

SOME RESULTS ON TEICHMÜLLER SPACES OF KLEIN SURFACES

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1. Background and statement of main results. The deformation theory of non-orientable surfaces deals with the problem of studying parameter spaces for the different dianalytic structures that a surface can have. It is an extension of the classical theory of Teichmüller spaces of Riemann surfaces, and as such, it is quite rich. In this paper we study some basic properties of the Teichmüller spaces of non-orientable surfaces, whose parallels in the orientable situation are well known. More precisely, we prove an uniformization theorem, similar to the case of Riemann surfaces, which shows that a non-orientable compact surface can be represented as the quotient of a simply connected domain of the Riemann sphere, by a discrete group of Möbius and anti-Möbius transformation (mappings whose conjugates are Möbius transformations). This uniformization result allows us to give explicit examples of Teichmüller spaces of non-orientable surfaces, as subsets of deformation spaces of orientable surfaces. We also prove two isomorphism theorems: in the first place, we show that the Teichmüller spaces of surfaces of different topological type are not, in general, equivalent. We then show that, if the topological type is preserved, but the signature changes, then the deformation spaces are isomorphic. These are generalizations of the Patterson and Bers-Greenberg theorems for Teichmüller spaces of Riemann surfaces, respectively.

A Riemann surface (Σ, X) is a topological surface Σ with a complex structure X , that is, a covering of Σ by charts with holomorphic changes of coordinates. Since holomorphic functions have positive Jacobian, it turns out that Riemann surfaces are orientable. The natural generalisation to the case of non-orientable surfaces is that of a *dianalytic structure*, where we require that the changes of coordinates are either holomorphic or anti-holomorphic (the complex conjugate is holomorphic). A pair (Σ, X) , where Σ is a surface and X is a dianalytic structure, is called a *Klein surface*. In particular, Riemann surfaces are Klein surfaces. It is classical fact that any Klein surface can be represented as \tilde{X}/Γ , where \tilde{X} is either the Riemann sphere, the complex plane or the upper half plane, and Γ is a group of dianalytic bijections of \tilde{X} . Except for a finite number of cases, up to homeomorphism, Klein surfaces are uniformised by the upper half plane; these are called *hyperbolic surfaces*. A compact non-orientable surface Σ is the connected sum of g (real) projective planes; g is called the genus of the surface. Observe that here we use the genus in the topological sense; some authors (in particular, [17]) use the so-called arithmetic genus, which is equal to $g - 1$. A non-orientable surface is hyperbolic if and only if $g \geq 3$. In the first result of this paper, we prove a uniformization theorem, by groups which are more suitable for computations than groups acting on the upper half plane.

THEOREM 1.1. *Let Σ be a compact non-orientable surface of genus bigger than 2. Then there exists a Kleinian group G , acting discontinuously on a simply connected set Δ of $\hat{\mathbb{C}}$, and an antiholomorphic function r , such that*

1. $g(\Delta) = \Delta$ for all $g \in G$; $r(\Delta) = \Delta$;
2. Δ/G is isomorphic to the complex double Σ^c of Σ ;

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- 3. r is of the form $r:z \rightarrow \frac{a\bar{z} + b}{c\bar{z} + d}$, with $ad - bc \neq 0$;
- 4. $\Delta/\Gamma \cong \Sigma$, where Γ is the group generated by G and r ;
- 5. Γ is unique up to conjugation by Möbius transformations.

Here by a Kleinian group we mean a group of Möbius transformations that acts discontinuously on a non-empty open set of the Riemann sphere. The *complex double* of Σ is a Riemann surface Σ^c , together with a unramified double cover $\pi:\Sigma^c \rightarrow \Sigma$. If Σ is hyperbolic, then Σ^c is also hyperbolic (see §2 below).

Let $M(\Sigma)$ denote the set of dianalytic structures, on the non-orientable surface Σ , that are compatible with the differential structure induced by X . The quotient of $M(\Sigma)$ by the group of diffeomorphisms homotopic to the identity (acting by pullback, see §3), is the *Teichmüller space* $T(\Sigma)$ of Σ . It has a natural real analytic structure given by projecting the natural structure of $M(\Sigma)$. It is not hard to prove that $T(\Sigma)$ embeds in the Teichmüller space of Σ^c (see §3). Combining this embedding with Theorem 1.1 and the results of I. Kra in [12], we can give presentations for the deformation spaces of some non-orientable surfaces. As an example, we compute the Teichmüller space of a surface of genus 3.

THEOREM 1.2. *The space $T(\Sigma)$ of a non-orientable surface of genus 3 can be identified with the set of points (τ_1, τ_2, τ_3) of $T(\Sigma^c)$, such that*

$$\begin{cases} \operatorname{Re}(\tau_2) & = 0, \\ \operatorname{Re}(\tau_1) & = \operatorname{Im}(\tau_3), \\ \operatorname{Re}(\tau_1) + \operatorname{Re}(\tau_3) & = 0. \end{cases}$$

We introduce the concept of puncture on a non-orientable surface as a generalisation of the corresponding idea on Riemann surfaces; a puncture is a domain on Σ , homeomorphic to the unit disc minus the origin, that cannot be completed to be homeomorphic to the unit disc, and such that any change of coordinates in the domain is holomorphic. The above theorems extends easily to the case of surfaces with punctures. For example, we can identify the deformation space of a surface of genus 1 with two punctures.

THEOREM 1.3. *The space $T(\Sigma)$, where Σ is the (real) projective plane with two punctures, can be identified with the set of points of the upper half plane with imaginary part bigger than 1 and real part equal to zero.*

One can define a *Klein hyperbolic orbifold* as a non-orientable surface Σ , with finitely many (maybe zero) punctures, such that the covering from the upper half plane to Σ is ramified over a finiter number of points. The surface Σ^c carries an anticonformal involution σ , such that $\Sigma^c/\langle\sigma\rangle \cong \Sigma$. We have that $T(\Sigma)$ can be identified with the set of fixed points of the anti-conformal involution σ^* , induced by σ in $T(\Sigma^c)$. We say that the Teichmüller spaces of two non-orientable surface Σ_1 and Σ_2 are *real isomorphic*, if there exists a biholomorphic mapping $f:T(\Sigma_1^c) \rightarrow T(\Sigma_2^c)$, such that $f \circ \sigma_1^* = \sigma_2^* \circ f$. The following result is a generalisation of the Bers–Greenberg isomorphism for Riemann surfaces.

THEOREM 1.4 (Bers–Greenberg theorem for non-orientable surfaces). If $\Sigma_i, i = 1, 2$,

are two non-orientable hyperbolic orbifolds, with the same genus and number of ramification points, then the spaces $T(\Sigma_1)$ and $T(\Sigma_2)$ are real isomorphic.

This paper is organized as follows: in §2 we prove the uniformization theorem; §3 contains the proof of Theorem 1.4 and other results about isomorphisms of deformation spaces; finally, §4 has the examples.

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2. Uniformization. Classically, hyperbolic Klein surfaces are uniformized as the quotient of the upper half plane by a discrete group of dianalytic self-homeomorphisms (Möbius and anti-Möbius transformations with real coefficients), known as NEC (non-euclidean crystallographic) groups. In this section, we will prove a uniformisation theorem by a different type of groups, which are more suitable for computations. We will use these groups, in §4, to produce some *explicit* examples of deformation spaces of non-orientable surfaces.

We start by recalling some facts of uniformization of Riemann surfaces. A *partition* \mathcal{C} on a Riemann surface Σ , of genus $g \geq 2$, is a collection of simple closed disjoint curves, such that no curve of \mathcal{C} is homotopically trivial, and no two curves of \mathcal{C} are freely homotopically equivalent. A partition consists of at most $3g - 3$ curves; if this bound is attained, we say that the partition is *maximal*. See [18] for the proof of the existence of partition on surfaces.

THEOREM 2.1 (Maskit Uniformization Theorem, [13], [14]). *Given a Riemann surface Σ and a maximal partition $\mathcal{C} = \{a_1, \dots, a_{3g-3+n}\}$, there exists a Kleinian group G , known as a terminal regular b -group, such that:*

1. *there is a unique maximal simply connected set Δ of the Riemann sphere, where G acts discontinuously, and $g(\Delta) = \Delta$ for all $g \in G$;*
2. $\Delta/\Gamma \cong \Sigma$;
3. *to each curve of \mathcal{C} corresponds a maximal conjugacy class of cyclic subgroups of G generated by a parabolic transformation;*
4. *besides Σ , the group G uniformizes the $2g - 2$ thrice punctured spheres obtained from squeezing each curve of \mathcal{C} to a puncture;*
5. *G is unique up to conjugation by Möbius transformations.*

A *symmetry* σ on a Riemann surface is an anticonformal involution. If $F(\sigma)$ denotes the set of fixed points of σ , then we have that $\Sigma - F(\sigma)$ consists of at most two components. It is a well known fact that $\Sigma/\langle\sigma\rangle$ is orientable if and only if $\Sigma - F(\sigma)$ is not connected ([7], [17]). The classical result about the structure of $F(\sigma)$ is the following.

THEOREM 2.2 (Harnack). *If σ is a symmetry on a compact surface Σ of genus g , then $F(\sigma)$ is either empty or consists of s simple disjoint curves δ_j , with $s \leq g + 1$.*

This theorem can be improved as follows.

THEOREM 2.3 (Kra-Maskit). *In addition to the curves (if any) $\delta_1, \dots, \delta_s$, there exists closed curves $\delta_{s+1}, \dots, \delta_t$, such that:*

1. $\{\delta'_j\}_{j=1}^t$ is a collection of disjoint curves;

2. $\sigma(\delta_j) = \delta_j$, for all j ;
3. $\Sigma - \bigcup_{j=1}^l \delta_j$ consists of two components, Σ_1 and Σ_2 ;
4. σ interchanges Σ_1 and Σ_2 .

The existence of maximal partitions invariant under symmetries is a well known fact; but our proof is different from those in the literature (see, for example [17, 117–120]), but we include it here for the sake of completeness.

LEMMA 2.4. *Let Σ be a Riemann surface of genus $g \geq 2$, and let σ be a symmetry on Σ . Then there exists a maximal partition \mathcal{C} invariant under σ , that is, $\sigma(\mathcal{C}) = \mathcal{C}$.*

Proof. Let \mathcal{C}_1 denote the set of curves given by the Harnack–Kra–Maskit theorems. We claim that \mathcal{C}_1 is a partition on Σ . In fact, we have that if a curve of \mathcal{C}_1 is homotopically trivial, then Σ_1 and Σ_2 are discs, and therefore, Σ will be homeomorphic to the Riemann sphere. Similarly, if two curves of \mathcal{C}_1 are freely homotopic, we get that Σ is a torus.

If \mathcal{C}_1 is maximal, we are done. If not, let a be a curve such that $\mathcal{C}_2 = \mathcal{C}_1 \cup \{a\}$ is a partition. We claim that $\mathcal{C}_2 \cup \{\sigma(a)\}$ is a partition. This can be seen in three easy steps:

1. $\sigma(a)$ is not homotopically trivial, since σ is a homeomorphism, and a is not trivial (being a curve in a partition);
2. $\sigma(a)$ is not (freely) homotopically equivalent to any curve of \mathcal{C}_1 . If there is a curve δ in \mathcal{C}_1 , freely homotopic to $\sigma(a)$, then, applying σ , we would get that a is freely homotopic to δ , contradicting the fact that \mathcal{C}_2 is a partition;
3. $\sigma(a)$ is not freely homotopic to a . If these two curves are freely homotopic, then we have that a and $\sigma(a)$ bound a cylinder in Σ . Since these curves lie in different components of $\Sigma - \mathcal{C}_1$, we get that there is a curve, δ in \mathcal{C}_1 , in that cylinder. But this implies that a is homotopically equivalent to δ , which is again not possible.

Any non-orientable surface Σ has a double unramified cover by a Riemann surface Σ^c , called the *complex double* ([2, 37–40]). If Σ has genus g , then Σ^c has genus $g - 1$. Σ has a symmetry σ , such that $\Sigma^c / \langle \sigma \rangle \cong \Sigma$. We have now all the necessary tools to prove Theorem 1.1.

Proof of Theorem 1.1. Let Σ^c be the complex cover of Σ , and let σ be the symmetry on Σ^c such that $\Sigma^c / \langle \sigma \rangle \cong \Sigma$. By our hypothesis, Σ^c has genus greater than 1, so applying the Lemma 2.4 we obtain a σ -invariant maximal partition \mathcal{C} on Σ^c . Using the Maskit Uniformization Theorem, we get a Kleinian group G , uniformizing Σ^c in the invariant simply connected component Δ . We only need to show that the symmetry σ lifts to an anti-Möbius transformation, in the covering determined by G (i.e., it is of the form given in the statement of the theorem). For simplicity, assume first that σ is orientation preserving. Then σ induces a set of conformal mappings, $\sigma_j: S_j \rightarrow S_k$, among the parts S_1, \dots, S_{2g-2} of $\Sigma - \mathcal{C}$. The infinite Nielsen extension, \hat{S}_j , of S_j is a thrice punctured sphere, obtained from S_j by completing the holes to punctured discs. It is a classical fact that σ_j extends to a quasiconformal mapping, denoted by $\bar{\sigma}_j$, from \hat{S}_j to \hat{S}_k , with maximal dilatation $1 \leq K(\bar{\sigma}_j) \leq K(\sigma_j)$ ([4], [1]). Since σ_j is conformal, we have that its dilatation is equal to 1, and therefore $K(\bar{\sigma}_j) = 1$, that is, $\bar{\sigma}_j$ is also conformal. Let U_j be a component of $\pi^{-1}(S_j)$, where $\pi: \Delta \rightarrow \Sigma$ is the natural quotient mapping from Δ onto Σ , and let $G_j = \text{stab}(G, U_j) := \{g \in G; g(U_j) = U_j\}$. We have that the G_j 's are triangle groups with two invariant components; let U'_j be the component that does not contain Δ . Then, the

mapping σ_j induces a conformal mapping between U'_j and U'_k , for a proper choice of U_k . This can be done with all the components of $\pi^{-1}(S_j)$, and all the $j = 1, \dots, 2g - 2$, obtaining in this way a conformal self-mapping $\bar{\sigma}$, of $\Delta \cup g(U'_j)$. But this set is the region of discontinuity of G (that is, the set of points of the Riemann sphere where G acts discontinuously). Since G is finitely generated, we have that the complement of the region of discontinuity has measure zero. Therefore, the classical theory of quasiconformal mappings gives us a conformal automorphism of the Riemann sphere that extends $\bar{\sigma}$. Such mapping should be a Möbius transformation.

To complete the proof of the theorem it suffices to observe the following two facts, which are easy to prove:

1. the theory of quasiconformal mappings has a natural extension to cover the orientation reversing mappings [17]; and
2. Bers' results on Nielsen extensions can be applied to orientation reversing mappings.

We define a *ramification point* x on a Klein surface as a point such that the universal covering looks like $z \mapsto z^n$, in a neighborhood of x , (which corresponds to the points $z = 0$) for some finite positive integer n . The number n is called the ramification value of x . Ramification points correspond to fixed points of orientation preserving transformations, of finite order.

DEFINITION 2.5. *A puncture is a domain D in Σ satisfying the following conditions:*

1. D is homeomorphic to $\mathbb{D}^* = \{z \in \mathbb{C}; 0 < |z| < 1\}$;
2. for any sequence of points in \mathbb{D}^* converging to the origin, the corresponding sequence in the surface diverges;
3. if there are two patches on Σ , whose images contain some sets of the form $\{z \in \mathbb{C}; 0 < |z| < r \leq 1\}$ (that is, neighborhoods of the "missing point"), then the change of coordinates is holomorphic.

Given a Klein surface with ramification points and/or punctures, called a *Klein orbifold*, we define its *signature* as a collection of numbers (and a symbol) of the form $(g, \pm, n; \nu_1, \dots, \nu_n)$, where g is the genus of the surface, n is the number of special points, and ν_1, \dots, ν_n are the ramification values, with punctures having ramification value equal to ∞ . If the orbifold is orientable, then we take the symbol $+$, while $-$ is used for non-orientable surfaces. If all the ramification values are equal to ∞ , then we will write the signature as (g, \pm, n) . It is not difficult to see that if Σ has signature $(g, -, n; \nu_1, \dots, \nu_n)$, then the signature of Σ^c must be $(g - 1, +, 2n; \nu_1, \nu_1, \dots, \nu_n, \nu_n)$. A Klein orbifold Σ , is hyperbolic if and only if $kg - 2 + n - \sum_{j=1}^n \frac{1}{\nu_j}$ is positive, where $k = 1$ if Σ is not orientable, and $k = 2$ in the orientable case. Since the Maskit Uniformization Theorem and the theory of quasiconformal mappings extend to the case of surfaces with ramification points, we have that Theorem 1.1 can be applied also to hyperbolic Klein orbifolds.

3. Isomorphisms between Teichmüller spaces. A natural problem in deformation theory is to study which properties of a surface are determined by its Teichmüller space, and vice versa. More precisely, in this section we will see that, if the Teichmüller spaces

of two surfaces are equivalent, then the surfaces are homeomorphic. Reciprocally, if two Klein orbifolds have the same genus and number of ramification points, we will prove that their deformation spaces are isomorphic.

We start by recalling the definition of the modular group, and some basic facts about hyperelliptic surfaces. The *modular group* $\text{Mod}(\Sigma)$ of a non-orientable surface is the quotient of the group of diffeomorphisms, by those homotopic to the identity (in the case of Riemann surfaces, one takes only the orientation preserving diffeomorphisms). We have that $\text{Mod}(\Sigma)$ acts on $T(\Sigma)$ by pullback: given a mapping f , and a real analytic structure X , we define $f^*(X)$ as the unique structure on Σ that makes $f: (\Sigma, f^*(X)) \rightarrow (\Sigma, X)$ dianalytic. The mapping $f^*: [X] \rightarrow [f^*(X)]$ becomes dianalytic in the natural structure of $T(\Sigma)$.

We say that a non-orientable surface Σ is *hyperelliptic* if it is a double cover of the (real) projective plane (respectively, the Riemann sphere, in case of orientable surfaces). Hyperelliptic surfaces carry the so-called *hyperelliptic involution*, which is a dianalytic involution (holomorphic, in the case of orientable surfaces) α , such that $\Sigma/\langle\alpha\rangle$ is the projective plane. It is not hard to see that if Σ is hyperelliptic, so is its complex cover Σ^c , and that α lifts to the hyperelliptic involution j of Σ^c . Since the hyperelliptic involution on a Riemann surface is unique, we obtain the reciprocal result: if Σ^c is the complex double of a surface Σ , and Σ^c is hyperelliptic, then Σ is also hyperelliptic; moreover, the involution j can be pushed down to the hyperelliptic involution α in Σ . Hyperelliptic Klein surfaces were introduced in [9]; for general background and more results on this topic, see [6].

As in the case of Riemann surfaces, we have that the modular group acts effectively on Teichmüller space, except for a finite number of cases. See also [5, Theorem 8.1], where some of the results of the following proposition are proven.

PROPOSITION 3.1. *$\text{Mod}(\Sigma)$, for a hyperbolic non-orientable surface Σ , compact with finitely (maybe zero) punctures, acts effectively on $T(\Sigma)$, except for the following cases: the projective plane with two punctures, the Klein bottle with one puncture, or the connected sum of three projective planes.*

Proof. As usual, let Σ^c denote the complex double of Σ , and let σ be the involution associated to such covering. It is a well known fact ([17, p. 149] or below) that $T(\Sigma)$ can be identified with the set of fixed points $T(\Sigma^c)_{\sigma^*}$ of σ^* in $T(\Sigma^c)$. If f is a diffeomorphism of Σ , there is a unique orientation preserving lift, $F: \Sigma^c \rightarrow \Sigma^c$ (see [16, 20] and [2, 39]). By uniqueness, we have that F satisfies $F^* \circ \sigma^* = \sigma^* \circ F^*$. This proves that $\text{Mod}(\Sigma)$ embeds into the set $A = \{h^* \in \text{Mod}(\Sigma^c); h^* \circ \sigma^* = \sigma^* \circ h^*\}$. The modular group of a hyperbolic Riemann surface acts properly on the corresponding Teichmüller space, unless the signature of the surface is $(0, +, 4)$, $(1, +, 2)$, $(2, +, 0)$ or $(1, +, 1)$. Since complex covers have an even number of punctures, we get that only the first three signatures can give non-orientable surfaces. We therefore obtain that the cases where $\text{Mod}(\Sigma)$ may fail to act effectively correspond to the signatures $(1, -, 2)$, $(2, -, 1)$ and $(3, -, 0)$. This proves the first part of the proposition.

If, in the other hand, Σ has signature in the above list, we have that the only elements of $\text{Mod}(\Sigma)$ that do not act properly are the classes of the identity and the hyperelliptic involution. By the remarks before the theorem, we also have that these classes are the only elements that act trivially on the Teichmüller space of hyperbolic Klein surfaces.

It would be interesting to know whether the image of $\text{Mod}(\Sigma)$ is equal to the whole set A described in the above proof.

The proof of the following result is straightforward from the parallel result for Riemann surfaces. Nevertheless, the proposition is interesting, because it shows the great similarity between the theory of deformation of orientable surfaces, and that of non-orientable ones.

THEOREM 3.2 (Patterson theorem for non-orientable surfaces). *Let $\Sigma_i, i = 1, 2$, be two hyperbolic compact, with finitely many (possibly zero) punctures, non-orientable surfaces, and suppose that either Σ_1 or Σ_2 has genus not equal to 3. If $T(\Sigma_1)$ is real isomorphic to $T(\Sigma_2)$, then Σ_1 is homeomorphic to Σ_2 .*

Proof. It suffices to observe that if $T(\Sigma_1)$ and $T(\Sigma_2)$ are real isomorphic, then the spaces $T(\Sigma_1^c)$ and $T(\Sigma_2^c)$ are biholomorphic, and therefore, Σ_1^c and Σ_2^c are homeomorphic.

In order to prove Theorem 1.4, we need to review a basic concept of Teichmüller theory: quadratic differentials and Beltrami coefficients. Let $Q(\Sigma)$ denote the space of *bounded quadratic differentials* on a Klein surface. These are simply quadratic differentials, regular on the surface, with at most simple poles at the punctures, and with zeros of certain order (determined by the ramification value) at the ramification points (see, for example [10]). It is easy to see that the set $Q(\Sigma)$, for a non-orientable surface, can be identified with the subspace of elements of $Q(\Sigma^c)$, that are preserved by σ , that is, $\bar{\phi} = (\phi \circ \sigma)(\bar{\partial}\sigma)^2$. Here by $\bar{\partial}\sigma$ we mean $\partial\sigma/\partial\bar{z} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \sigma$. The dimension of $Q(\Sigma)$ over \mathbb{R} is equal to the dimension of $Q(\Sigma^c)$ over \mathbb{C} . The embedding from $Q(\Sigma)$ into $Q(\Sigma^c)$ is an isometry in the norm,

$$\|\varphi\| = \sup |\varphi(x)| \lambda^{-2}(x), \quad f \in Q(\Sigma),$$

where λ is the metric obtained by pushing down the Poincaré metric of the upper half plane onto the corresponding surface (and the supremum is taking over the whole surface). The cotangent bundle of $T(\Sigma)$ can be naturally identified with the space $Q(\Sigma)$.

Let Σ be a Klein surface, uniformized by the NEC group Γ . Let $M(\mathbb{H}, \Gamma)$ denote the space of *Beltrami differentials* for Γ . This set consists of (classes of) measurable functions μ , with support in the upper half plane, and L^∞ -norm less than one, satisfying $(\mu \circ \gamma)\bar{\gamma}'/\gamma = \mu$, if $\gamma \in \Gamma$ is orientation preserving, or $(\mu \circ \gamma)\bar{\gamma}'/\bar{\partial}\gamma = \bar{\mu}$, if $\gamma \in \Gamma$ reverses the orientation. For each $\mu \in M(\mathbb{H}, \Gamma)$, there is a unique quasiconformal homeomorphism w_μ , of the upper half plane, with dilatation μ , that fixes 0, 1 and ∞ . Two Beltrami coefficients, μ and ν , are *equivalent* if $w_\mu = w_\nu$ on the real line. The space of Beltrami differentials, quotiented by the above equivalence relation is the Teichmüller space $T(\Gamma)$ of the group Γ . It can be proven that $T(\Gamma)$ is naturally isomorphic to $T(\Sigma)$, where $\Sigma \cong \mathbb{H}/\Gamma$. It is easy to see that, if G is the subgroup of Γ consisting of the orientation preserving mappings, then $\mathbb{H}/G \cong \Sigma^c$. We can identify the Beltrami differentials for Γ with those Beltrami differentials for G , that are invariant under σ , that is, $(\mu \circ \sigma)\bar{\partial}\sigma/\bar{\partial}\sigma = \mu$. This allows us to identify the deformation space $T(\Sigma)$ with the set of fixed points of σ^* , $T(\Sigma^c)_{\sigma^*}$, in the deformation space of the complex double $T(\Sigma^c)$. In

this way we obtain a Teichmüller's lemma for non-orientable surfaces: on each equivalence class of Beltrami differentials there is a unique mapping with minimal dilatation, which is of the form $\mu = k\bar{\varphi}/|\varphi|$, with $\varphi \in Q(\Sigma)$, k a real number. With this background, we can provide two proofs of Theorem 1.4 of §1.

First proof of Theorem 1.4. Let Σ be a hyperbolic surface of signature $(g, -, n; \nu_1, \dots, \nu_n)$, where we assume that at least one of the ramification values is finite. Let Σ_0 be the surface of signature $(g, -, n; \infty, \dots, \infty)$, obtained by removing from Σ all the points with finite ramification value. Let Γ and Γ_0 be NEC groups uniformizing Σ and Σ_0 respectively. Define \mathbb{H}_Γ as $\mathbb{H} - \{\text{fixed points of elliptic elements of } \Gamma\}$. Then, by our hypothesis we have that $\mathbb{H} \neq \mathbb{H}_\Gamma$. Since $\Sigma_0 \cong \mathbb{H}_\Gamma/\Gamma$, we have a covering map $h: \mathbb{H} \rightarrow \mathbb{H}_\Gamma$; that makes the following diagram commutative.

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{h} & \mathbb{H}_\Gamma \\ & \searrow \pi_0 & \swarrow \pi \\ & & \Sigma_0 \end{array}$$

The function h induces a group homomorphism $\chi: \Gamma_0 \leftarrow \Gamma$, defined by the rule $h \circ \chi(\gamma) = \gamma \circ h$. The mapping h induces a mapping $h^*: M(\mathbb{H}, \Gamma_0) \rightarrow M(\mathbb{H}, \Gamma)$, between Beltrami coefficients, given by the expression $(h^*\mu) \circ h = \mu h' / \bar{h}'$ (see below for a proof of the fact that h is a holomorphic function). It is not hard to see, using the same arguments that in the orientable case, that h^* induces an real analytic bijection between the spaces $T(\Sigma_0)$ and $T(\Sigma)$. See, for example, [8] (or [3], for more details). By analytic continuation, we can extend h^* to a biholomorphic function between $T(\Sigma_0^c)$ and $T(\Sigma^c)$, that obviously commutes with the involutions σ_0 and σ , giving the desired isomorphism.

Second proof of Theorem 1.4. In this case, we will use the Bers–Greenberg theorem for Riemann surfaces. Consider the same setting as in the first proof. Let G and G_0 be the orientation preserving subgroups of Γ and Γ_0 respectively. Then we have $\mathbb{H}/G \cong \Sigma^c$ and $\mathbb{H}/G_0 \cong \Sigma_0^c/G \cong \Sigma_0^c$. We get that the following commutative diagram is commutative:

$$\begin{array}{ccc} & & \mathbb{H}_\Gamma \\ & \nearrow h & \downarrow \pi \\ & & \Sigma_0^c \\ & \nearrow f & \downarrow \rho \\ \mathbb{H} & \xrightarrow{\pi_0} & \Sigma_0 \end{array}$$

where π , π_0 and ρ are the natural projections. The mapping h is defined as in the first proof. The function f is the unique holomorphic mapping that makes the lower triangle commutative. We have that π is holomorphic (a covering of a Riemann surface by an open set of the complex plane), so the function h is holomorphic. This implies that the group homomorphism χ , of the first proof, takes the subgroup G_0 onto G . Since the surfaces Σ and Σ^c have the same genus and number of ramification points/punctures, we have that the spaces $T(\Sigma_0^c)$ and $T(\Sigma^c)$ are isomorphic, via the function h^* induced by h , as in the first proof. To prove Theorem 1.4 it suffices to show that h commutes with the

antiholomorphic involutions σ and σ_0 (that “produce” the surfaces Σ and Σ_0 , respectively). In other words, we have to show $h \circ \sigma_0 = \sigma \circ h$. But we have that $\pi \circ h \circ \sigma_0 = \pi \circ \sigma \circ h$, so $h \circ \sigma_0$ is equal to either $\sigma \circ h$ or $\sigma \circ h \circ \sigma$. Since this last function is holomorphic, we must have $h \circ \sigma_0 = \sigma \circ h$, as claimed. Identifying $T(\Sigma_0)$ and $T(\Sigma)$ with the set of fixed points of σ_0^* and σ^* in $T(\Sigma_0^c)$ and $T(\Sigma^c)$, respectively, we get the Bers–Greenberg theorem for non-orientable surfaces. \square

4. Examples. In this section, we will show with two examples, how the techniques of Kra of [12] can be applied to the case of non orientable surfaces. We will work with deformation spaces of Kleinian groups, which are equivalent (if the groups are chosen properly, for example, groups given by Theorem 1) to deformation spaces of Riemann or Klein surfaces (see [11] or [17] for more details).

In our first example, we consider a Klein surface, Σ , of signature, $(1, -, 2)$. Its complex double, Σ^c , has signature $(0, +, 4)$. A Kleinian group, G_α , uniformizing Σ^c is generated by the transformations

$$A = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}, \quad B_\alpha = \begin{bmatrix} -1 + 2\alpha & -2\alpha^2 \\ 2 & -1 - 2\alpha \end{bmatrix},$$

where $\text{Im}(\alpha) > 1$ (see above reference). The coordinate of G_α in the Teichmüller space $T(0, +, 4)$ (notation should be obvious) is given by the expression

$$\alpha = \text{cr}(f(A), f(B), f(AB), f(B_\alpha)).$$

Here cr denotes the cross ratio of four points in the Riemann sphere, chosen so that $\text{cr}(\infty, 0, 1, z) = z$, and $f(T)$ denotes the unique fixed point of the parabolic transformation T .

A maximal partition in Σ^c consists of a simple closed curve, say a_1 . We can assume that the punctures P_1 and P_2 lie on the same component of $\Sigma^c - a_1$. The Möbius transformation A corresponds to the partition curve. Let $\gamma_j, j = 1, \dots, 4$, be a small simple loop around the puncture P_j , oriented such that the puncture lies to the left of γ_j . The parabolic elements $T_1 = B, T_2 = (AB)^{-1}, T_3 = B_\alpha^{-1}$ and $T_4 = B_\alpha A$, correspond to these four loops. Without loss of generality, we can assume that T_j corresponds to γ_j . Consider on Σ^c the involution $\sigma = r \circ R$, where R is a rotation of 180 degrees on the axis of Figure 1, and r is an anticonformal reflection on a_1 . The anticonformal mapping σ has not fixed points, and the quotient $\Sigma^c / \langle \sigma \rangle$ has signature $(1, -, 2)$.

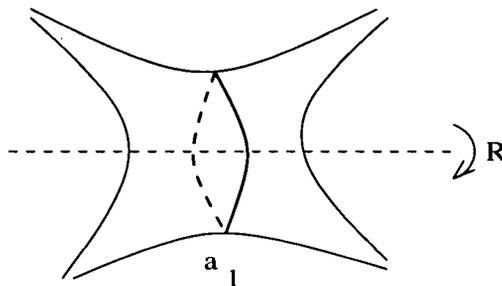


Figure 1. A sphere with four punctures.

In order to identify $T(\Sigma)$ in $T(\Sigma^c)$, we have to study the action of the mappings R and r on the group G_α . The transformation R interchanges the punctures that lie on the same component of $\Sigma^c - \{a_1\}$, that is, R sends γ_1 to γ_2 and γ_3 to γ_4 (up to free homotopy). The function r , not only interchanges the two components of $\Sigma^c - \{a_1\}$, but also changes the orientation of the loops, sending γ_1 to γ_4^{-1} and γ_2 to γ_3^{-1} . We have that R lifts to $A_1^{1/2}$, while r lifts to $\bar{r}(z) = z + \mu$, $\mu \in \mathbb{C}$, in the covering determined by the group G_α . Observe that, although a Möbius transformation may have many square roots, parabolic elements have only one, and therefore, $A_1^{1/2}$ is well defined.

From these observations we can compute the action of σ^* on $T(\Gamma_\alpha)$ as follows. First of all, observe that the group G_α is generated by A, B and B_α , with the property that AB and $A^{-1}B_\alpha^{-1}$ are parabolic elements. We will use the notation $G(A, B, B_\alpha)$ to emphasise this fact. Observe also that $AB_\alpha^{-1} = B_{\alpha-1}$. The mapping R sends $G(A, B, B_\alpha)$ to $G' = G(A, A^{1/2}BA^{-1/2}, A^{1/2}B_\alpha A^{-1/2}) = G(A, B^{-1}A^{-1}, B_{\alpha+1}^{-1})$. Since the transformation r is orientation reversing, we have that its action is given by conjugating the group G' into $G(A, \bar{r}B_{\alpha+1}\bar{r}^{-1}, \bar{r}AB\bar{r}^{-1})$. The mapping σ^* has therefore the form

$$\sigma^*(\alpha) = \text{cr}(\infty, \bar{\alpha} + 1 + \mu, \bar{\alpha} + 2 + \mu, 1 + \mu) = -\bar{\alpha}.$$

Thus, the Teichmüller space $T(\Sigma)$ can be identified with the set of points $\alpha \in T(\Sigma^c)$ such that $\text{Re}(\alpha) = 0$. By the work of Kra, we have that $T(\Sigma)$ is precisely the set $\{z \in \mathbb{C}; \text{Re}(z) = 0, \text{Im}(z) > 1\}$.

Consider now the case of a surface Σ of signature $(3, -, 0)$. The complex double of Σ is a Riemann surface Σ^c , of genus 2 without punctures. In [12] we can find a group G_τ uniformizing Σ^c , generated by the Möbius transformations:

$$\begin{aligned} A_1 &= \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix}, & A_2 &= \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}, & C_1 &= i \begin{bmatrix} \tau_1 & 1 \\ 1 & 0 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} -1 - 2\tau_2(1 - \tau_2) & -2(1 - \tau_2)^2 \\ 2\tau_2^2 & -1 + 2\tau_2(1 - \tau_2) \end{bmatrix}, \\ C_3 &= \begin{bmatrix} \tau_3\tau_2^2 + 2(1 - \tau_3)\tau_2 + \tau_3 - 2 & -\tau_3\tau_2^2 + (3\tau_3 - 2)\tau_2 - 2\tau_3 + 3 \\ \tau_3\tau_2^2 + (2 - \tau_3)\tau_2 - 1 & -\tau_3\tau_2^2 + 2(1 - \tau_3)\tau_2 + 2 \end{bmatrix}. \end{aligned}$$

The mapping A_j correspond to the curves a_j of the partition of Σ^c of Figure 2.

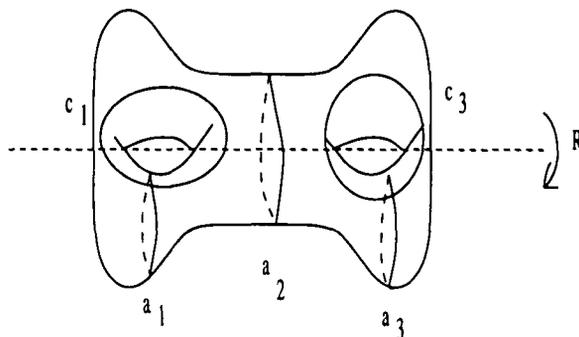


Figure 2. A Riemann surface of genus 2.

The C_j are loxodromic elements with the property that $B_1^{-1} := C_1^{-1}A_1C_1$ and $B_3 := C_3^{-1}A_3C_3$ are parabolic elements. The coordinates on $T(\Sigma^c)$ are given by the cross ratios

$$\begin{cases} \tau_1 = \text{cr}(f(A_1), f(B_1), f(A_2), C_1(f(A_1))), \\ \tau_2 = \text{cr}(f(A_2), f(A_1), f(B_1), f(A_3)), \\ \tau_3 = \text{cr}(f(A_3^{-1}), f(A_3A_2), f(A_2), C_3(f(A_3^{-1}))). \end{cases}$$

Let $\sigma = rR$ be defined in a similar way as in the previous example: R is a rotation by 180 degrees on the line of Figure 2, and r is an antiholomorphic reflection on the curve a_2 . The computation of the case $(1, -, 2)$ applies to the coordinate τ_2 , since the part corresponding to it (that is, $\Sigma - \{a_1, a_3\}$) is a surface of signature $(0, +, 4)$. So we get that the action of σ^* on τ_2 is $\tau_2 \mapsto -\bar{\tau}_2$. We have that the mapping R lifts to $A_2^{1/2}$, and the lift of r is

$$\tilde{r}(z) = \frac{(1 - \mu)\bar{z} + \mu}{\mu\bar{z} + 1 + \mu}.$$

Computing as in the previous example, and taking care of the fact that r reverses orientation, we see that $\sigma^*(\tau_1)$ is given by the cross ratio of the points

$$\begin{aligned} & f(\tilde{r}A_2^{1/2}A_2^{-1}A_3^{-1}A_2^{-1/2}\tilde{r}^{-1}), \quad f(\tilde{r}A_2^{1/2}A_3A_2^{-1/2}\tilde{r}^{-1}), \\ & f(A_2), \quad \tilde{C}_3(f(\tilde{r}A_2^{1/2}A_2^{-1}A_3^{-1}A_2^{-1/2}), \tilde{r}^{-1}), \end{aligned}$$

where $\tilde{C}_3 = \tilde{r}A_2^{1/2}C_3^{-1}A_2^{-1/2}\tilde{r}^{-1}$. This cross ratio gives $\sigma^*(\tau_1) = 1 - \bar{\tau}_3$. Similarly, one gets $\sigma^*(\tau_3) = 1 - \bar{\tau}_1$. Therefore, the Teichmüller space $T(\Sigma)$ can be identified with the set of points $(\tau_1, \tau_2, \tau_3) \in T(2, +, 0)$ such that

$$\begin{cases} \text{Re}(\tau_2) = 0 \\ \text{Re}(\tau_1) = 1 - \text{Re}(\tau_3) \\ \text{Im}(\tau_1) = \text{Im}(\tau_3), \end{cases}$$

which proves Theorem 1.2 of the introduction.

The above computations give us some other isomorphisms, different from those of the previous section. Observe that the transformation R is just the hyperelliptic involution on Σ^c . It is not hard to see that R^* acts like the identity in $T(\Sigma^c)$ ([15, 126]). The mapping r has a curve of the partition as the set of fixed points. We have that $\Sigma^c/\langle r \rangle$ is a sphere with one hole and two punctures, in the first example, or a torus with a hole in the second example. Let us denote this surfaces by S_1 and S_2 , respectively.

COROLLARY 4.1. *The spaces $T(S_1)$ and $T(S_2)$ are isomorphic to $T(1, -, 2)$ and $T(3, -, 0)$ respectively.*

Isomorphisms between deformation spaces of orientable and non-orientable surfaces, as those of the above corollary, do not happen if the genus is bigger than 2 [17, 152]).

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