A NOTE ON REGULAR LOCAL NOETHER LATTICES II

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Let \((R, M)\) be a local ring and let \(R^*\) be the \(M\)-adic ring completion of \(R\). It is well
known that \(R\) is a regular local ring if and only if \(R^*\) is a regular local ring. The purpose of
the note is to show that this result is essentially a consequence of a more general theory concerning
local Noether lattices which was developed in [6].

By a multiplicative lattice we will mean a complete lattice on which there is defined a
commutative, associative, totally join distributive multiplication for which the unit element
of the lattice is an identity for multiplication (written juxtaposition). Let \(\mathcal{L}\) be a multiplicative
lattice. An element \(P\) of \(\mathcal{L}\) is said to be meet principal if \(AP \land B = A \land (B : P)\), for all \(A\) and \(B\)
in \(\mathcal{L}\); \(P\) is said to be join principal if \((A \lor BP) : P = B \lor (A : P)\), for all \(A\) and \(B\) in \(\mathcal{L}\); and \(P\) is said to be principal if \(P\) is both meet and join principal. \(\mathcal{L}\) will be called principally generated
if each element of \(\mathcal{L}\) is a join (finite or infinite) of principal elements of \(\mathcal{L}\). \(\mathcal{L}\) is called a
Noether lattice in case \(\mathcal{L}\) is modular, principally generated, and satisfies the ascending chain
condition on elements. A Noether lattice \(\mathcal{L}\) is said to be local if it has a unique maximal
(proper) prime \(M\). In this case we shall write \((\mathcal{L}, M)\). In general we adopt the lattice termin-
ology of [2] and [6].

Let \((\mathcal{L}, M)\) be a local Noether lattice. As in section 2 of [3] we let \(\mathcal{L}^*\) be the collection of
all formal sums \(\sum_{n=1}^{\infty} A_n\) of elements of \(\mathcal{L}\) such that \(A_n = A_{n+1} \lor M^n\), for \(n = 1, 2, \ldots\). On
\(\mathcal{L}^*\) define
\[
\sum_{n=1}^{\infty} A_n \leq \sum_{n=1}^{\infty} B_n \quad \text{if and only if} \quad A_n \leq B_n, \quad n = 1, 2, \ldots
\]

so that \(\mathcal{L}^*\) becomes a multiplicative lattice satisfying the ascending chain condition [3,
Theorem 2.1]. For each element \(A\) in \(\mathcal{L}\), set \(A^* = \sum_{n=1}^{n} A_n\), where \(A_n = A \lor M^n\) and note that
\(A^* \in \mathcal{L}^*\).

Let \(\mathcal{L}\) be a Noether lattice, \(D \in \mathcal{L}\), and set \(\mathcal{L}/D = \{A \in \mathcal{L} \mid A \geq D\}\). If we define \(A \circ B = \)
\(AB \lor D\), for all \(A, B \in \mathcal{L}/D\), then \(\mathcal{L}/D\) becomes a Noether lattice [2, Lemma 4.1]. If \((\mathcal{L}, M)\) is
a local Noether lattice, \(D \in \mathcal{L}\), \(D \leq M\), then \(\mathcal{L}/D\) is a local Noether lattice with maximal
element \(M\) [2, Corollary 4.1]. A local Noether lattice \((\mathcal{L}, M)\) is called \(M\)-complete if, given
any decreasing sequence \(<A_i>\) of elements of \(\mathcal{L}\) and any \(n \geq 1\), it follows that \(A_j \leq \bigcap_i A_i \lor M^n\),
for all large integers \(j\).

REMARK 1. We will require the following known properties of \(\mathcal{L}^*\). We refer the reader to
[3, p. 331] for their proof.

(i) \( \mathcal{L}^* \) is a local Noether lattice with maximal element \( M^* = \sum_{n=1}^{\infty} M \).

(ii) For each natural number \( n \), the map \( A \mapsto A^* \) from \( \mathcal{L}/M^* \to \mathcal{L}^*/M^* \) is a multiplicative lattice isomorphism.

(iii) \( \mathcal{L}^* \) is \( M^* \)-complete.

The following result will be needed in the proof of Theorem 3 [3, Corollary 1.3].

**Lemma 2.** Let \( (\mathcal{L}_1, M_1) \) and \( (\mathcal{L}_2, M_2) \) be local Noether lattices and \( \{ \varphi_i : \mathcal{L}_1/M_1^i \to \mathcal{L}_2/M_2^i \} \) a sequence of multiplicative lattice homomorphisms of \( \mathcal{L}_1/M_1^i \) onto \( \mathcal{L}_2/M_2^i \) such that \( \varphi_{i+1} \) extends \( \varphi_i \) for all \( i \). If \( \mathcal{L}_2 \) is \( M_2 \)-complete, then \( \mathcal{L}_1 \) is embeddable in \( \mathcal{L}_2 \). If also \( \mathcal{L}_1 \) is \( M_1 \)-complete, then \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are isomorphic as multiplicative lattices.

We shall in general adopt the ring terminology of [7]. In particular, a local ring is commutative, Noetherian, and has an identity. If \( R \) is a ring, we denote the multiplicative lattice of ideals of \( R \) by \( \mathcal{L}(R) \). If \( R \) is a local ring, \( \mathcal{L}(R) \) is a local Noether lattice [2, p. 486].

**Theorem 3.** Let \( (R, M) \) be a local ring with \( M \)-adic completion \( (R^*, MR^*) \). Then \( \mathcal{L}(R^*) \) and \( \mathcal{L}(R)^* \) are isomorphic as multiplicative lattices.

**Proof.** For each \( i, i = 1, 2, \ldots \), define

\[
\lambda_i : \mathcal{L}(R)/M^i \to \mathcal{L}(R^*)/(MR^*)^i
\]

by \( \lambda_i : A \mapsto AR^* \)

so that \( \lambda_i \) is the canonical multiplicative ideal lattice isomorphism. For each \( i \), define

\[
\alpha_i : \mathcal{L}(R)^*/M^* \to \mathcal{L}(R)/M^i
\]

by \( \alpha_i : \sum_{n=1}^{\infty} A_n \mapsto \bigcap_{n=1}^{\infty} A_n \)

\[
\psi_i : \mathcal{L}(R)/M^i \to \mathcal{L}(R^*)/M^* \]

by \( \psi_i : A \mapsto A^* \).

For each \( i \), \( \psi_i \) is a multiplicative lattice isomorphism (Remark 1) and by [6, p. 160, Remark 1] \( \alpha_i \) is the inverse of \( \psi_i \), thus \( \alpha_i \) is a multiplicative lattice isomorphism. For each \( i \), set \( \varphi_i = \lambda_i \alpha_i \)

so that

\[
\varphi_i : \mathcal{L}(R)^*/M^* \to \mathcal{L}(R^*)/(MR^*)^i
\]

and \( \varphi_i : \sum_{n=1}^{\infty} A_n \mapsto \bigcap_{n=1}^{\infty} A_n R^* \).

Thus each \( \varphi_i \) is a multiplicative lattice isomorphism and \( \varphi_{i+1} \) extends \( \varphi_i \). Since \( \mathcal{L}(R)^* \) is \( M^* \)-complete (Remark 1) and \( \mathcal{L}(R)^* \) is \( MR^* \)-complete [8, Theorem 1] it follows that \( \mathcal{L}(R)^* \) and \( \mathcal{L}(R^*) \) are isomorphic as multiplicative lattices by Lemma 2.

The **height** of a prime element \( P \) of a Noether lattice \( \mathcal{L} \) is defined to be the supremum of all integers \( n \) for which there exists a prime chain \( P_0 < P_1 < \ldots < P_n = P \) in \( \mathcal{L} \), and the **altitude** of \( \mathcal{L} \) is defined to be the supremum of the heights of the prime elements of \( \mathcal{L} \). A local Noether lattice \( (\mathcal{L}, M) \) of altitude \( k \) is said to be regular if in case \( M \) is the join of \( k \) principal elements.

**Lemma 4.** Let \( (R, M) \) be a local ring. Then \( R \) is a regular local ring if and only if \( \mathcal{L}(R) \) is a regular local Noether lattice.

**Proof.** Clearly the altitudes of \( R \) and \( \mathcal{L}(R) \) are the same. Let \( d \) be their common altitude. If \( R \) is regular, there exist \( d \) elements \( a_1, a_2, \ldots, a_d \) in \( R \) such that \( M = a_1 R + \ldots + a_d R \). Since
each $a_i R$ is principal in $\mathcal{L}(R)$ [2, p. 482], $\mathcal{L}(R)$ is regular. Conversely, if $\mathcal{L}(R)$ is regular, there exist principal elements $A_1, A_2, \ldots, A_d$ in $\mathcal{L}(R)$ such that $M = A_1 \lor \cdots \lor A_d$. Since $\mathcal{L}(R)$ is local, for each $i, 1 \leq i \leq d$, there exists $a_i$ in $R$ such that $A_i = a_i R$ [5, Corollary 6] and so $R$ is regular.

The proof of the following theorem may be found in [6, Theorem 3].

**Theorem 5.** Let $(\mathcal{L}, M)$ be a local Noether lattice. Then $\mathcal{L}$ is a regular local Noether lattice if and only if $\mathcal{L}^*$ is a regular local Noether lattice.

By Lemma 4, $R$ is regular if and only if $\mathcal{L}(R)$ is regular and similarly for $R^*$ and $\mathcal{L}(R^*)$. By Theorem 5, $\mathcal{L}(R)$ is regular if and only if $\mathcal{L}(R)^*$ is regular. These results in conjunction with Theorem 3 immediately yield the following theorem.

**Theorem 6.** Let $(R, M)$ be a local ring and let $R^*$ be the $M$-adic completion of $R$. Then $R$ is a regular local ring if and only if $R^*$ is a regular local ring.

As we have seen (Lemma 4) the ideal lattice $\mathcal{L}(R)$ of a regular local ring $R$ is contained in class of regular local Noether lattices. The following example (due to Bogart [1]) shows the existence of regular local Noether lattices which are not the ideal lattice of any regular local ring. Let $F$ be a field, let $x_1, x_2, \ldots, x_n$ be indeterminates, and let $RL_n$ be the collection of elements of $\mathcal{L}(F[x_1, x_2, \ldots, x_n])$ which are joins of products of the principal ideals $(x_1), (x_2), \ldots, (x_n)$. $RL_n$ is a sublattice of $\mathcal{L}(F[x_1, x_2, \ldots, x_n])$ and is a regular local Noether lattice of altitude $n$. For $n \geq 2$, it can be shown $RL_n$ is not isomorphic to the ideal lattice of any ring. We refer the reader to [1, p. 169] for the details.

**REFERENCES**


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