2 The Role of Stochastic Geometry in HetNet Analysis

In Chapter 1, Section 1.8, we list several important and general analytical results for single-tier networks and multitier HetNets that were derived using results from stochastic geometry. Of course, the value of analytical results in yielding insights to help network design has long been recognized by the industry. Stochastic geometric approaches are only the latest, and by far the most successful, of a long line of analytical approaches that began almost immediately after the first “cellular” models for wireless communication were proposed. Before commencing a detailed study of stochastic geometry, it is useful to take a step back and examine competing analytical approaches to get a better idea of why stochastic geometric approaches achieved greater success. To do so, let us revisit the origins of the now-canonical hexagonal cell model.

2.1 The Hexagonal Cellular Concept

The earliest proposals for wireless coverage via hexagonal “cells” date back to 1947, even before the principles of information theory were published by Shannon in 1948. A more up-to-date overview of the cellular concept may be found in MacDonald’s 1979 paper (MacDonald 1979), applied to what is now considered the “first-generation” or “1G” cellular standard, advanced mobile phone service. MacDonald’s carefully reasoned rationale for the hexagonal cellular model bears reading even today:

1. He acknowledged that a practical cellular deployment would contain cells of different shapes and sizes (owing to topographic obstacles, restricted rights-of-way, etc.), but made the reasonable argument that systematization of the design and layout of a cellular system is greatly aided by modeling all cells of the system as having the same shape.
2. At the time MacDonald wrote his paper, cellular base stations transmitted using omnidirectional antennas. This in turn means that the boundary of the coverage region of a base station is circular. In other words, omnidirectional transmitting antennas at the base stations mean circular disk-shaped cells.

As an aside, although the paper is almost entirely about a single-tier macrocell layout, MacDonald even discussed a scenario with a tier of (also hexagonal) small cells overlaid on the macrocell tier, but only to propose a dedicated allocation of spectrum to the small-cell tier.
3. Unfortunately, circular disks do not tile (meaning, cover completely, without overlaps or gaps) the two-dimensional plane.

4. Three shapes that do tile the plane are equilateral triangles, squares, and (regular) hexagons.

5. For each of these three shapes, and with the base station located at the center of a cell with that shape, the farthest point within the cell is a vertex of that shape.

6. To assure satisfactory received signal strength at this farthest point (that is, “worst-case” or “cell-edge” coverage), the distance between the center of the cell and the vertex must be kept below a maximum.

7. For a fixed center-to-vertex distance, the hexagonal shape has a larger area than an equilateral triangle or a square.

8. This implies that a cellular deployment with hexagonal cells requires fewer base stations to cover the same region than either of the other shapes.

Notwithstanding its prescience on many matters, MacDonald’s paper contains almost no analytical results, except for some basic calculations of distances between points located on a hexagonal lattice. In particular, there is no quantitative analysis of spectral efficiency or signal-to-interference ratio (SIR) distributions, because closed-form analytical results for hexagonal lattices do not exist.

In fact, these arguments in favor of the hexagonal cell shape implicitly assume that the criterion for network design is the distribution of received signal power at a user location, and not the SIR distribution at the user terminal. However, the relationship between the data rate on an individual link to a user terminal (or, indeed, the network average throughput per unit area, which we identified as one of the key criteria for evaluating the performance of a network) and the distribution of SIR at this user’s location, means that the design of the network must be driven by the distribution of SIR rather than the distribution of signal power. Before we proceed further, let us take a moment to provide a clear definition of what we mean by the received signal power and the SIR (and its close relative, the SINR [signal-to-interference-plus-noise ratio]) of a link, and in the process introduce some basic notation that is used throughout the book.

### 2.2 Propagation, Fading, and SINR

When a transmitter (say, a base station) transmits with power $P_{tx}$, the received power $P_{rx}$ at a receiver (say, a user terminal) at a distance of $d$ from the transmitter is given by

$$P_{rx} = P_{tx} \times G_{tx} \times G_{rx} \times H \times \ell(d).$$  \hspace{1cm} (2.1)

Note that in Eq. (2.1), $G_{tx}$, $G_{rx}$, $H$, and $\ell(d)$ are all dimensionless.

The quantities $G_{tx}$ and $G_{rx}$ are the gains of the transmit and receive antennas, respectively.

The random variable $H$ is the fading coefficient on the link. Note that in industry terminology, the fading is described by the probability distribution of the amplitude of the equivalent baseband complex waveform that is actually transmitted over the wireless
channel. Thus, if the link is said to be subject to Rayleigh fading, then the distribution of $H$, which applies to the received power and not the received signal amplitude, is that of the square of a Rayleigh-distributed random variable, and is therefore exponential. Without loss of generality, the expected value of $H$ can be set to unity, and it is customary to do so. Thus, in the case of Rayleigh fading on this link, $H$ is exponentially distributed with unit mean, and its probability density function is given by

$$f_H(x) = \exp(-x), \quad x > 0. \tag{2.2}$$

In Eq. (2.1), $\ell(d)$ is called the \textit{path loss} over the link where $d$ is the transmitter-receiver distance.\(^2\) A widely used model has the form

$$\ell(d) = K\left(d/d_\text{unit}\right)^{-\alpha}, \tag{2.3}$$

where $d_\text{unit} = 1$ m and $\alpha$ is called the \textit{path loss exponent}. The free-space path loss exponent is $\alpha = 2$, so $\alpha > 2$ for nearly all links, except when the link is in the form of a waveguide (for example, a tunnel). In addition, $\alpha$ is dependent upon the frequency band. In Eq. (2.3), the dimensionless quantity $K$ is a function of the geometry of the link (mostly, the relative heights of the transmitter and receiver, with some dependence upon the frequency band of the transmission).

It is customary in the industry to work with powers and gains in decibels. Rewriting Eq. (2.3), we have

$$10 \log_{10}[\ell(d)] = -10\alpha \times \log_{10}(d/d_\text{unit}) + 10 \log_{10} K, \tag{2.4}$$

or equivalently in decibels,

$$(\ell(d))_\text{dB} = -\alpha \times (d/d_\text{unit})_\text{dB} + (K)_\text{dB}, \tag{2.5}$$

which is the equation of a straight line with slope $-\alpha$ and intercept $(K)_\text{dB}$. This is why Eq. (2.3) is often called the \textit{slope-intercept} model. For notational convenience in the future, we shall define the quantity

$$\kappa = K^{-1/\alpha}/d_\text{unit}, \tag{2.6}$$

so that Eq. (2.3) may now be written as

$$\ell(d) = (\kappa d)^{-\alpha}. \tag{2.7}$$

Note that unlike $K$, $\kappa$ is not dimensionless.

The receivers at the user terminals always have a certain level of background noise power, called \textit{thermal noise power}, which we denote by $N$.

Suppose a user terminal is receiving from its serving base station at a distance of $d_0$ while $n$ other base stations at distances $d_1, \ldots, d_n$ from this user location are simultaneously transmitting (to other users served by them). Assuming all base stations are identical and transmit using omnidirectional antennas with the same power $P_\text{tx}$, and the fading coefficients on the links from the $n$ interfering base stations to the user are iid

\(^2\) Note that, following the convention in the wireless literature, we call $\ell(d)$ a \textit{loss} even though it multiplies the transmit power, and is therefore actually a \textit{gain}, albeit of magnitude less than unity.
random variables \( H_1, \ldots, H_n \), assumed independent of the fading coefficient \( H_0 \) on the link between the user and its serving base station, then the received power at the user from its serving base station, called the \textit{received signal power}, is given by

\[
S = P H_0 \ell(d_0),
\]  

(2.8)

where

\[
P = P_{tx} G_{tx} G_{rx}.
\]

Note that the antenna gains \( G_{tx} \) and \( G_{rx} \) have no units, so \( P \) may be treated as the effective transmit power of each base station. The interference power at the user location is similarly given by

\[
I = \sum_{i=1}^{n} P H_i \ell(d_i).
\]

(2.9)

Finally, the SIR and SINR are defined by

\[
\text{SIR} = \frac{S}{I} \quad \text{and} \quad \text{SINR} = \frac{S}{I + N}.
\]

(2.10)

The distribution of SIR is the single most important quantity in the design and analysis of wireless communication networks because it simply determines the quality of communication channels, their sustainable bit rates, outage probabilities, etc. Specific relations between SIR and channel quality metrics depend on particular communication technology, such as the coding schemes, type of multiplexing, and so forth. Information theory provides theoretical upper bounds on these metrics as functions of SIR for several mathematical models of communication channel. For example, Shannon’s celebrated result (Cover & Thomas 1991) states that in the additive white Gaussian noise (AWGN) channel a given transmission bit rate is sustainable in the long term (that is, such transmissions over such an AWGN channel can be decoded without error) if and only if this transmission bit rate is smaller than \( W \log_2 (1 + \text{SNR}) \), where \( W \) is the channel bandwidth.

The distribution of SIR at the user location for the above scenario has an elegant form when the link to the serving base station at distance \( d_0 \) has Rayleigh fading. Then \( H_0 \) has the probability density (function) (Eq. [2.2]), and from Eqs. (2.8) and (2.10), the complementary cumulative distribution function (CCDF) of SIR can be derived as follows:

\[
\Pr \left\{ \frac{S}{I} > x \right\} = \Pr \left\{ H_0 > \frac{x}{P \ell(d_0)} I \right\} = \mathbb{E} \exp \left( - \frac{x}{P \ell(d_0)} I \right) = \mathcal{L}_I \left( \frac{x}{P \ell(d_0)} \right), \quad x > 0,
\]

(2.11)

where, for any random variable \( X \),

\[
\mathcal{L}_X(s) = \mathbb{E} \exp(-sX), \quad s \geq 0
\]

(2.12)

is the Laplace transform of \( X \). Note that from Eq. (2.11), the distribution of SIR is given in terms of the Laplace transform of interference power \( I \) at the user. We next discuss the circumstances under which \( \mathcal{L}_I(\cdot) \) can be derived analytically.
2.3 Base Station Locations Modeled by Point Processes

When we re-examine MacDonald’s arguments from the perspective of SIR distribution, it is obvious that we no longer require cells to have the same shape (that is, there is no need for a regular tiling of the plane). Instead, it is apparent that “irregular” layouts of base stations can still lead to the same cell-edge distribution of SIR. Moreover, such irregular base station layouts are a better match to real-world network deployments than a hexagonal lattice.

It is appealing to propose a “realistic” model of base station locations by “perturbing” the points of a hexagonal lattice in some way (see Banani, Eckford, & Adve 2014, for a modern example of this approach, including a comparison to the regular hexagonal lattice and the Poisson point process). Unfortunately, it turns out that perturbed hexagonal lattices are no easier to analyze exactly than the regular hexagonal lattice. However, the modeling of these “perturbations” by random variables means that the resulting base station locations are now random. This idea leads naturally to modeling base station locations as realizations of the class of stochastic processes called point processes.\(^3\)

Once we accept the idea of modeling base station locations by point processes, we can focus on the reasons for analytical tractability using results from stochastic geometry. As we discuss in Chapter 4, the main reasons are the following:

1. assumption of iid fading on all links to a user terminal from all points of the point process;
2. existence of Campbell theorems yielding analytical expressions for the probability generating functional of a point process, resulting in an analytical expression for the Laplace transform of the interference power at the user terminal; and
3. assumption of Rayleigh fading on the link between the user terminal and its serving base station, which allows us to write an expression for the CCDF of the SIR at the user terminal in terms of the Laplace transform of the interference power without needing to invert this Laplace transform.

It is not surprising that the first successful examples of such analysis were for base station locations modeled as points of homogeneous Poisson point processes, which are the most analytically tractable point processes. However, recent advances have extended the set of analytic results on user SIR distribution to encompass other point processes such as Ginibre and Cox point processes.

To proceed further, we require the contents of the next chapter, Chapter 3, which provides a rigorous and detailed introduction to the theory of point processes.

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\(^3\) A point process may be loosely described as a random pattern of points, and indeed a realization of a point process is called a point pattern.