A CHARACTERISATION RESULT FOR MATRIX RINGS

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For a ring $R$ with identity we show that the existence of certain nilpotent elements forces $R$ to be a matrix ring of size $\geq 2$.

In structure theory it is often useful to know whether a given ring $R$ with identity is isomorphic to a matrix ring over some ring $S$ with a more tractable structure. For instance, it is nice to have that $S$ is an integral domain.

A ring $R$ is isomorphic to a matrix ring of size $n$ if and only if there exists a set of matrix units $\{e_{ij} | 1 \leq i, j \leq n\} \subseteq R$, that is, $\sum_{i=1}^{n} e_{ii} = 1$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$ where $\delta_{jk}$ is the Kronecker delta. In this case $R \cong M_n(S)$, where $S \cong e_iRe_i$ for all $1 \leq i \leq n$. For this and other well-known facts concerning matrix rings see [2]. In this note we prove the following criterion for $R$ to be a matrix ring. Ann$(s)$ shall denote the left annihilator of an element $s \in R$.

**Theorem 1.** For a ring $R$ with identity 1 the following are equivalent:

1. $R \cong M_n(S)$ for some ring $S$ and some positive integer $n \geq 2$.
2. For some positive integer $n \geq 2$, there exist elements $x, y \in R$ such that $x^{n-1} \neq 0$, $x^n = y^2 = 0$, $x + y$ is invertible and Ann$(x^{n-1}) \cap Ry = (0)$.

Moreover if (2) holds and $r$ is the inverse of $x + y$, then $\{e_{ij} | 1 \leq i, j \leq n\}$, where $e_{ij} = r^{n-i}(ry)x^{n-j}$, is a set of matrix units for $R$. Thus, if $e$ denotes the idempotent $ry = e_{nn}$, then $R \cong M_n(S)$, where $S \cong eRe$.

Condition 2 in Theorem 1 can often be easily verified or rejected. Also, once the inverse $r$ of $x + y$ is known, the matrix units for $R$ are given explicitly. For the proof of Theorem 1 we need two propositions in which we keep the notation of Theorem 1 and assume that condition 2 holds.

**Proposition 2.** $yr^ky = 0$ for all $2 \leq k \leq n$.

**Proof:** Since $rx + ry = 1$, $y = yrx + yry$, hence $yrx \in Rx \cap Ry$. But $Rx \subseteq$ Ann$(x^{n-1})$, so $Rx \cap Ry = (0)$, that is $yrx = 0$. Since $x^{n-1} = yrx^{n-1}$, $yr^2y = yrx^{n-1} = 0$; thus $yr^2y \in$ Ann$(x^{n-1}) \cap Ry = (0)$. Consequently $yr = yrx + yr^2y = \ldots$

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\[ yr^2 x \text{, and hence } yr^2 x^2 = yr x = 0. \text{ Now let } 2 < k \leq n \text{ and suppose we have shown that } yr^i y = 0, yr^i x^j = 0 \text{ for all } 2 \leq i < k \text{ and } i \leq j. \text{ Then } yr^k y z^{n-1} = yr^{k-1} x^{n-1} = 0; \text{ hence } yr^k y = 0. \text{ Thus } yr^{k-1} = yr^k x + yr^k y = yr^k x. \text{ By our induction hypothesis } yr^{k-1} x^j = 0 \text{ for all } j \geq k - 1; \text{ hence } yr^k x^j = 0 \text{ for all } j \geq k. \text{ Our claim now follows.} \]

**Proposition 3.** \( ry^i r^j r y = \delta_{ij} r y \) for all \( 0 \leq i, j \leq n - 1. \)

**Proof:** Let \( 0 \leq i \leq n - 1. \) We show that \( ry^i r^j r y = r y. \) In Proposition 2 we have seen that \( yr x = 0; \) hence \( y = y r y, \) that is, \( ry \) is idempotent. So our claim is true for \( i = 0. \) Since \( r x + r y = 1, \) we have \( x^{i-1} r^{i-1} = x^i r + x^{i-1} y r r^{-1} \) for \( 1 \leq i \leq n - 1, \) and hence \( x^{i-1} r^{i-1} y r = x^i r y r + x^{i-1} y r r r^{-1} y r. \) Since \( i + 1 \leq n, \) \( y r r r^{-1} y r = 0 \) by Proposition 2, thus \( x^{i-1} r^{i-1} y r = x^i r y r. \) Consequently \( x^i r^j y r = r y \) for all \( 0 \leq i \leq n - 1. \)

Now let \( i \neq j. \) If \( i > j, \) then \( x^i r^j y r = x_i-1 x^j r^j y r = x^i r^j y r. \) But since \( x = x r x + x r y, x r y \in R x \cap R y = (0). \) Finally suppose that \( i < j. \) If \( i = 0, \) then \( r y x^{i-1} r^j r y = r y r r r r^{-1} y r = 0, \) since \( j + 1 \leq n. \)

Now let \( i > 0 \) and \( 0 < k \leq i. \) Then \( x^{i-k} r^{i-k} r^{-1} r y = x^{i-k} r^{i-k-1} r^i r y + x^{i-k} y r r^{-1} r^{-1} r y. \) But since \( k > 0 \) and \( j \leq n - 1, j - k + 2 \leq n; \) hence \( y r j k + 2 y r = 0 \) and therefore \( x^{i-k} r^{i-k} r^{-1} r y = x^{i-k} r^{i-k+1} r^i r y. \) It now follows that \( x^i r^j r y = x^i r^j r^i r y = r y r^j r y = r y r^j r y = 0. \)

We are now ready to prove Theorem 1.

**Proof of Theorem 1:** \( 2 \Rightarrow 1. \) For \( 1 \leq i, j \leq n \) let \( e_{ij} = r^{n-i} (r y) x^{n-j}. \) It is then immediate from Proposition 3 that \( e_{ij} e_{kl} = \delta_{jk} e_{ii}. \) It remains to show that \( \sum_{i=1}^{n} e_{ii} = 1. \) Since \( r x + r y = 1, r^{n-i} x^{n-i} = r^{n-i+1} x^{n-i+1} + r^{n-i} (r y) x^{n-i} \) for all \( 1 \leq i \leq n - 1. \) But \( r^n x^n = 0, \) thus \( \sum_{i=1}^{n-1} r^{n-i} (r y) x^{n-i} = \sum_{i=1}^{n-1} e_{ii} = r x. \) Since \( r y = e_{nn}, \) \( \sum_{i=1}^{n} e_{ii} = 1. \)

\( 1 \Rightarrow 2. \) Let \( \{ e_{ij} | 1 \leq i, j \leq n \} \) be a set of matrix units for \( R. \) If \( x := e_{12} + \ldots + e_{n-1 n}, y := e_{n 1}, \) then \( x^{n-1} \neq 0, x^n = y^2 = 0 \) and \( x + y \) is invertible with inverse \( r = e_{11} + e_{22} + \ldots + e_{n n}. \) Moreover one checks that \( \text{Ann}(x^{n-1}) = R x \) and that \( R x \cap R y = (0). \)

The following special case of Theorem 1 has also been stated in [1] (Theorem III.2).

**Corollary 4.** For a ring \( R \) with identity the following are equivalent:

1. \( R \) is a ring of \( 2 \times 2 \) matrices over some ring \( S. \)
2. There exist elements \( x, y \in R \) such that \( x^2 = y^2 = 0 \) and \( x + y \) is invertible.

Corollary 4 follows immediately from Theorem 1 for if \( 0 \neq s y \in \text{Ann}(x) \cap R y, \) then \( s y \in \text{Ann}(x + y) \) which is a contradiction to the fact that \( x + y \) is invertible.
In a forthcoming paper we give some applications of Theorem 1 in structure theory. For instance, if $R$ is a prime Goldie ring we show how elements $x, y$ satisfying condition 2 can be constructed in the quotient ring of $R$. In this manner we obtain another proof for the fact that the quotient ring of $R$ is isomorphic to a matrix ring over a division ring.

**REFERENCES**
