a cycle. Otherwise, $\sigma_0 = \mu(\sigma_0) \setminus (A_{n+i} \uplus B_i)$ for some $m > i \geq 1$. Since σ_1 is matched upwards, it is easy to see, that σ_1 should be $\mu(\sigma_0) \setminus (A_{n+i+1} \uplus B_{i+1})$. We see that in σ_1 the number of vertices which are subsets of $\mathrm{CN}^2(A_n) \uplus \mathrm{CN}(A_n)$ is increased by 1 compared to σ_0. Repeating this argument, we see that σ_t has t vertices more, therefore $\sigma_0 \neq \sigma_t$. This leads to the conclusion that μ is acyclic.

The critical simplices form a subcomplex $\operatorname{sd}(\operatorname{Hom}(K_2, G))$, which completes the proof. ∎

Theorem 10 $\operatorname{ssd}(\mathrm{B}(G))$ *$\mathbb{Z}_2$-collapses to* $\mathrm{L}(G)$.

Proof First we show that $\operatorname{ssd}(\mathrm{B}(G))$ $\mathbb{Z}_2$-collapses onto $\mathrm{CN}^2(\operatorname{ssd}(\mathrm{B}(G)))$. This follows from Theorem 4, since CN^2 is a $\mathbb{Z}_2$-descending closure operator.

Next we show that $\mathrm{CN}^2(\operatorname{ssd}(\mathrm{B}(G)))$ $\mathbb{Z}_2$-collapses onto $\mathrm{L}(G)$. We will define simplicial complexes

$$\mathrm{CN}^2\big(\operatorname{ssd}(\mathrm{B}(G))\big) =: \mathrm{S}_0 \supset \mathrm{S}_1 \supset \cdots \supset \mathrm{S}_{N+1} = \mathrm{L}(G)$$

such that S_i $\mathbb{Z}_2$-collapses to S_{i+1}. Assume that S_i is already defined. To define S_{i+1}, we choose a vertex $X \uplus \varnothing \in \mathrm{S}_i$ such that

(i) $\varnothing \uplus \mathrm{CN}(X) \in \mathrm{S}_i$,
(ii) $|X| \geq |\mathrm{CN}(X)|$,

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(iii) there is no Y such that $Y \uplus \varnothing \in \mathsf{S}_i$, $\varnothing \uplus \mathrm{CN}(Y) \in \mathsf{S}_i$, $|Y| \geq |\mathrm{CN}(Y)|$ and $|Y| > |X|$.

The maximality of X implies that a maximal simplex which contains $X \uplus \varnothing$ (resp. $\varnothing \uplus X$) also contains $\varnothing \uplus \mathrm{CN}(X)$ (resp. $\mathrm{CN}(X) \uplus \varnothing$). Now we will just work with the first shore vertex $X \uplus \varnothing$. In order to obtain a $\mathbb{Z}_2$-collapse at each step, a $\mathbb{Z}_2$-pair should be done as well.

We define an acyclic matching on the face poset of S_i. Let $\sigma \in \mathsf{S}_i$ such that $X \uplus \varnothing$ is its vertex. If $\varnothing \uplus \mathrm{CN}(X)$ is a vertex of σ, then we match σ with the simplex $\sigma \setminus (\varnothing \uplus \mathrm{CN}(X))$. Otherwise, we match σ with $\sigma \cup (\varnothing \uplus \mathrm{CN}(X))$.

Next we show that the obtained matching μ is acyclic. Assume that there exists a sequence $\sigma_0, \dots, \sigma_t \in \mathsf{S}_i$ such that all σ_i are different, with the exception $\sigma_0 = \sigma_t$, and such that $\mu(\sigma_i) \succ \sigma_{i+1}$ for $0 \leq i \leq t-1$. Then $\mu(\sigma_0) = \sigma_0 \cup (\varnothing \uplus \mathrm{CN}(X))$. We must obtain σ_1 from $\mu(\sigma_0)$ by deleting one vertex in such a way that it matches upwards. This is possible if and only if we delete the vertex $\varnothing \uplus \mathrm{CN}(X)$, and therefore $\sigma_0 = \sigma_1$. This leads to the conclusion that μ is acyclic.

The critical simplices always form a subcomplex. At the end of this process we arrive at a simplicial complex, that is $\mathbb{Z}_2$-isomorphic to $\mathrm{L}(G)$. This $\mathbb{Z}_2$-isomorphism was proven in [4, Theorem 1]. This completes the proof. ■

4 Simple $\mathbb{Z}_2$-universality of Graph Complexes

It is known that graph complexes are universal up to $\mathbb{Z}_2$-homotopy type.

Theorem 11 **([3])** *Given a free $\mathbb{Z}_2$-simplicial complex (K, ν), there is a graph G such that its graph complex is $\mathbb{Z}_2$-homotopy equivalent to the given complex.*

Now we show the simple homotopy type extension. First we start with the neighborhood complex $\mathrm{N}(G)$.

Theorem 12 *Given a free $\mathbb{Z}_2$-simplicial complex (K, ν), there is a graph G such that its neighborhood complex $\mathrm{N}(G)$ is simple homotopy equivalent to the given complex.*

We will use the construction from [3].

Construction 13 $(\mathsf{K} \to G_{\mathsf{K}})$ *Let K be a $\mathbb{Z}_2$-simplicial complex. The vertices of G_{K} are the vertices of K, and each vertex is connected to its $\mathbb{Z}_2$-pair and the neighbors (neighbors in the 1-skeleton of K) of the $\mathbb{Z}_2$-pair. Thus if $x, y \in V(G_{\mathsf{K}}) = V(\mathsf{K})$ then there is an edge between them if and only if $\nu(x) = y$ or $\{x, \nu(y)\} \in \mathsf{K}$ (or $\{y, \nu(x)\} \in \mathsf{K}$). See Figure 1.*

Proof of Theorem 12 For technical reasons we need the first barycentric subdivision $\mathrm{sd}(\mathsf{K})$ of K. The free simplicial $\mathbb{Z}_2$-action on $\mathrm{sd}(\mathsf{K})$ will be denoted by ν as well. There is no free $\mathbb{Z}_2$-action on the neighborhood complex $\mathrm{N}(G_{\mathrm{sd}(\mathsf{K})})$ in general. But now ν acts freely on $\mathrm{N}(G_{\mathrm{sd}(\mathsf{K})})$. Then $\mathrm{sd}(\mathsf{K})$ and $\mathrm{N}(G_{\mathrm{sd}(\mathsf{K})})$ have the same vertex set. If $A_1 \subset A_2 \subset \cdots \subset A_m$ is a simplex in $\mathrm{sd}(\mathsf{K})$, then in $G_{\mathrm{sd}(\mathsf{K})}$ they have a common neighbor, *e.g.*, $\nu(A_1)$, so it is a simplex in $\mathrm{N}(G_{\mathrm{sd}(\mathsf{K})})$ as well. This means that $\mathrm{sd}(\mathsf{K})$ is a (not induced) subcomplex of $\mathrm{N}(G_{\mathrm{sd}(\mathsf{K})})$. In order to show that $\mathrm{N}(G_{\mathrm{sd}(\mathsf{K})})$ collapses

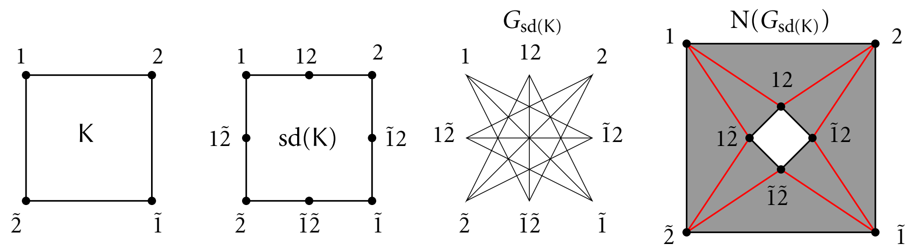

Figure 1: Example for K, sd(K), $G_{\mathrm{sd(K)}}$ and $\mathrm{N}(G_{\mathrm{sd(K)}})$. The $\mathbb{Z}_2$-action is the antipodal map.

to sd(K) we define an acyclic matching on the face poset of $\mathrm{N}(G_{\mathrm{sd(K)}})$ such that its critical cells are exactly the simplices of sd(K). (See Figure 1.)

Let $\sigma = \{X_1, X_2, \dots, X_n\}$ be a simplex of $\mathrm{N}(G_{\mathrm{sd(K)}}) \setminus \mathrm{sd(K)}$. Since it is not a simplex of sd(K) we must have at least two vertices X_i, X_j, such that $X_i \not\subset X_j$ and $X_i \not\supset X_j$. We define a graph H_σ with vertex set $\{X_1, X_2, \dots, X_n\}$. There is an edge between the two vertices X_i, X_j if and only if $X_i \not\subset X_j$ and $X_i \not\supset X_j$. We take the non-trivial connected components of H_σ, *i.e.*, those which contain at least an edge, and denote them by $\{X_{i_1}, X_{i_2}, \dots, X_{i_k}\}, \{X_{j_1}, X_{j_2}, \dots, X_{j_l}\}, \dots$. (See Figure 2.)

Since σ is a simplex, there exist Y (common neighbor) such that $Y \subseteq X_i$ or $Y \supseteq X_i$ for any i. Now for each non-trivial connected component of H_σ we have $\bigcup_{t=1}^{k} X_{i_t}$ and $\bigcap_{t=1}^{k} X_{i_t}$, and we call them *special*. For each component at least one of the special

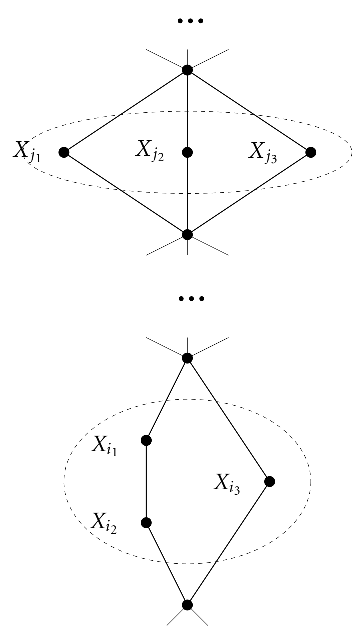

Figure 2: The inclusion poset of $X_1, \dots, X_n$.

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vertices must be a vertex of $\mathrm{N}(G_{\mathrm{sd}(\mathrm{K})})$. If $Y \subseteq X_{i_t}$ for any t, then the intersection is a vertex. If $Y \supseteq X_{i_t}$ for any t, then the union is a vertex. If there exist s and t such that $X_{i_s} \subseteq Y \subseteq X_{i_t}$, then the vertices of the component above Y and the vertices of this component below Y were in different connected components. So for a simplex $\sigma \in \mathrm{N}(G_{\mathrm{sd}(\mathrm{K})}) \setminus \mathrm{sd}(\mathrm{K})$ we have the special vertices assigned to σ and we denote the smallest by X_σ. Now we are ready to define our matching μ:

$$\mu(\sigma) := \begin{cases} \sigma \setminus X_\sigma & \text{if } X_\sigma \in \sigma, \\ \sigma \cup X_\sigma & \text{otherwise.} \end{cases}$$

This matching is well defined since for any i, $X_\sigma \subseteq X_i$ or $X_\sigma \supseteq X_i$. This means the non-trivial components of H_σ and $H_{\mu(\sigma)}$ are the same, $\mu(\mu(\sigma)) = \sigma$.

This matching is acyclic. If $\mu(\sigma) \supset \sigma$ (we went up by the matching), then we have to delete some vertex of $\mu(\sigma)$ to go down. H_σ and $H_{\mu(\sigma)}$ have the same number of edges, so we have to delete a vertex $X_\sigma \neq X_i \in \sigma$ such that for any j $X_i \subset X_j$ or $X_i \supset X_j$. But now the connected components of H_σ, $H_{\mu(\sigma)}$ and $H_{\mu(\sigma)\setminus X_i}$ are the same, which means that $\mu(\sigma) \setminus X_i$ is matched down so we cannot get back to $\mu(\sigma)$.

The critical cells of μ are the simplices of $\mathrm{sd}(\mathrm{K})$, which completes the proof. ■

Remark 14. Since this matching respects the $\mathbb{Z}_2$-action ν, we have actually shown that (K, ν) and $(\mathrm{N}(G_{\mathrm{sd}(\mathrm{K})}), \nu)$ are simple $\mathbb{Z}_2$-homotopy equivalent.

Theorem 15 *Given a free $\mathbb{Z}_2$-simplicial complex (K, ν), there is a graph G such that its graph complex ($\mathrm{L}(G)$, $\mathrm{B}(G)$, $\mathrm{Hom}(K_2, G)$) is simple $\mathbb{Z}_2$-homotopy equivalent to the given complex.*

Proof Using the same construction as before, we will show that $(\mathrm{sd}(\mathrm{K}), \nu)$ and one of the graph complexes, $\mathrm{B}(G_{\mathrm{sd}(\mathrm{K})})$ are simple $\mathbb{Z}_2$-homotopy equivalent, where the $\mathbb{Z}_2$-action is to interchange the shores. First we define a $\mathbb{Z}_2$-embedding of $\mathrm{sd}(\mathrm{K})$ into $\mathrm{B}(G_{\mathrm{sd}(\mathrm{K})})$. For each pair of vertices $(A, \nu(A))$ of $\mathrm{sd}(\mathrm{K})$ we have a choice. We can map A into $A \uplus \varnothing$ and $\nu(A)$ into $\varnothing \uplus A$, or we can map A into $\varnothing \uplus \nu(A)$ and $\nu(A)$ into $\nu(A) \uplus \varnothing$. So it is not a canonical embedding, since we have two choices for each vertex pair. We show that this map defined on the vertex set is simplicial. A simplex $A_1 \subset A_2 \subset \cdots \subset A_m$ of $\mathrm{sd}(\mathrm{K})$ is mapped to a subsimplex of $(A_1 \uplus \varnothing) \subset (A_2 \uplus \varnothing) \subset \cdots \subset (A_m \uplus \varnothing), (\varnothing \uplus \nu(A_1)) \subset (\varnothing \uplus \nu(A_2)) \subset \cdots \subset (\varnothing \uplus \nu(A_m))$ which is a simplex of $\mathrm{B}(G_{\mathrm{sd}(\mathrm{K})})$. We consider $\mathrm{sd}(\mathrm{K})$ as the image of this embedding.

We will collapse $\mathrm{B}(G_{\mathrm{sd}(\mathrm{K})})$ to $\mathrm{sd}(\mathrm{K})$ in two steps.

STEP1. We pick a simplex $\sigma \in \mathrm{B}(G_{\mathrm{sd}(\mathrm{K})}) \setminus \mathrm{sd}(\mathrm{K})$,

$$\sigma = \{A_1 \uplus \varnothing, A_2 \uplus \varnothing, \dots, A_l \uplus \varnothing; \varnothing \uplus B_1, \varnothing \uplus B_2, \dots, \varnothing \uplus B_k\}.$$

We define the following two simplices of $(\mathrm{N}(G_{\mathrm{sd}(\mathrm{K})}))$: $\sigma_1 = \{A_1, A_2, \dots, A_l\}$ and $\sigma_2 = \{B_1, B_2, \dots, B_k\}$. We will exploit the notations of the proof of Theorem 12. We use the graphs H_{σ_1} and H_{σ_2} and we define the vertex $V_\sigma := A_{\sigma_1} \uplus \varnothing$ if A_{σ_1} exists (the smallest special vertex assigned to σ_1). If it does not exist let $V_\sigma := \varnothing \uplus B_{\sigma_2}$ if it exists. If that does not exist either, it means that H_{σ_1} and H_{σ_2} contain no edge, and thus the vertices of σ_1 and σ_2 form a chain, so they are simplices of $\mathrm{sd}(\mathrm{K})$ as well. Those will be the critical cells of this first matching. Now we are able to define the matching μ. If $V_\sigma \neq \varnothing$, then

$$\mu(\sigma) := \begin{cases} \sigma \setminus V_\sigma & \text{if } V_\sigma \in \sigma, \\ \sigma \cup X_\sigma & \text{otherwise.} \end{cases}$$

This matching is well defined and acyclic as before. Its critical cells form a subcomplex $\mathrm{C} \subseteq \mathrm{B}(G_{\mathrm{sd(K)}})$.

STEP2. Now we will collapse down C to sd(K). When we constructed the embedding $\mathrm{sd(K)} \hookrightarrow \mathrm{B}(G_{\mathrm{sd(K)}})$, we had to choose for every $\mathbb{Z}_2$-pair $(A, \nu(A))$ whether to map it to $(A \uplus \varnothing, \varnothing \uplus A)$ or $(\varnothing \uplus \nu(A), \nu(A) \uplus \varnothing)$. We will refer to the vertices of $\mathrm{B}(G_{\mathrm{sd(K)}}) \setminus \mathrm{sd(K)}$ as *bad* vertices. We pick a simplex $\sigma \in \mathrm{C}$ such that it contains a bad vertex. Let $\sigma = \{(A_1 \uplus \varnothing) \subset (A_2 \uplus \varnothing) \subset \cdots \subset (A_l \uplus \varnothing), (\varnothing \uplus B_1) \subset (\varnothing \uplus B_2) \subset \cdots \subset (\varnothing \uplus B_k)\}$. Now we define W_σ to be the smallest bad vertex of σ in the form $A_i \uplus \varnothing$. If they are all good, then we define W_σ to be the smallest bad vertex of σ in the form $\varnothing \uplus B_j$. Now we can define a matching μ:

$$\mu(\sigma) := \begin{cases} \sigma \setminus (\varnothing \uplus \mu(A_i)) & \text{if } W_\sigma = A_i \uplus \varnothing \in \sigma \text{ and } (\varnothing \uplus \mu(A_i)) \in \sigma, \\ \sigma \cup (\varnothing \uplus \mu(A_i)) & \text{if } W_\sigma = A_i \uplus \varnothing \in \sigma \text{ and } (\varnothing \uplus \mu(A_i)) \notin \sigma, \\ \sigma \setminus (\mu(B_j) \uplus \varnothing) & \text{if } W_\sigma = \varnothing \uplus B_j \in \sigma \text{ and } (\mu(B_j) \uplus \varnothing) \in \sigma, \\ \sigma \cup (\mu(B_j) \uplus \varnothing) & \text{if } W_\sigma = \varnothing \uplus B_j \in \sigma \text{ and } (\mu(B_j) \uplus \varnothing) \notin \sigma. \end{cases}$$

Since we add/delete a good vertex $W_\sigma = W_{\mu(\sigma)}$, this matching is well defined. The acyclicity easily follows from the fact that σ and $\mu(\sigma)$ have the same bad vertex set. The critical cells of this matching are exactly the simplices of sd(K).

Since our matching respects the $\mathbb{Z}_2$-actions, we have completed the proof. ■

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Department of Mathematics, Middlesex College, The University of Western Ontario, London, Ontario N6A 5B7
e-mail: pcsorba@uwo.ca