A BANACH LATTICE NOT WEAKLY PROJECTABLE

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(Received 1 June 1971)
Communicated by B. Mond

In [4] a concept of a weakly projectable vector lattice has been introduced. Stone vector lattices [3] and thus all special types of them, like Riesz [5], \( \sigma \)-complete and complete vector lattices are weakly projectable. Moreover \( C[0,1] \) is weakly projectable but not Stone [4]. As we see the collection \( W \) of weakly projectable vector lattices is quite large. This explains to some extent the difficulty in producing examples of vector lattices which do not belong to \( W \). In this note an example of a Banach lattice [1] which is not weakly projectable is described.

DEFINITION. A vector lattice \( E \) is said to be weakly projectable if for any \( x, y \in E \) there exists \( z \in x^\perp \) such that \( y \in (|x| + |z|)^\perp \).

(For definitions of symbols used above we refer e.g. to [2]).

EXAMPLE. Let \( F[0,1] \) denote the space of bounded real valued functions defined on \( [0,1] \) and discontinuous at most at a countable set of points. Addition and multiplication by scalars are introduced in the usual way. The order is defined by: \( x \geq 0 \) if and only if \( x(t) \geq 0 \) for each \( t \in [0,1] \). Define also \( \| x \| = \sup_{0 \leq t \leq 1} |x(t)| \).

Using standard methods it is easy to prove that \( F[0,1] \) is a Banach lattice and thus an Archimedean vector lattice [1]. We shall prove that \( F[0,1] \notin W \).

Let \( \{r_n\} \) be a sequence dense in the interval \([0,1]\). Denote by \( A \) the set

\[
A = [0,1] \cap \left( \bigcup_{n=1}^{\infty} (r_n - 4^{-n}, r_n + 4^{-n}) \right).
\]

\( A \) is open in \([0,1]\) and \( \text{mes} \ A \leq \frac{1}{2} \), thus \( A' = [0,1] \setminus A \) is a closed uncountable set. We have also \( \overline{A} = [0,1] \). Define \( x: [0,1] \to \mathbb{R} \) by

\[
x(t) = \text{distance from } t \text{ to } A'.
\]

\( x \) is continuous on \([0,1]\), and so \( x \in F[0,1] \). To show that \( F[0,1] \notin W \) it is sufficient to prove that for any \( z \in x^\perp \), the identity function \( e: e(t) = 1 \) for all \( t \in [0,1] \) does not belong to \((|x| + |z|)^\perp \). Take any \( z \in x^\perp \). Then \( z(t) = 0 \) for all \( t \in A \).
Moreover, since $A$ is dense in $[0,1]$, any point $t_0$ of $[0,1]$ is a limit of a sequence \( \{t_n\} \) of points in $A$. Therefore if $z$ is continuous at $t_0$ then $z(t_0) = \lim_{n \to \infty} z(t_n) = 0$. Since $z \in F[0,1]$, it is discontinuous at most at a countable set. On the other hand $A'$ is not countable. Consequently, there exists a point $\tau \in A'$ such that $z(\tau) = 0$. Since $\tau \in A'$, we have also $x(\tau) = 0$. Thus $w = |z| + |x|$ vanishes at $\tau$. Hence the function $u$ defined by

$$u(t) = \begin{cases} 1 & \text{if } t = \tau, \\ 0 & \text{if } t \neq \tau \end{cases}$$

belongs to $F[0,1]$ and $u \perp w$.

Let $v \in w^\perp$. Since $u \in w^\perp$ and $u(\tau) \neq 0$, it follows that $v(\tau) = 0$. On the other hand $e(\tau) = 1$ and thus $e \notin w^\perp$. This concludes the proof that $F[0,1]$ is not weakly projectable.

References


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