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ANALOGUES OF ENTIRE FUNCTION INEQUALITIES FOR AN ANALYTIC FUNCTION

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1. Let $f(z) = \sum_{k=1}^{\infty} c_k z^{\lambda_k}$ be an analytic function with radius of convergence R $(0 < R < \infty)$. Set

$$M(r) = M(r,f) = \max_{|z|=r} |f(z)|, \quad m(r) = \max_{k \geqq 0} \{|c_k| r^{\lambda_k}\},$$

$$\nu(r) = \max \{\lambda_k | m(r) = |c_k| r^{\lambda_k}\},$$

and let the order ρ and lower order λ of $f(z)$ be defined by

$$(1.1) \quad \begin{matrix} \rho \\ \lambda \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} [\log x]^{-1} \log^+ \log^+ m(r),$$

where $x = Rr/(R - r)$. If $0 < \rho < \infty$, we define the type T and lower type t of $f(z)$ by

$$(1.2) \quad \begin{matrix} T \\ t \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} [x^{-\rho} \log^+ m(r)].$$

Also, if $0 < \rho < \infty$, define the "growth numbers" γ and δ by

$$(1.3) \quad \begin{matrix} \gamma \\ \delta \end{matrix} = \lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \left\{ \frac{\nu(r)}{x^{\rho+1}} \right\}.$$

The purpose of our discussion will be to obtain some inequalities involving the growth constants defined above. Similar inequalities hold in the case where $f(z)$ is an entire function.

2. If $f(z)$ is an entire function of order ρ $(0 < \rho < \infty)$ and if T, γ, δ are defined by

$$T = \limsup_{r \to \infty} \frac{\log m(r)}{r^\rho}, \quad \begin{matrix} \gamma \\ \delta \end{matrix} = \lim_{r \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\nu(r)}{r^\rho},$$

we have that $\delta \leqq \rho T \leqq \gamma \leqq e\rho T$ and $\gamma + \delta \leqq e\rho T$. (See Shah [**3**] and Singh [**4**]). In our case we have the following:

THEOREM 1. *Let $f(z)$ be an analytic function with radius of convergence R*

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$(0 < R < \infty)$ *and order* ρ $(0 < \rho < \infty)$. *Then*

$$(2.1)\quad \delta R \leqq \rho t \leqq \rho T \leqq \gamma R \leqq \rho T\left(1 + \frac{1}{\rho}\right)^{1+\rho}\left[1 + \frac{\delta}{\gamma\rho}\right]^{-\rho-1},$$

where ρ, γ, δ, t, T *are as in* (1.1), (1.2), *and* (1.3).

Proof. We have that (Sons [**5**])

$$(2.2)\quad \log m(r) = O(1) + \int_{r_0}^{r} \frac{\nu(s)}{s}\, ds.$$

Thus, for $k \geqq 1$, we have

$$(2.3)\quad \log m(r) = O(1) + \int_{r_0}^{R(r/R)^k} \frac{\nu(s)}{s}\, ds + \int_{R(r/R)^k}^{r} \frac{\nu(s)}{s}\, ds.$$

Now, from (1.3), we have for $r \geqq r_0(\epsilon)$ $(r_0 < R)$,

$$(2.4)\quad (\gamma + \epsilon)x^{\rho+1} > \nu(r) > (\delta - \epsilon)x^{\rho+1}.$$

Since $(R/\log (R/r)) \sim x$, as $r \to R$, we can obtain, for $r \geqq r_0$,

$$(2.5)\quad (\gamma + \epsilon)\left(R/\log \frac{R}{r}\right)^{\rho+1} > \nu(r) > (\delta - \epsilon)\left(R/\log \frac{R}{r}\right)^{\rho+1}$$

Thus, from (2.3) and (2.5) we get

$$\begin{aligned}\log m(r) &\geqq O(1) + \int_{r_0}^{R(r/R)^k} \frac{(\delta - \epsilon)R^{\rho+1}ds}{s[\log (R/s)]^{\rho+1}} + \nu\left(R\left(\frac{r}{R}\right)^k\right)\int_{R(r/R)^k}^{r} \frac{ds}{s}\\ &= O(1) + \frac{(\delta - \epsilon)R}{\rho k^\rho}\left(\frac{R}{\log (R/r)}\right)^\rho + (k-1)\,\nu\left(R\left(\frac{r}{R}\right)^k\right)\log\frac{R}{r}\\ &\sim O(1) + \frac{(\delta - \epsilon)Rx^\rho}{\rho k^\rho} + \frac{(k-1)R\nu(R(r/R)^k)}{x}.\end{aligned}$$

Thus,

$$T \geqq \frac{\delta R}{\rho k^\rho} + (k-1)R \limsup_{r\to R} \frac{\nu(R(r/R)^k)}{[Rr/(R-r)]^{\rho+1}}.$$

But

$$\begin{aligned}&\limsup_{r\to R} \frac{\nu(R(r/R)^k)}{\{[R(R(r/R)^k)]/[R - R(r/R)^k]\}^{\rho+1}}\cdot\frac{\{[R(R(r/R)^k)]/[R - R(r/R)^k]\}^{\rho+1}}{[Rr/(R-r)]^{\rho+1}}\\ &\qquad\qquad \geqq \gamma \liminf_{r\to R}\left\{\frac{r^{k-1}(R-r)}{(R^k - r^k)}\right\}^{\rho+1} = \frac{\gamma}{k^{\rho+1}}.\end{aligned}$$

Hence,

$$T \geqq \frac{\delta R}{\rho k^\rho} + \frac{R(k-1)\gamma}{k^{\rho+1}},$$

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so that

$$(2.6)\quad \rho T \geqq \frac{R[\delta k + (k-1)\rho\gamma]}{k^{\rho+1}}.$$

The right-hand side of (2.6) is maximized for

$$k = \frac{\gamma\rho + \gamma}{\delta + \gamma\rho}.$$

Substituting this value for k in (2.6) yields

$$\rho T \geqq \gamma R \left(\frac{\rho}{1+\rho}\right)^{1+\rho} \left(1 + \frac{\delta}{\gamma\rho}\right)^{1+\rho},$$

which gives the right-most inequality of (2.1). The other inequalities in (2.1) follow readily from (1.2), (2.2), and (2.4).

3. We now establish some inequalities involving certain mean moduli of an analytic function and its derivative proceeding along the lines of Lakshminarasimhan [**2**]. First of all, let

$$A^*(r,f) = \max_{|z|=r} |\mathrm{Re} f(z)|,$$

and define $I_\delta(r,f)$ and $J_\delta(r,f)$ for $\delta > 0$ by

$$I_\delta(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta\right)^{1/\delta}$$

$$J_\delta(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |\mathrm{Re} f(re^{i\theta})|^\delta d\theta\right)^{1/\delta}.$$

Lakshminarasimhan [**2**] has shown that $I_\delta(r,f)$, $J_\delta(r,f)$, and $A^*(r,f)$ are monotonic increasing functions of r, that $\lim_{\delta\to\infty} I_\delta(r,f) = M(r,f)$, $\lim_{\delta\to\infty} J_\delta(r,f) = A^*(r,f)$, and that

$$\log J_\delta(r,f) \sim \log I_\delta(r,f) \sim \log A^*(r,f) \sim \log M(r,f)$$

provided $\log m(r) \sim \log M(r)$. He also gives the following two lemmas (given as Lemma 1 and Lemma 6 in [**2**]) which we shall need.

LEMMA 1. *Suppose $P(z)$ is a polynomial of degree n having derivative $P'(z)$. Then for any constant δ such that $1 \leqq \delta \leqq \infty$ and $z = re^{i\theta}$,*

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |P'(re^{i\theta})|^\delta d\theta\right)^{1/\delta} \leqq A_\delta \frac{n}{r} \left(\frac{1}{2\pi}\int_0^{2\pi} |\mathrm{Re}\, P(re^{i\theta})|^\delta d\theta\right)^{1/\delta}$$

where

$$A_\delta^\delta = \sqrt{\pi}\,\frac{\Gamma(1+\delta/2)}{\Gamma(1/2+\delta/2)}, \quad A_\delta \to 1 \text{ as } \delta \to \infty.$$

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LEMMA 2. *Define* $I_\delta(r, f)$ *and* $J_\delta(r, f)$ *for a* $\delta \geqq 1$. *Then, for almost all* r,

$$\text{(i) } I_\delta(r, f') \geqq \frac{d}{dr} I_\delta(r, f)$$

$$\text{(ii) } J_\delta(r, zf') \geqq r \frac{d}{dr} J_\delta(r, f)$$

$$\text{(iii) } A^*(r, zf') \geqq r \frac{d}{dr} A^*(r, f).$$

Lakshminarasimhan gives the preceding lemma for an entire function but the same proofs hold in the case where $f(z)$ is an analytic function with a finite radius of convergence.

The next lemma is analogous to Lemma 2 of Lakshminarasimhan.

LEMMA 3. *Let* $f(z) = \sum_{k=0}^{\infty} c_k z^{\lambda_k}$ *be an analytic function with radius of convergence* R $(0 < R < \infty)$, *order* ρ $(0 < \rho < \infty)$ *and type* T $(0 \leqq T < \infty)$. *Let* β *be chosen so that* $0 < \beta < 1$. *Then there exists a positive integer* $K(\beta)$ *tending to* ∞ *as* β *tends to* 0 *such that*

$$g(z) = \sum_{k=K(\beta)+1}^{\infty} c_k z^{\lambda_k}$$

where

$$|z| = R \exp\left[-\lambda_{K(\beta)}^{-1/(1+\rho)} (L + \beta)^{1/(1+\rho)}/(1 - \beta)\right],$$

$$L = T(1 + \rho)^{1+\rho} (R/\rho)^\rho$$

satisfies the inequalities

$$|g(z)| \leqq B(\beta), \quad |zg'(z)| \leqq A(\beta),$$
$$B(\beta) < A(\beta) \to 0 \quad \text{as} \quad K \to \infty \text{ and } \beta \to 0.$$

Proof. The type T can be given by

$$T = \limsup_{k \to \infty} \frac{[\log^+ (|c_k| R^{\lambda_k})]^{\rho+1}}{(1 + \rho)^{1+\rho} (R/\rho)^\rho \lambda_k^{\rho}}.$$

Thus, for $k \geqq K(\beta)$ we have

$$|c_k| < R^{-\lambda_k} \exp\left[(L + \beta)\lambda_k^{\rho}\right]^{1/(\rho+1)}.$$

Thus,

$$|g(z)| < \sum_{k=K(\beta)+1}^{\infty} |c_k| r^{\lambda_k},$$

$$\text{where } r = R \exp\left[-\lambda_k^{-1/(1+\rho)} (L + \beta)^{1/(1+\rho)}/(1 - \beta)\right],$$

so that

$$|g(z)| < \sum_{k=K(\beta)+1}^{\infty} \exp\left[-\lambda_k^{1/(1+\rho)} B (L + \beta)^{1/(1+\rho)}/(1 - \beta)\right] = B(\beta) < \infty.$$

Also,

$$|zg'(z)| \leqq \sum_{k=K(\beta)+1}^{\infty} \lambda_k |c_k| r^{\lambda_k}$$

$$< \sum_{k=K(\beta)+1}^{\infty} \lambda_k \exp\left[-\beta(L+\beta)^{1/(1+\rho)} \lambda_k^{1/(1+\rho)}/(1-\beta)\right] = A(\beta)$$

$$< \infty .$$

We are now able to prove the following theorem, which is analogous to Theorem 1 of Lakshminarasimhan [**2**].

THEOREM 2. *Let* $f(z) = \sum_{k=0}^{\infty} c_k z^{\lambda_k}$ *have radius of convergence* R $(0 < R < \infty)$, *order* ρ $(0 < \rho < \infty)$, *and type* T $(0 \leqq T < \infty)$. *Then, if* $\log m(r) \sim \log \mathrm{M}(r)$.

$$\text{(3.1)}\quad \rho R^\rho T \leqq \limsup_{r \to R} \frac{r I_\delta(r, f')}{I_\delta(r, f)(\log R/r)^{-\rho-1}} \leqq A_\delta \rho R^\rho T \left(1 + \frac{1}{\rho}\right)^{1+\rho},$$

where A_δ *is as in Lemma* 1.

Proof. We first prove the left-hand inequality.

In view of Lemma 2(i) it suffices to prove that

$$\text{(3.2)}\quad \limsup_{r \to R} \left[\frac{r(d/dr) I_\delta(r, f)}{I_\delta(r, f)(\log R/r)^{-\rho-1}}\right] \geqq \rho T R^\rho .$$

Suppose now that inequality (3.2) does not hold. Then we must have for some h $(h > 0)$ that

$$\text{(3.3)}\quad \frac{d}{dr} I_\delta(r, f) < \frac{1}{r} I_\delta(r, f) \left(\log \frac{R}{r}\right)^{-\rho-1} \rho T R^\rho (1 - h),$$

for all $r \geqq r_0(h)$ for which the left-hand side of (3.3) exists. Thus,

$$\log I_\delta(r, f) - \log I_\delta(r_0, f) < \rho T R^\rho (1-h) \int_{r_0}^{r} \frac{1}{r} \left(\log \frac{R}{r}\right)^{-\rho-1} dr$$

$$= T R^\rho \left(\log \frac{R}{r}\right)^{-\rho} (1-h) + O(1).$$

Since $(\log R/r)^{-\rho} \sim R^{-\rho}(Rr/(R-r))^\rho$ we obtain that $T \leqq (1-h)T$, a contradiction for $T \neq 0$. Thus, the left-hand inequality of (3.1) is established.

Now let $P(re^{i\theta})$ and $g(re^{i\theta})$ be respectively the sum of powers of $re^{i\theta}$ up to $K(\beta)$ and the sum of powers from $K(\beta) + 1$ onwards in

$$f(re^{i\theta}) = \sum_{k=0}^{\infty} c_k (re^{i\theta})^{\lambda_k},$$

$K(\beta)$ and r being chosen with reference to an arbitrary β as in Lemma 3. Then, using Lemma 3 and Minkowski's inequality we get

$$I_\delta(r,f') = \left(\frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|^\delta d\theta\right)^{1/\delta}$$
$$\leqq \left(\frac{1}{2\pi}\int_0^{2\pi} |P'(re^{i\theta})|^\delta d\theta\right)^{1/\delta} + \frac{A(\beta)}{r}.$$

Applying Lemma 1 we have

$$I_\delta(r,f') \leqq A_\delta \frac{\lambda_{K(\beta)}}{r}\left(\frac{1}{2\pi}\int_0^{2\pi} |\mathrm{Re}\,\{P(re^{i\theta})\}|^\delta d\theta\right)^{1/\delta} + \frac{A}{r}.$$

Since $|\mathrm{Re}\{P(re^{i\theta})\}| \leqq |\mathrm{Re}\ f(re^{i\theta})| + |\mathrm{Re}\ g(re^{i\theta})|$, we obtain (again using Minkowski's inequality and Lemma 3)

$$(3.4)\quad rI_\delta(r,f') \leqq A_\delta\lambda_{K(\beta)}J_\delta(r,f) + \lambda_{K(\beta)}BrA_\delta + A.$$

Now $J_\delta(r,f)$ is monotonic increasing and thus has a positive lower bound, so that, since from Lemma 3

$$\lambda_{K(\beta)} = \frac{L+\beta}{(1-\beta)^{1+\rho}(\log R/r)^{\rho+1}},$$

We obtain

$$\limsup_{r\to R}\left\{\frac{rI_\delta(r,f')(1-\beta)^{1+\rho}}{J_\delta(r,f)(L+\beta)(\log R/r)^{-\rho-1}}\right\} \leqq A_\delta.$$

Letting $\beta \to 0$ and using the fact that $J_\delta(r,f) \leqq I_\delta(r,f)$, we obtain

$$\limsup_{r\to R}\left\{\frac{rI_\delta(r,f')}{I_\delta(r,f)(\log R/r)^{-\rho-1}}\right\} \leqq LA_\delta.$$

COROLLARY. *For $f(z)$ as in Theorem 2 we have*

$$(3.5)\quad \rho R^\rho T \leqq \limsup_{r\to R}\frac{rM(r,f')}{M(r,f)[\log R/r]^{-\rho-1}} \leqq \rho R^\rho T\left(1+\frac{1}{\rho}\right)^{1+\rho}.$$

Proof. Letting $\delta \to \infty$ in (3.4), we have

$$rM(r,f') \leqq \lambda_{K(\beta)}A^*(r,f) + \lambda_{K(\beta)}B_r + A$$
$$\leqq \lambda_{K(\beta)}M(r,f) + \lambda_{K(\beta)}Br + A.$$

Thus,

$$\limsup_{r\to R}\frac{rM(r,f')}{M(r,f)[\log R/r]^{-\rho-1}} \leqq \rho R^\rho T\left(1+\frac{1}{\rho}\right)^{1+\rho}.$$

The left-hand inequality of (3.5) is proven in the same manner as in the theorem.

In a similar manner we can show that, if $\log m(r) \sim \log M(r)$,

$$\rho R^\rho T \leqq \limsup_{r\to R} \frac{J_\delta(r, zf')}{J_\delta(r,f)[\log R/r]^{-\rho-1}} \leqq A_\delta \rho R^\rho T\left(1+\frac{1}{\rho}\right)^{1+\rho}$$

and

$$\rho R^\rho T \leqq \limsup_{r\to R} \frac{A^*(r, zf')}{A^*(r,f)[\log R/r)^{-\rho-1}} \leqq \rho R^\rho T\left(1+\frac{1}{\rho}\right)^{1+\rho}.$$

From inequality (3.5) we now obtain the following theorem which is analogous to a result of G. H. Fricke, S. M. Shah, and W. C. Sisarcick [**1**].

THEOREM 3. *Let $f(z)$ be as in Theorem 2. Then, given R_1 $(0 < R_1 < R)$, we have that there exists a constant $K > 0$ such that for $r \geqq R_1$,*

$$(3.6)\quad M(r,f') \leqq \frac{K\rho R^\rho}{r}\left(\log\frac{R}{r}\right)^{-\rho-1} M(r).$$

Furthermore, if inequality (3.6) holds for a given R_1 and K, $r \geqq R_1$, then $T \leqq K$. Also, given $\epsilon > 0$, there exists $R_0(\epsilon)$ such that for $r \geqq R_0(\epsilon)$ and $n = 1, 2, 3, \ldots$

$$(3.7)\quad M_n(r) \leqq \frac{n!}{r_1^{\,n}} M(r_1+r) \leqq \left\{\frac{n!}{r_1^{\,n}}\exp\left[dR^\rho\left(\log\frac{R}{r+r_1}\right)^{-\rho}\right]\right\} M(r),$$

where $d = T(1+1/\rho)^{1+\rho}+\epsilon$, $M_n(r) = M(r, f^{(n)})$, and r_1 is chosen so that $r < r_1 < R$. (Choosing r_1 to satisfy

$$[n(r_1+r)/r_1]\left[\log\frac{R}{r+r_1}\right]^{1+\rho} = dR^\rho\rho$$

will minimize the bracketed expression in (3.7).)

Proof. Let $R_1 > 0$ be given. Then, given $\theta > 0$, we have from (3.5) that there exists $R_0(\epsilon) < R$ such that for $r \geqq R_0(\epsilon)$

$$(3.8)\quad M(r,f') \leqq \left(T\left(1+\frac{1}{\rho}\right)^{1+\rho}+\epsilon\right)\frac{\rho R^\rho}{r}\left[\log\frac{R}{r}\right]^{-\rho-1} M(r).$$

Letting

$$K_1 = \max\left\{\frac{M(r,f')}{M(r)}\,\middle|\, R_1 \leqq r \leqq R_0(\epsilon)\right\} \text{ and}$$

$$K = \max\left\{\left(T\left(1+\frac{1}{\rho}\right)^{1+\rho}+\epsilon\right), K_1\right\},$$

we have inequality (3.6).

Now suppose there exist two positive numbers K and R_1 such that (3.6)

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holds if $r \geqq R_1$. Then, letting $M'(r) = (d/dr)M(r)$ where it exists, we have

$$\log \frac{M(r)}{M(R_1)} = \int_{R_1}^{r} \frac{M'(t)}{M(t)}\,dt \leqq \int_{R_1}^{r} \frac{M(t, f')}{M(t)}\,dt$$

$$\leqq K\rho R^{\rho} \int_{R_1}^{r} \frac{1}{t}\left[\log \frac{R}{t}\right]^{-\rho-1} dt = KR^{\rho}\left[\log \frac{R}{r}\right]^{-\rho} + O(1).$$

Since $x = Rr/(R - r) \sim R/\log R/r$, we have that $T \leqq K$.

Finally, for $d = T(1 + 1/\rho)^{1+\rho} + \epsilon$ and $r \geqq R_0(\epsilon)$ we have from (3.8) that

$$\frac{M(r, f')}{M(r)} \leqq \frac{d\rho R^{\rho}}{r}\left[\log \frac{R}{r}\right]^{-\rho-1}.$$

Hence, for $r \geqq R_0(\epsilon)$, $r < r_1 < R$, we find that

$$\log \frac{M(r + r_1)}{M(r)} < dR^{\rho}\left[\log \frac{R}{r + r_1}\right]^{-\rho}.$$

Using Cauchy's integral formula we have

$$M_n(r) \leqq \frac{n!}{r_1^{\,n}} M(r + r_1) \leqq \left\{\frac{n!}{r_1^{\,n}} \exp\left[dR^{\rho}\left(\log \frac{R}{r + r_1}\right)^{-\rho}\right]\right\} M(r),$$

establishing (3.7).

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