IMMERSIONS OF METRIC SPACES INTO EUCLIDEAN SPACES

TAKEO AKASAKI

1. Introduction. In a recent paper on isotopy invariants (1), S. T. Hu defined the enveloping space $E_m(X)$ of any given topological space $X$ for each integer $m > 1$. By an application of the Smith theory to the singular cohomology of the enveloping space $E_m(X)$, he obtained his immersion classes $\Psi_m^n(X)$ for every $n = 1, 2, 3, \ldots$ and proved (3) the main theorem that a necessary condition for a compact metric space $X$ to be immersible into the $n$-dimensional Euclidean space $R^n$ is $\Psi_2^n(X) = 0$. This theorem was proved earlier by W. T. Wu (4) for finitely triangulable spaces $X$ using purely combinatorial methods.

The objective of the present paper is to prove the above-mentioned theorem for arbitrary metric spaces. Our treatment follows that of S. T. Hu (3) in which he considers a homotopically equivalent subspace of $E_m(X)$. By a further localization of the situation, we obtain a homotopically equivalent subspace of $E_m(X)$ for locally finite open coverings of $X$. This enables us to remove the compactness condition.

The reader is referred to (1) and (3) for definitions and notation.

2. The map $\delta$ and the subspace $E_m(X, \delta)$. Let $\mathfrak{g}$ be a given locally finite open covering of an arbitrary metric space $X$ with a distance function $d : X^m \times X^m \to R$ in the topological power $X^m$. Define a positive real-valued function $\delta$ on $X$ as follows. Let $x$ be an arbitrary point of $X$. Since $\mathfrak{g}$ is a locally finite open covering of $X$, the point $x$ meets only a finite number of members of $\mathfrak{g}$, say $V_1, V_2, \ldots V_q$. Then define

$$\delta(x) = \max_{\delta < \epsilon} [d(x, X \setminus V_\delta)].$$

Continuity of $\delta$ is obvious. Call $\delta$ the canonical map of the given covering $\mathfrak{g}$. Next, for any path $\sigma \in E_m(x)$, $\sigma(0)$ is a point of the diagonal $X$ of the $m$th power $X^m$ and thus $\delta(\sigma(0))$ is a well-defined positive real number. Let $E_m(X, \delta)$ denote the subspace of the $m$th enveloping space $E_m(x)$ which consists of all paths $\sigma \in E_m(X)$ satisfying the condition

$$d[\sigma(0), \sigma(t)] < \frac{1}{2} \delta[\sigma(0)]$$

for every $t \in I$. Since

$$d[\sigma(0), \sigma(t)] = d[\xi[\sigma(0)], \xi[\sigma(t)]],$$

Received July 20, 1964. This research was supported in part by the Air Force Office of Scientific Research.

1015
\(\xi\) sends \(E_m(X, \delta)\) onto itself. Therefore, we have the orbit space
\[
E_m^*(X, \delta) = E_m(X, \delta)/G.
\]
Since the canonical map \(\delta\) is continuous and positive for all points of \(X\), the following theorem holds as a result of the proof of (3, 4.1).

**Theorem 2.1.** There exists a homotopy
\[
h_t: E_m(X) \to E_m(X) \quad (t \in I)
\]
satisfying the following conditions:

1. (2.1A) \(h_0\) is the identity map on \(E_m(X)\).
2. (2.1B) \(h_1\) sends \(E_m(X)\) into \(E_m(X, \delta)\).
3. (2.1C) For every \(t \in I\), \(h_t\) sends \(E_m(X, \delta)\) into itself.
4. (2.1D) For every \(t \in I\), \(h_t \circ \xi = \xi \circ h_t\).

**Corollary 2.2.** The inclusion map
\[
i^*: E_m^*(X, \delta) \subset E_m^*(X)
\]
is a homotopy equivalence.

### 3. The main theorem.

We are concerned here with an arbitrarily given immersion \(j: X \to Y\) of a metric space \(X\) into any topological space \(Y\).

For each point \(x\) of \(X\), choose an open neighbourhood \(U_x\) of \(x\) in \(X\) such that \(j|U_x\) is an imbedding. Since every metric space is paracompact, the open cover \(\mathcal{C} = \{U_x|x \in X\}\) has a locally finite open refinement \(\mathcal{G} = \{V_\mu|\mu \in M\}\) (\(M\) an index set) which covers \(X\). Let \(\delta\) denote the canonical map of the covering \(\mathcal{G}\), and consider the subspace \(E_m(X, \delta)\) of the \(m\)th enveloping space \(E_m(X)\) of the metric space \(X\) as defined in §2.

Let \(\sigma \in E_m(X, \delta)\) be arbitrarily given. Since \(\sigma: I \to X^m\) is a path in the \(m\)th topological power \(X^m\) of \(X\), we may compose \(\sigma\) with the \(m\)th topological power \(j^m: X^m \to Y^m\) of the given immersion \(j: X \to Y\) and obtain a path \(j^m \circ \sigma: I \to Y^m\).

**Lemma 3.1.** For every \(\sigma \in E_m(X, \delta)\), we have
\[
j^m \circ \sigma \in E_m(Y).
\]

**Proof.** Let \(\sigma\) be an arbitrary path from \(E_m(X, \delta)\). We must show that \(j^m[\sigma(t)]\) is a point on the diagonal \(Y\) of \(Y^m\) if and only if \(t = 0\). If \(t = 0\), the result follows immediately. On the other hand, suppose that \(j^m[\sigma(t)]\) is a point on the diagonal \(Y\) for some \(t \in I\). In order to conclude that \(t = 0\), it suffices to show that \(\sigma(t)\) is a point on the diagonal \(X\) of \(X^m\). Let
\[
\sigma(t) = (x_1, x_2, \ldots, x_m) \in X^m \quad \text{and} \quad j^m[\sigma(t)] = (y, y, \ldots, y)
\]
where \(y\) is a point of the space \(Y\). Then
\[
j(x_1) = j(x_2) = \ldots = j(x_m) = y.
\]
Let $V_1, V_2, \ldots, V_q$ be the members of $\mathcal{F}$ which contain the point $\sigma(0)$. Since $\sigma \in E_m(X, \delta)$, it follows that
\[
d[\sigma(0), \sigma(t)] < \frac{1}{2}\delta[\sigma(0)] = \frac{1}{2} \max_{1 \leq i \leq q} [d(\sigma(0), X \setminus V_i)] = \frac{1}{2} d[\sigma(0), X \setminus V_k]\]
for some $k = 1, 2, \ldots, q$, and thus the set of points \{x_1, x_2, \ldots, x_m\} is in $V_k$. Since $\mathcal{F}$ is a refinement of $\mathcal{C}$, there is an open neighbourhood $U$ of $\mathcal{C}$ containing $V_k$. But the restriction $j|U$ is an imbedding, and hence $x_1 = x_2 = \ldots = x_m$. This completes the proof.

According to 3.1, $j^*$ defines an imbedding
\[
E_m(j): E_m(X, \delta) \to E_m(Y).
\]
By means of the induced isomorphism
\[
i^{**}: H^n[E_m^*(X); G] \to H^n[E_m^*(X, \delta); G]
\]
of the homotopy equivalence $i^*$ in (2.2) and the map $E_m(j)$, one can define a homomorphism
\[
E_m^{**}(j): H^n[E_m^*(Y); G] \to H^n[E_m^*(X); G]
\]
for each dimension $n$ and every abelian coefficient group $G$ using methods analogous to (3). Routine verification shows that $E_m^{**}(j)$ is independent of the choice of the locally finite open refinement $\mathcal{F}$ of $\mathcal{C}$; that is to say, if $\mathcal{F}'$ is another locally finite open refinement of $\mathcal{C}$ and $\delta'$ is the canonical map of $\mathcal{F}'$, then the diagram
\[
\begin{array}{ccc}
H^n[E_m^*(Y); G] & \to & H^n[E_m^*(X, \delta); G] \\
\downarrow & & \downarrow \\
H^n[E_m^*(X, \delta'); G] & \to & H^n[E_m^*(X); G]
\end{array}
\]
is commutative. Furthermore, one obtains the following proposition.

**Proposition 3.2.** For each $n = 1, 2, \ldots$, we have
\[
E_m^{**}(j)[\Psi_m^n(Y)] = \Psi_m^n(X).
\]

Because of (3, 5.1), (3.2), and the fact that $\Phi_2^n(R^n) = 0$ (3; 4), we are able to state the main theorem.

**Theorem 4.3.** If a metric space $X$ can be immersed in the $n$-dimensional Euclidean space $R^n$, then $\Psi_2^n(X) = 0$.

https://doi.org/10.4153/CJM-1965-095-6 Published online by Cambridge University Press

*University of California, Los Angeles*
*Rutgers, The State University*