MATRIX LINKS, AN EXTREMIZATION PROBLEM, AND THE REDUCTION OF A NON-NEGATIVE MATRIX TO ONE WITH PRESCRIBED ROW AND COLUMN SUMS

M. V. MENON*

1. Definitions and preliminaries. The word matrix will, in this paper, always connote a matrix with non-negative elements, having in general m rows and n columns. In conformity with the definition in **(4)**, two matrices will be said to have the same pattern if the entry in any row or column is zero or not according as the corresponding entry of the other is zero or not. The symbols ρ_i and σ_j will stand for the respective phrases "ith row-sum" and "jth column-sum" of the matrix under consideration. $r_1, \ldots, r_m$; $c_1, \ldots, c_n$ is a set of positive numbers. It will be said to be consistent for the pattern of an $m \times n$ matrix, if there exists a matrix of that pattern for which $\rho_i = r_i$ and $\sigma_j = c_j$ for all i and j.

Given a matrix $A = (a_{i,j}) = (a_{ij})$, a link L between any row and column, say, between the first row and the first column, is defined to be a_{11}, if $a_{11} > 0$, or as any set of positive elements $a_{1j_1}, a_{i_1j_1}, a_{i_1j_2}, a_{i_2j_2}, \ldots, a_{i_kj_k}, a_{i_k1}$, where there are either no elements or exactly two elements of the set in each row or column, except in the first row and the first column which have only one each.

A set S of matrix-positions (i,j) will be said to form a complete set of link positions between the first row and the first column if the set $\{a_{i,j}\colon (i,j) \in S\}$ constitutes a link for the matrix A, between the first row and column, if the a_{ij} were positive. The elements of a link or of a complete set of link positions, between the first row and column, are called vertices, and numbering them in order starting from the vertex in the first row, we have a succession of odd (numbered) and even (numbered) vertices. A link operation $L(\theta)$ on the matrix will consist of the replacement of the odd vertices a_{ij} of the link L by $a_{ij} + \theta$, and the even vertices a_{uv} by $a_{uv} - \theta$. It is understood that θ so chosen that $L(\theta)$ does not give rise to a matrix with negative entries.

We observe that if L links the ith row and the jth column, then $L(\theta)$ changes the ith row-sum r_i to $r_i + \theta$ and the jth column-sum c_j to $c_j + \theta$ but leaves all other row- and column-sums unaltered.

If $\theta > 0$, $L(\theta)$ is called an increasing link operation, and if $\theta < 0$, a decreasing one.

Received March 30, 1966. Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin, under Contract No. DA31-124-ARO-D-462.

*Present address: Department of Statistics, University of Missouri, Columbia, Missouri.

Next, if θ equals the smallest of the even vertices, $L(\theta)$ results in a matrix for which the link L is "broken" at one or more of the even vertices, i.e., there are now zero entries at one or more of the positions occupied by the even vertices. The odd vertex-positions still have positive entries in them. Similarly, if $-\theta$ equals the smallest of the odd vertices, $L(\theta)$ produces a matrix for which L is broken at one or more of the odd vertex-positions. For a value of θ in between these two extremes, $L(\theta)$ preserves the link, and yields a matrix of the same pattern.

2. Lemmas. This section contains two lemmas which are used in the proofs of the theorems stated in the next section. *Throughout the rest of this paper, the word "link" is used to denote a link between the first row and the first column of the matrix under consideration.*

LEMMA 2.1. *Let A and B be two $m \times n$ matrices with $a_{11} = b_{11} = 0$. Let the sums of the elements in the first row and column of A be less than the corresponding sums for B. Further, let $\rho_i = r_i$, $i > 1$, and $\sigma_j = c_j$, $j > 1$, for both the matrices. Then there exists a complete set of link positions between the first row and the first column such that $a_{ij} < b_{ij}$ at every odd vertex and $a_{ij} > b_{ij}$ at every even vertex.*

Proof. Let S_0 consist of all elements a_{i1} such that $a_{i1} < b_{i1}$. T_0 is the set whose sole members are the rows containing an element of S_0. Let (c_{ij}) denote $A - B$. For $k \geqslant 1$, define sets S_k and T_k by the following recursive procedure. S_{2k-1} is the set of all elements a_{ij} for which $c_{ij} > 0$ and which are contained in the set of rows T_{2k-2}, T_{2k-1} is the set of all columns containing an element of S_{2k-1}, S_{2k} is the set of all elements a_{ij} for which $c_{ij} < 0$ and which are contained in the set of columns T_{2k-1}, and T_{2k} is the set of all rows containing an element of S_{2k}. Let p be the least integer for which

$$\bigcup_0^p S_i = \bigcup_0^{p+1} S_i.$$

Let s stand for $\sum c_{ij}$, where the summation is taken over all i and j for which

$$a_{ij} \in \bigcup_0^p S_i.$$

We shall now show that the set $T = T_0 \cup T_1 \cup \ldots \cup T_p$ contains the first row. Suppose that this is not the case.

We observe that if the expression for s as a sum of certain of the c_{ij} contains any particular c_{uv}, then it also contains each c_{uj} in the uth row for which $c_{uj} > 0$. Hence, since $\sum_j c_{ij} = 0$, $i \neq 1$, we have $s \geqslant 0$.

On the other hand, if the expression for s contains any elements in the vth column, it also contains all the elements c_{iv} in the vth column, for which $c_{iv} < 0$. Hence, since $\sum_i c_{ij} = 0$, $j \neq 1$, and $\sum_i c_{i1} < 0$, we have $s < 0$.

The contradictory values of s thus arrived at show that T contains the first row. If i_0 is the smallest integer for which T_i contains the first row, then one sees easily that elements a_{ij} can be chosen from $S_0, S_1, \ldots, S_{i_0}$, the set of whose subscripts (i, j) constitutes a complete set of link positions between the first row and the first column, with the property that $c_{ij} < 0$ at every odd vertex and $c_{ij} > 0$ at every even one.

LEMMA 2.2. *Let A be a matrix one of whose rows is not linked to one of its columns. Then there exist permutations of its rows and of its columns which will transform it either to the direct sum of two matrices P and Q, i.e., to the form*

$$\begin{pmatrix} P & Z_1 \\ Z_2 & Q \end{pmatrix},$$

or to the form

$$\begin{pmatrix} Z_3 \\ Q \end{pmatrix},$$

or (Z_3, Q), where Z_1, Z_2, and Z_3 are matrices all of whose elements are zero. In the first of these two forms, the row and the column which are not linked may be assumed to be contained in the submatrices $(P\ Z_1)$ and

$$\begin{pmatrix} Z_1 \\ Q \end{pmatrix},$$

respectively.

Proof. We assume, without any loss of generality, that the first row of A is not linked to its first column, and hence, in particular, that $a_{11} = 0$. If the first row (column) consists entirely of zeros, then A obviously has the second (third) of the three forms given in the lemma. Suppose, then, that there is at least one non-zero element in the first row and column. Define a set S of rows and columns as follows. S contains all the columns of A whose first elements are non-zero. It contains all the rows which intersect these columns at non-zero elements, all the columns which have non-zero elements in common with these rows, and so on.

The columns contained in S consist of *all* the columns of A which are linked to its first row. The rows contained in S are *all* the rows of A which are linked to these columns. Hence, if we move the rows and columns of S to the initial positions, $1, 2, \ldots$, by means of permutations, A is transformed to the first of the two forms given in the statement of the lemma.

COROLLARY. *Let the ith row of A be not linked to its jth column. Then the equations $\rho_u = r_u$, $u \neq i$, $\sigma_v = c_v$, $v \neq j$, determine ρ_i and σ_j as linear combinations of the other row- and column-sums.*

3. Theorems.

THEOREM 1. *Let $\mathfrak{A}$ be the class of matrices A of a given pattern, assumed non-empty, with $\rho_i = r_i$, $i > 1$, and $\sigma_j = c_j$, $j > 1$, and with $a_{11} = 0$. If A is any*

https://doi.org/10.4153/CJM-1968-021-9 Published online by Cambridge University Press

member of $\mathfrak{A}$, *and finite sequences of increasing link operations are applied to* A *until all the links between the first row and the first column are broken—each link being broken at least at one of the even vertex-positions—then, for the resulting matrix, the first row-sum* $= \sup_{A\in\mathfrak{A}} (\rho_1)$ *and the first column-sum* $= \sup_{A\in\mathfrak{A}} (\sigma_1)$. *An analogous result holds if in the preceding sentence, one replaces "increasing" by "decreasing," "sup" by "inf," and "even" by "odd."*

Proof. Let $\tilde{A}$ be a matrix obtained from $A \in \mathfrak{A}$ by applying increasing link operations until all links are broken, and let $\tilde{B}$ be a matrix similarly obtained from $B \in \mathfrak{A}$. The first part of the theorem is clearly proved if we can show that the first row-sums $\rho_1(\tilde{A})$ and $\rho_1(\tilde{B})$ of $\tilde{A}$ and $\tilde{B}$ are the same, as are the first column-sums $\sigma_1(\tilde{A})$ and $\sigma_1(\tilde{B})$.

Suppose, if possible, that $\rho_1(\tilde{A}) < \rho_1(\tilde{B})$. Now, since $\rho_i = r_i$, $i > 1$, and $\sigma_j = c_j, j > 1$, for both $\tilde{A}$ and $\tilde{B}$, it follows that $\rho_1(\tilde{B}) - \rho_1(\tilde{A}) = \sigma_1(\tilde{B}) - \sigma_1\tilde{A}$, and hence $\sigma_1(\tilde{A}) < \sigma_1(\tilde{B})$.

But, then, applying Lemma 2.1, one concludes that there is a complete set S of link positions between the first row and the first column with the property that at every even vertex $\tilde{a}_{ij} > \tilde{b}_{ij}$ and at every odd vertex $\tilde{a}_{ij} < \tilde{b}_{ij}$. This, however, means that the set $\{a_{ij}\colon (i, j) \in S\}$ is a link for A, and furthermore that in transforming A to $\tilde{A}$ by increasing link operations, we have in fact not broken this link at any of its even vertices, contrary to our assumption.

Hence $\rho_1(\tilde{A}) = \rho_1(\tilde{B})$ and $\sigma_1(\tilde{A}) = \sigma_1(\tilde{B})$, as was to be proved. The proof of the second part of the theorem is along exactly the same lines as the one above, and is omitted.

This theorem, which deals with a mathematical (linear) programming problem and for the solution of which it provides an easy algorithm, is used in the proof of the next theorem.

THEOREM 2. (i) *Let* $\tilde{A}_{m\times n}$ *be a given non-negative matrix with such a pattern that there do not exist any permutations of the rows and independent permutations of the columns which will transform* $\tilde{A}$ *into the direct sum of two or more matrices. Let* $r_1, \ldots, r_m$; $c_1, \ldots, c_n$ *be consistent for the pattern of* $\tilde{A}$. *Let* $\mathfrak{A}$ *be the class of matrices having the same pattern that* $\tilde{A}$ *does, and with* $\rho_i = r_i$, $\sigma_j = c_j$, *for all* i *and* j. *Then there exists a uniquely determined matrix* A, $A \in \mathfrak{A}$ (*called the associated matrix of* $\tilde{A}$) *and two diagonal matrices* $U_{m\times m} = \operatorname{diag}(u_1, \ldots, u_m)$ *and* $V_{n\times n} = \operatorname{diag}(v_1, \ldots, v_n)$ *which are such that* $UAV = \tilde{A}$. *For any* i *and* j, $u_i v_j$ *is uniquely determined, and is a continuous function of the* r_i *and the* c_j.

(ii) $u_i v_j$ *is a monotone strictly decreasing, and hence* a_{ij}, *if it is non-zero, is a monotone strictly increasing function of* r_i *and* c_j, *the other row- and column-sums being kept fixed. If* U_1 *and* V_1 *are diagonal matrices such that* $\tilde{A} = U_1 A V_1$, *then* $U_1 = \delta U$ *and* $V = \delta V_1$, *for some* $\delta > 0$.

COROLLARY. *Let* $\tilde{A}$ *be a given square, non-negative matrix. There exists a doubly stochastic matrix* A *and two diagonal matrices* U *and* V *such that* $UAV = \tilde{A}$

if and only if after independent permutations of the rows and columns, $\tilde{A}$ can be written as the direct sum of fully indecomposable, square matrices. Furthermore, A is unique.

The special case of the corollary when $\tilde{A}$ is a (strictly) positive matrix was proved first in **(5)**. Later, a simpler proof using a fixed point theorem appeared in **(2)**. The special case of Theorem 2 (without the assertions about continuity and monotonicity) when $\tilde{A}$ is a positive matrix is contained in **(6; 3)**. Independent proofs of the corollary are contained in **(1; 7)**. For references to other work related to it, the reader is referred to **(1)**.

We make use of the following lemma in the proof of Theorem 2.

LEMMA 3.1. *Let $\tilde{A}$ be a non-negative matrix with $\tilde{a}_{11} = 0$. Let $r_1, \ldots, r_m$; $c_1, \ldots, c_n$ be consistent for the pattern of $\tilde{A}$. Let A be the associated matrix (assumed to exist) of $\tilde{A}$ with $\rho_i = r_i$ and $\sigma_i = c_i$, for all i. Thus, there exist diagonal matrices U and V such that $\tilde{A} = UAV$.* (i) *If the first row of $\tilde{A}$ is not linked to its first column, then $u_1 v_1$ is undetermined and can be chosen to be any positive value.* (ii) *If the first row of $\tilde{A}$ is linked to its first column, then $u_1 v_1$, which is a function $u_1 v_1\ (r_1, c_1)$ of r_1 and c_1, has the following strict monotonicity property. If $\theta > 0$ is such that $r_1 + \theta, r_2, \ldots, r_m$; $c_1 + \theta, c_2, \ldots, c_n$ are consistent for the pattern of $\tilde{A}$, and if $\tilde{A}$ has an associated matrix with these corresponding row- and column-sums, then $u_1 v_1\ (r_1, c_1) > u_1 v_1\ (r_1 + \theta, c_1 + \theta)$.*

Proof. (i) When the first row of $\tilde{A}$ is not linked to the first column, there exist by Lemma 2.2 permutation matrices π_1 and π_2 such that

$$\pi_1 \tilde{A} \pi_2 = \tilde{B} = \begin{pmatrix} \tilde{P} & Z_1 \\ Z_2 & \tilde{Q} \end{pmatrix},$$

where Z_1 and Z_2 are zero matrices and

$$(*)\begin{cases} \text{the first row of } \tilde{A} \text{ is that of } (\tilde{P}\ \ Z_1) \\ \text{and the first column of } \tilde{A} \text{ is that of } \begin{pmatrix} Z_1 \\ \tilde{Q} \end{pmatrix}. \end{cases}$$

Let

$$\pi_1 A \pi_2 = B = \begin{pmatrix} P & Z_1 \\ Z_2 & Q \end{pmatrix}.$$

Then, since $\tilde{A} = UAV$, we have $\tilde{B} = (\pi_1 U \pi_1^{-1}) B (\pi_2^{-1} V \pi_2)$. Hence $\tilde{P} = RPS$ and $\tilde{Q} = XQY$, where R, S, X, and Y are diagonal matrices, and

$$\pi_1 U \pi_1^{-1} = \operatorname{diag}(R, X) \quad \text{and} \quad \pi_2^{-1} V \pi_2 = \operatorname{diag}(S, Y).$$

Furthermore, because of (*), we have

$$u_1 v_1 = R_1 y_1, \tag{1}$$

where R_1 and y_1 are the first diagonal elements of R and Y respectively. But, for any $\theta \neq 0$ and $\phi \neq 0$, $\tilde{P} = (\theta R) P (\theta^{-1} S)$ and $\tilde{Q} = (\phi X) Q (\phi^{-1} Y)$. Hence, (1) shows that we may take $u_1 v_1$ to be $\theta \phi^{-1} R_1 y_1$. Since θ and ϕ are arbitrary non-zero quantities, part (i) of the lemma follows.

Next, suppose that the first row of $\tilde{A}$ is linked to its first column. Let B be the associated matrix of $\tilde{A}$, with $\rho_1 = r_1 + \theta$, $\sigma_1 = c_1 + \theta$, and $\rho_i = r_i$, $\sigma_i = c_i$, $i \geqslant 2$. It follows from Lemma 2.1 that there exists a link determined by $a_{1j}, a_{ij}, a_{ik}, \ldots, a_{u,v}, a_{u,1}$ for the matrix A, and by $b_{1j}, b_{ij}, b_{ik}, \ldots, b_{u,v}, b_{u,1}$ for B with the property that the odd (even) vertices in the second set are greater (less) than the corresponding vertices of the first set. But $u_i a_{ij} v_j = \tilde{a}_{ij}$, for all i and j. Therefore

$$u_1 v_1(r_1, c_1) = (\tilde{a}_{1j}/a_{1j})(a_{ij}/\tilde{a}_{ij}) \ldots (\tilde{a}_{u1}/a_{u1}), \tag{2}$$

whereas $u_1 v_1(r_1 + \theta, c_1 + \theta)$ is obtained from (2) by replacing the a_{ij} by the b_{ij}. The monotonicity property mentioned in (ii) of the lemma now follows.

Note. Let $\tilde{A} = UAV$ as in the statement of Theorem 2. Suppose that $a_{ij} = 0$ but that there is a link between the ith row and the jth column of A. Let $S = \{(\alpha, \beta): a_{\alpha,\beta}$ is an odd vertex of the link$\}$ and $T = \{(\alpha, \beta): a_{\alpha,\beta}$ is an even vertex of the link$\}$. Then

$$u_i v_j = \prod\nolimits_{(\alpha,\beta)\in S} u_\alpha v_\beta \Big/ \prod\nolimits_{(\alpha,\beta)\in T} u_\alpha v_\beta. \tag{3}$$

This fact was already used in (2) above and will be used also during the course of the proof of Theorem 2 that follows.

Proof of Theorem 2. We prove part (i) first, using a double induction on the rows of A and on the number of non-zero elements in any row.

(i) clearly holds if $\tilde{A}$ has just one row. Suppose that it is true when the number of rows is not greater than $m - 1$.

Let ν denote the number of non-zero elements in the first row of $\tilde{A}$. We shall prove that the theorem holds if $\nu = 1$. We may clearly assume that the only non-zero element in the first row is $\tilde{a}_{11}$. Let $\tilde{B}_{(m-1)\times n}$ be the matrix consisting of the last $m - 1$ rows of $\tilde{A}$. Then $r_2, \ldots, r_m$; $c_1 - r_1, c_2, \ldots, c_n$ is a set which is consistent for the pattern of $\tilde{B}$. By the induction hypothesis, (i) holds for $\tilde{B}$. Hence $\tilde{B} = XBV$, where X and V are diagonal matrices, and B is the associated matrix of $\tilde{B}$ with $\rho_i = r_{i+1}$, for $1 \leqslant i \leqslant m - 1$, $\sigma_1 = c_1 - r_1$, and $\sigma_i = c_i$, $i > 1$. But, now, let A be the matrix whose first row has the single non-zero element $a_{11} = r_1$, and whose other rows form a matrix identical with B. Let $U = \operatorname{diag}(\tilde{a}_{11}/r_1 v_1, X)$. Then A is the associated matrix of $\tilde{A}$, with $\rho_i = r_i$, $\sigma_i = c_i$, for all i, and $\tilde{A} = UAV$. We have thus shown that for every i and j such that $a_{ij} \neq 0$, $u_i v_j$ is unique and continuous. But, by Lemma 2.2, every row is connected to every column. Hence, by (3), every $u_i v_j$ is uniquely determined and is continuous. Thus, (i) is established for the case $\nu = 1$.

To continue the induction on ν, assume that (i) is true when $1 \leqslant \nu \leqslant k < n$. We now prove that it is true when $\nu = k + 1$. We may, and shall, assume that $\tilde{a}_{11} \neq 0$.

https://doi.org/10.4153/CJM-1968-021-9 Published online by Cambridge University Press

Now, there exists a matrix of the same pattern as that of $\tilde{A}$ and with $\rho_i = r_i$ and $\sigma_i = c_i$, for all i. If θ is the non-zero element in the first row and the first column of this matrix, $r_1 - \theta, r_2, \ldots, r_m$; $c_1 - \theta, c_2, \ldots, c_n$ are consistent for the pattern of $\tilde{B}$, where $\tilde{B}$ is the matrix defined by $\tilde{b}_{11} = 0$, $\tilde{b}_{ij} = \tilde{a}_{ij}$, $i \neq 1$ or $j \neq 1$. For $\tilde{B}$, $\nu = k$. Hence, by the induction assumption, $\tilde{B} = UBV$, where B is the associated matrix of $\tilde{B}$, and for B, $\rho_i = r_i$, $\sigma_i = c_i$, $i \geqslant 2$, $\rho_1 = r_1 - \theta$, $\sigma_1 = c_i - \theta$.

Now, if the first row of B is not linked to its first column, then by the corollary to Lemma 2.2, the first row sum of B, which is $r_1 - \theta$, is fixed, and is a linear combination of the r_i and the c_i, $i > 1$. Hence θ is a continuous function of, and is uniquely determined by, the r_i and the c_i. But, by (i) of Lemma 3.1, $u_1 v_1$ can be made to assume any value. Hence, in particular, it can be chosen to equal $\tilde{a}_{11}/\theta$. Thus the associated matrix of $\tilde{A}$ is the uniquely determined matrix which differs from B only in that the element in the first row and the first column is θ, and not 0 as it is for B. And now, just as for the case $\nu = 1$ above, it follows from Lemma 2.2 and equation (3) that the $u_i v_j$ are uniquely determined and continuous.

Suppose, on the other hand, that the first row of B is linked to its first column. Let $\mathfrak{B}$ be the class of matrices of the same pattern as B and with $\rho_i = r_i$, $\sigma_i = c_i$, $i > 1$. $\mathfrak{B}$ is not empty. Let $\Lambda = \sup_{\mathfrak{B}} (\sigma_1)$ and $\lambda = \inf_{\mathfrak{B}} (\rho_1)$. Then

$$\lambda < r_1 - \theta < \Lambda. \tag{4}$$

Because of the induction assumption, we have, in particular, that for each (i, j) for which $b_{i,j} \neq 0$, $u_i v_j$ is a uniquely determined, continuous function of the row- and column-sums of B. Hence by (3) (with i and j replaced by unity), and by Lemma 3.1 (ii), we conclude that

(5) $u_1 v_1$ is a monotonically strictly increasing continuous function of θ.

If $r_1 \leqslant \Lambda$, it follows from (5) that $\theta u_1 v_1 \downarrow 0$, and also (4) is satisfied, as $\theta \downarrow 0$.

If $r_1 > \Lambda$, on the other hand, let $r_1 - \theta_0 = \Lambda$. Then $\theta_0 > 0$. Let $\theta \downarrow \theta_0$. Then it follows from Theorem 1, Lemma 2.1, and relations (2) and (4), that $u_1 v_1 \downarrow 0$.

Lastly, if $r_1 - \theta_1 = \lambda$, then $\theta_1 > 0$, and we have for the same reasons as those in the preceding paragraph, that $u_1 v_1 \uparrow \infty$ as $\theta \uparrow \theta_1$. Thus, $u_1 v_1 \theta$ is a continuous monotonically strictly increasing function of θ varying from 0 to ∞ as θ varies in a suitable interval, and in a manner satisfying (4). Hence, there exists a unique value of θ for which $u_1 v_1 \theta = \tilde{a}_{11}$. Regarded as a function of the r_i and the c_j, θ is continuous. Again, as in the case $\nu = 1$, it follows, by Lemma 2.2 and equation (3), that the $u_i v_j$ are uniquely determined and are continuous. The induction is thus complete and (i) of the theorem is established.

The first part of (ii) follows from the fact that every row is linked to every column (by Lemma 2.2), and by making use of the same argument as is used to establish Lemma 3.1 (ii). The second part of (ii) is merely another way of stating the fact, already established, that the $u_i v_j$ are uniquely determined, for all i and j.

Remark. The theorem holds with the obvious modifications needed, when $\tilde{A}$ is transformed—as it always can be—by independent permutations of its rows and columns into a direct sum of matrices.

Proof of the corollary. It is proved in **(4)** that a doubly stochastic matrix of the same pattern as that of a given square matrix A exists if and only if A can be transformed by independent permutations of its rows and columns into the direct sum of fully indecomposable matrices. This result, together with Theorem 2, establishes the corollary.

Acknowledgment. It is a pleasure for the author to thank the referee for his careful reading of the paper and courteous and highly useful suggestions.

References

1. Richard A. Brualdi, Seymour V. Parter, and Hans Schneider, *The diagonal equivalence of a non-negative matrix to a stochastic matrix*, J. Math. Anal. Appl., *16*, 1 (1966), 31–50.

2. M. V. Menon, *Reduction of a matrix with positive elements to a doubly stochastic matrix*, Proc. Amer. Math. Soc., *18*, 2 (1967), 244–247.

3. ——— *Reduction of a matrix with positive elements to a doubly stochastic matrix*, Tech. Rept. 658, Math. Res. Center, U.S. Army, Univ. of Wis., Madison, Wis. (1966).

4. H. Perfect and L. Mirsky, *The distribution of positive elements in doubly stochastic matrices*, J. London Math. Soc., *40* (1965), 688–698.

5. R. Sinkhorn, *A relationship between arbitrary positive matrices and doubly stochastic matrices*, Ann. Math. Statist., *35* (1964), 876–879.

6. R. Sinkhorn, *Diagonal equivalence to matrices with prescribed row and column sums*, Amer. Math. Monthly, *74*, 4 (1967), 402–405.

7. R. Sinkhorn and P. Knopp, *Concerning non-negative matrices and doubly stochastic matrices*, Pac. J. Math., *21*, No. 2, 343–348.

University of Missouri,
Columbia, Missouri