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A CHARACTERISATION RESULT FOR MATRIX RINGS

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For a ring R with identity we show that the existence of certain nilpotent elements forces R to be a matrix ring of size $\geqslant 2$.

In structure theory it is often useful to know whether a given ring R with identity is isomorphic to a matrix ring over some ring S with a more tractable structure. For instance, it is nice to have that S is an integral domain.

A ring R is isomorphic to a matrix ring of size n if and only if there exists a set of matrix units $\{e_{ij} \mid 1 \leqslant i, j \leqslant n\} \subseteq R$, that is, $\sum_{i=1}^{n} e_{ii} = 1$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$ where δ_{jk} is the Kronecker delta. In this case $R \cong M_n(S)$, where $S \cong e_{ii}Re_{ii}$ for all $1 \leqslant i \leqslant n$. For this and other well-known facts concerning matrix rings see [**2**]. In this note we prove the following criterion for R to be a matrix ring. $\operatorname{Ann}(s)$ shall denote the left annihilator of an element $s \in R$.

Theorem 1. *For a ring R with identity 1 the following are equivalent:*

(1) *$R \cong M_n(S)$ for some ring S and some positive integer $n \geqslant 2$.*

(2) *For some positive integer $n \geqslant 2$, there exist elements $x, y \in R$ such that $x^{n-1} \neq 0$, $x^n = y^2 = 0$, $x + y$ is invertible and $\operatorname{Ann}(x^{n-1}) \cap Ry = (0)$.*

Moreover if (2) holds and r is the inverse of $x + y$, then $\{e_{ij} \mid 1 \leqslant i, j \leqslant n\}$, where $e_{ij} = r^{n-i}(ry)x^{n-j}$, is a set of matrix units for R. Thus, if e denotes the idempotent $ry = e_{nn}$, then $R \cong M_n(S)$, where $S \cong eRe$.

Condition 2 in Theorem 1 can often be easily verified or rejected. Also, once the inverse r of $x + y$ is known, the matrix units for R are given explicitly. For the proof of Theorem 1 we need two propositions in which we keep the notation of Theorem 1 and assume that condition 2 holds.

Proposition 2. *$yr^ky = 0$ for all $2 \leqslant k \leqslant n$.*

Proof: Since $rx + ry = 1$, $y = yrx + yry$, hence $yrx \in Rx \cap Ry$. But $Rx \subseteq \operatorname{Ann}(x^{n-1})$, so $Rx \cap Ry = (0)$, that is $yrx = 0$. Since $x^{n-1} = ryx^{n-1}$, $yr^2yx^{n-1} = yrx^{n-1} = 0$; thus $yr^2y \in \operatorname{Ann}(x^{n-1}) \cap Ry = (0)$. Consequently $yr = yr^2x + yr^2y =$

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yr^2x, and hence $yr^2x^2 = yrx = 0$. Now let $2 < k \leqslant n$ and suppose we have shown that $yr^iy = 0$, $yr^ix^j = 0$ for all $2 \leqslant i < k$ and $i \leqslant j$. Then $yr^kyx^{n-1} = yr^{k-1}x^{n-1} = 0$; hence $yr^ky = 0$. Thus $yr^{k-1} = yr^kx + yr^ky = yr^kx$. By our induction hypothesis $yr^{k-1}x^j = 0$ for all $j \geqslant k-1$; hence $yr^kx^j = 0$ for all $j \geqslant k$. Our claim now follows. ∎

PROPOSITION 3. *$ryx^ir^jry = \delta_{ij}ry$ for all $0 \leqslant i, j \leqslant n-1$.*

PROOF: Let $0 \leqslant i \leqslant n-1$. We show that $ryx^ir^iry = ry$. In Proposition 2 we have seen that $yrx = 0$; hence $y = yry$, that is, ry is idempotent. So our claim is true for $i = 0$. Since $xr + yr = 1$, we have $x^{i-1}r^{i-1} = x^ir^i + x^{i-1}yrr^{i-1}$ for $1 \leqslant i \leqslant n-1$, and hence $x^{i-1}r^{i-1}ry = x^ir^iry + x^{i-1}yr^{i+1}y$. Since $i+1 \leqslant n$, $yr^{i+1}y = 0$ by Proposition 2, thus $x^{i-1}r^{i-1}ry = x^ir^iry$. Consequently $x^ir^iry = ry$ for all $0 \leqslant i \leqslant n-1$.

Now let $i \neq j$. If $i > j$, then $x^ir^jry = x^{i-j}x^jr^jry = x^{i-j}ry$. But since $x = xrx + xry$, $xry \in Rx \cap Ry = (0)$. Finally suppose that $i < j$. If $i = 0$, then $ryx^ir^jry = ryr^{j+1}y = 0$, since $j+1 \leqslant n$.

Now let $i > 0$ and $0 < k \leqslant i$. Then $x^{i-k}r^{i-k}r^{j-i}ry = x^{i-k+1}r^{i-k+1}r^{j-i}ry + x^{i-k}yrr^{i-k}r^{j-i}ry$. But since $k > 0$ and $j \leqslant n-1$, $j-k+2 \leqslant n$; hence $yr^{j-k+2}y = 0$ and therefore $x^{i-k}r^{i-k}r^{j-i}ry = x^{i-k+1}r^{i-k+1}r^{j-i}ry$. It now follows that $x^ir^jry = x^ir^ir^{j-i}ry = r^{j-i}ry$, and thus $ryx^ir^jry = ryr^{j-i+1}y = 0$. ∎

We are now ready to prove Theorem 1.

PROOF OF THEOREM 1: $2 \Rightarrow 1$. For $1 \leqslant i, j \leqslant n$ let $e_{ij} = r^{n-i}(ry)x^{n-j}$. It is then immediate from Proposition 3 that $e_{ij}e_{kl} = \delta_{jk}e_{il}$. It remains to show that $\sum_{i=1}^{n} e_{ii} = 1$. Since $rx + ry = 1$, $r^{n-i}x^{n-i} = r^{n-i+1}x^{n-i+1} + r^{n-i}(ry)x^{n-i}$ for all $1 \leqslant i \leqslant n-1$. But $r^nx^n = 0$, thus $\sum_{i=1}^{n-1} r^{n-i}(ry)x^{n-i} = \sum_{i=1}^{n-1} e_{ii} = rx$. Since $ry = e_{nn}$, $\sum_{i=1}^{n} e_{ii} = 1$.

$1 \Rightarrow 2$. Let $\{e_{ij} \mid 1 \leqslant i, j \leqslant n\}$ be a set of matrix units for R. If $x := e_{12} + \ldots + e_{n-1n}$, $y := e_{n1}$, then $x^{n-1} \neq 0$, $x^n = y^2 = 0$ and $x+y$ is invertible with inverse $r = e_{1n} + e_{21} + \ldots + e_{nn-1}$. Moreover one checks that $\operatorname{Ann}(x^{n-1}) = Rx$ and that $Rx \cap Ry = (0)$. ∎

The following special case of Theorem 1 has also been stated in [1] (Theorem III.2).

COROLLARY 4. *For a ring R with identity the following are equivalent:*

(1) *R is a ring of 2×2 matrices over some ring S.*
(2) *There exist elements $x, y \in R$ such that $x^2 = y^2 = 0$ and $x + y$ is invertible.*

Corollary 4 follows immediately from Theorem 1 for if $0 \neq sy \in \operatorname{Ann}(x) \cap Ry$, then $sy \in \operatorname{Ann}(x+y)$ which is a contradiction to the fact that $x+y$ is invertible.

In a forthcoming paper we give some applications of Theorem 1 in structure theory. For instance, if R is a prime Goldie ring we show how elements x, y satisfying condition 2 can be constructed in the quotient ring of R. In this manner we obtain another proof for the fact that the quotient ring of R is isomorphic to a matrix ring over a division ring.

References

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